Unpredictable Iterations

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Lothar Collatz has defined the function

$$g(n) = \begin{cases} n/2 & (n \text{ even}) \\ 3n+1 & (n \text{ odd}) \end{cases}$$

and conjectured that for any positive integer n, there exists k such that

$$g^k(n) = 1.$$

The conjecture is still unproven, but has been verified for $n \leq 10^9$ by D.H. and Emma Lehmer and J.L.Selfridge. It prompts consideration of more general functions

$$g(n) = a_i n + b_i \quad (n \equiv i \mod P)$$

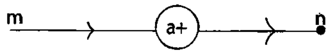
where $a_0, b_0, \ldots, a_{P-1}, b_{P-1}$ are rational numbers chosen so that $g(n)$ is always integral. What can be predicted about the iterates $g^k(n)$? Here we show that even when $b_i = 0$, the behavior is unpredictable, in general.

THEOREM. *If f is any computable function, there is a function g such that*

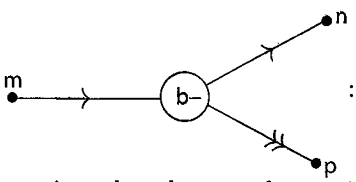
(1) *$g(n)/n$ is periodic (with rational values)*
(2) *$2^{f(n)} = g^k(2^n)$, where k is minimal positive subject to $g^k(2^n)$ a power of 2.*

COROLLARY. *There is no algorithm, which, given a function g with $g(n)/n$ periodic, and given a number n, determines whether or not there is k with $g^k(n) = 1$.* *The word "computable" will mean "computable by a Minksy program", as defined below. This is equivalent to (partial) recursive.*

Minsky machines. These have *registers* $a, b, c, \ldots$ capable of holding arbitrary non-negative integers, and two types of *order*:

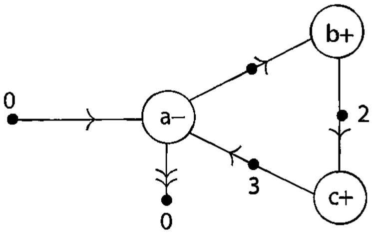


: at the point m in the program, we add 1 to register a, and proceed to n.



: at the point m of the program, we subtract 1 from register b and proceed to n, if $b > 0$, while if $b = 0$ we simply proceed to p.

A *Minsky program* consists of such orders:



All entry and exit points are conventionally labelled 0. Other labels are distinct positive integers.

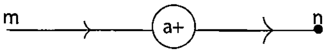
The example is a program to add the contents of register a to each of registers b and c, and clear a to 0. It is easy to write Minsky programs to simulate the orders of more conventional computing machines (multiplication, etc.) so that functions computable by such machines are computable in our sense. Equally, all (partial) recursive functions are Minsky computable in the sense that there is a Minsky program which started with register contents $n, 0, 0, \dots$ ends with register contents $f(n), 0, 0, \dots$.

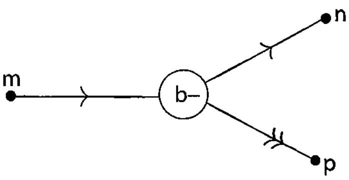
Vector games. Suppose wo are given a finite list of vectors with integer coordinates and the same dimension, e.g.:

$$
\begin{array}{c}
(0,0,0 \mid 1,-1) \\
(-1,0,0 \mid -3,1) \\
(0,0,0 \mid -3,0) \\
(0,0,1 \mid -2,3) \\
(0,1,0 \mid -1,2) \\
(-1,0,0 \mid -0,1) \\
(0,0,0 \mid -0,0)
\end{array}
$$

Then we can play the following game. Starting with a vector v with *non-negative* integer coordinates, add to v the first vector from the list which preserves this property. What happens when we repeat this indefinitely?

We show that for any computable f, there is a vector game, which when started at $(n, 0, 0, \dots)$ reaches $(f(n), 0, 0, \dots)$. In fact we use vector games to simulate Minsky programs, as follows.

We have one coordinate for each register, and two more coordinates (after the vertical bar). For an order 

we have a vector $(\overset{a}{1}, 0, \dots \mid -m, n)$ and for an order 

we have a pair of vectors $(0, \overset{b}{-1}, 0, \dots \mid -m, n)$, $(0, \overset{b}{0}, 0, \dots \mid -m, p)$ in that order. These vectors are listed in *decreasing* order of m, and preceded by the vector $(0, 0, \ldots \mid 1, -1)$. In this way, our example of a Minsky program yields the given vector-list.

It is easy to see how the vector game so defined simulates the corresponding Minsky program. Maybe some industrious reader will produce a short prime-generating vector game.

Rational games. Suppose we have a finite list of rational numbers $r_1, r_2, \ldots, r_s$. Then we can play another game. Starting with an integer n, replace n by $g(n) = r_i \cdot n$ for the least i for which this is an integer. What happens when we iterate?

Obviously, if we replace a vector $(a, b, c, \ldots)$ by the number $2^a 3^b 5^c \ldots$, we obtain this game from our previous one. So we have proved that for any computable function f, there is a rational game, which started at 2^n, gets to $2^{f(n)}$ without finding any intermediate power of 2.

Our main theorem now follows from the observation that for this function $g(n)$, $g(n)/n$ is periodic, with period dividing the least common denominator of the r_i. Since it is undecidable whether or not a given partial recursive function is everywhere defined and identically zero, we obtain also the Corollary.

Of course, particular games of this type can still have predictable properties, so that (for instance) our theorem says nothing about the Collatz game. But it does prohibit any general solution to games of this type, and also shows that there exist special cases for which the prediction problem is unsolvable.

It is amusing to note that the Theorem contains the Kleene Normal Form Theorem for recursive functions, since the functions $g(n)$, 2^n, etc., are obviously primitive recursive.

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Editorial Commentary

(1) This paper was one of the earliest mathematical papers on the $3x + 1$ problem. The results here grew out of J. H. Conway's interests in the theory of computation and games. Some of his early studies in computation appear in his 1971 book *Regular Algebra and Finite Machines* [**2**]. In this paper Conway challenges some industrious reader to produce a prime-generating vector game; this challenge was answered in 1983 by R. K. Guy [**12**]. The computational model underlying this paper using fractions was later formalized as FRACTRAN by Conway [4] in 1987. This name was clearly intended as a pun on FORTRAN. Just as FORTRAN has since become Fortran, FRACTRAN has since become Fractran.

(2) The *Minsky machines* defined in this paper are the counter machines described in the textbook of Marvin Minsky [**15**, Sect. 11.1]. Minsky originally introduced them to show the unsolvability of Post's problem of 'tag' in 1961 ([**14**]).

(3) The *Kleene Normal Form theorem* for partial recursive functions (Kleene [**13**, Sec. 63, Theorem XIX]) states, when specialized to the one-variable case, the following:

Theorem. *There are primitive recursive functions $U(y)$ and $T_1(y,e,x)$ such that for any one-variable partial recursive function $f(x)$ there is some positive integer e ("Gödel number") such that*

$$f(x) = U(\mu y T_1(y,e,x) = 0)$$

where μy is a quantifier that enotes the least positive integer y such that $T_1(y,e,x) = 0$, and is undefined otherwise ([**13**, Sec. 57]). *Furthermore the primitive recursive function $U(y)$ has the additional property that*

$$f(x) = U(y), \text{ for any } y \text{ such that } T_1(y,e,x) = 0.$$

Kleene [**13**, Sec. 58, Theorem IX] gives a construction of suitable primitive recursive functions U and $T_1(y,e,x)$.

(3) John Horton Conway (1937–) is well known for his work in geometry, group theory, algebra and various other subjects of his own invention. He was educated in Cambridge at Gonville and Caius College, where he received his BA in 1959. He completed a PhD at Cambridge with supervisor Harold Davenport in 1964 on Waring's problem for degree five, showing that each integer can be written as a sum of 37 fitth powers. This result was not published because it was independently proved and published first by Chen Jing-run. He is most well known for the construction of several of the sporadic finite simple groups, the *Conway groups*, which are obtained from automorphism groups of the Leech lattice. With Simon P. Norton he proposed conjectures giving connections between the Monster simple group and modular forms, termed "monstrous moonshine". These conjectures were eventually proved by his student Richard Borcherds, who was awarded a Fields medal in part for this work. Some of his work on groups and lattices appears in the book with N. J. A. Sloane on *Sphere Packings, Lattices and Groups* ([**10**]). In algebra he studied Quaternions and Icosians, both introduced by William Rowan Hamilton (see [**11**]). He is known for inventing a new system of numbers, the field of "surreal numbers", which include all ordinal numbers, and for an extended theory of combinatorial mathematical games (see [**3**], [**1**]). The set of all (equivalence classes of) games form a group containing the field of surreal numbers. This theory includes as a special case, the game of *Life*, which is a 2-dimensional cellular automaton, now proved to be universal. He has published on a wide range of subjects, on each of which he has his own viewpoint, full of stimulating ideas. These include books on algebra and computation ([**2**]), on numbers (with Richard Guy [**8**]) and on integral quadratic forms ([**7**]). Two of his more remarkable results involving recursive constructions are his study of "audioactive decay" sequences (Conway [**5**]) and constructions of lexicographic codes ([**9**], [**6**]).

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