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PORTFOLIO TURNPIKE THEOREMS FOR CONSTANT POLICIES

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This paper develops general overtaking techniques for studying the asymptotic properties of portfolio policies optimal with respect to a terminal utility valuation. For a restricted class of utility functions the sequence of optimal constant (non-revised) portfolio policies formed as the horizon recedes into the future is shown to converge. Furthermore, for utility functions unbounded above and below, this turnpike policy need not be the policy associated with the minimal constant relative risk aversion function that bounds the valuation function from above. Finally, an analogy between the portfolio turnpike problem and the turnpike problem of growth theory is studied.

Introduction

The interplay between uncertainty and dynamics provides some of the most difficult and important problems in financial economics. The dynamics of portfolio policies is at the core of this area and has been examined by a number of authors.[1] This paper will study a subclass of dynamic portfolio problems coming under the heading of turnpike theory.

In its most general form, the portfolio turnpike problem seeks the accumulation of wealth so as to maximize some criteria applied to wealth at a terminal date. The path of approach to the terminal wealth is of concern only in so far as it leads to a higher terminal valuation. The virtue of the turnpike problem lies largely in the fact that it is the simplest context within which serious dynamic problems about optimality under uncertainty can be posed and its solution often provides the basic intuition for more complex problems.

To explicate all of this, the general portfolio turnpike problem takes the following form. We seek to maximize the expected (von Neumann–Morgenstern)

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[1]Referencing selectively but not exhaustively, Breiman (1961) and Latane (1957) and more recently Hakansson (1971b), Markowitz (1972), Samuelson (1971) and Samuelson and Merton (1972) have studied the properties of the policy which maximize the expected log return, the so-called maximal growth criteria. Mossin (1968) and Hakansson (1971a) have looked at stationary portfolio policies and Merton (1971) has analyzed the optimal consumption withdrawal policy when returns are governed by diffusion processes.

utility, $U_T(\cdot)$, of random terminal wealth, $\tilde{w}_T$, over a horizon of T periods. Thus, if A_T denotes the feasible set of portfolio policies, $\langle\alpha\rangle$, available in the T period problem, we seek

$$\max_{\langle\alpha\rangle \in A_T} E\{U_T(\tilde{w}_T)\}. \quad (1)$$

Turnpike theory is largely devoted to the study of the asymptotic behavior and properties of the solution to (1) for large T, and that will be the central focus of this paper. Sect. 1 specializes our problem still further. In particular, we will be exclusively concerned with constant portfolio policies, i.e., policies that are unchanging in each period for a given horizon, and we will also assume that the stochastic environment is independently and identically distributed across periods. In a second paper, Ross (1974), we develop the general portfolio turnpike problem. Sect. 2 studies some familiar stationary solutions to the general turnpike problem that coincide with the constant solutions of our problem. Sect. 3 develops the central theorems of the paper and sect. 4 relates our findings to the turnpike literature of growth theory. Sect. 5 summarizes and concludes the paper and describes some areas of future research.

Section 1

The problem we will concern ourselves with is that of examining the asymptotic properties of the solution to (1) for large T. To facilitate this, we will begin by assuming that (up to a cardinal equivalence)

$$U_T(w) = U(w),$$

so that the terminal criterion can alter by at most some discount factor and a location factor as the horizon is altered. The stochastic investment environment is conceived of as follows. There are two assets, one of which is riskless and offers a return factor of

$$r > 1,$$

so that $r-1$ is the interest rate. The other asset is risky and offers a random return of

$$r+\tilde{x},$$

i.e., $\tilde{x}$ is the random return premium over the riskless asset. The individual is assumed able to form portfolios of arbitrary amounts in these two assets, although generally no borrowing (or short sales) will be permitted. The random return is then given by

$$(1-\alpha)r+\alpha(r+\tilde{x}) = r+\alpha\tilde{x},$$

where α is the proportion of wealth placed at risk.

Proceeding recursively, we have

$$\tilde{w}_\tau = w_{\tau-1}[r+\alpha_\tau^T \tilde{x}_\tau],$$

where subscripts indicate time periods, and the horizon is superscripted.

The general portfolio turnpike problem can now be written as

$$\max_{\langle\alpha_\tau\rangle \in A_T} E\{U(\tilde{w}_\tau)\}, \tag{2}$$

where

$$\tilde{w}_T = \tilde{w}_{T-1}[r+\alpha_T^T \tilde{x}_T]$$
$$= w \prod_1^T (r+\alpha_\tau^T \tilde{x}_\tau).$$

The two basic simplifying assumptions of this paper are:

(i) $\langle \tilde{x}_\tau \rangle$ is a vector of independently and identically distributed random variables,

(ii) the set of policies we shall deal with are constant portfolio policies, i.e., $\alpha_\tau^T = \alpha^T$, a predetermined constant for each horizon.

Assumption (i) needs little explanation. Its principal weakness is that it abstracts from intertemporal dependence and we shall have more to say about it further on. Assumption (ii) abstracts from a somewhat more serious induced dependence. In general, we would expect that for any finite horizon α_τ^T, the optimal policy for the τth period, will be dependent on the wealth inherited from the previous period. Thus,

$$\alpha_\tau^T = \alpha_\tau^T(w_{\tau-1}).$$

(With a stochastically interdependent environment α_τ^T would also depend on the current state of the world but assumption (i) permits us to ignore any such dependence). Nevertheless, it is our contention that assumption (ii) is a useful starting point for the development of a complete turnpike theory. For one thing, in a complex problem, where the computational costs associated with finding the true optimal policy sequence might be exorbitant, this simpler problem is of interest. More importantly, it permits us to develop a class of policies analogous to the proportionate turnpike paths of economic growth theory. In effect, we might conjecture that solving the portfolio turnpike problem under assumption (ii) is a technique for finding the turnpike.[2]

The central problem of turnpike theory is that of analyzing the behavior of α_1^T for large T, and in our problem we examine the optimal α^T for large T. As we shall see, the solution to this latter problem, as for turnpike analysis in general, is

[2]Of course, this conjecture rests heavily on the assumption that for very long periods the optimal initial policy for the T period problem $\alpha_1{}^T$ should be quite insensitive to the current wealth level. Unfortunately, this need not be the case in the neighbourhoods of the origin.

a sufficient justification for posing the problem in the first place; the solution will provide us with some deep insights into the dynamic behavior of optimal stochastic programs.

We conclude this section, then, with our specific problem

$$\max_{\alpha} E\{U(\tilde{w}_T)\}, \tag{3}$$

where

$$\tilde{w}_T = w \prod_{1}^{T} (r+\alpha\tilde{x}_\tau). \tag{4}$$

The solution to (3) is denoted α^T, and we will study how α^T behaves for large T.

Section 2

There are a number of specific cases for which the solution to (2) is known. In particular, if the utility function is a power function of the form

$$U(w) = (1/\beta)w^\beta;\ \beta < 1,$$

or

$$U(w) = \log w,$$

then [see Mossin (1968)] the constant solution is unaltered by the horizon and is, in fact, the solution to the general turnpike problem.

To see this, we have

$$\begin{aligned} E\{U(\tilde{w}_T)\} &= (1/\beta)w^\beta E\{[\prod_{1}^{T} (r+\alpha_\tau^T\tilde{x}_\tau)]^\beta\} \\ &= (1/\beta)w^\beta E\{\prod_{1}^{T} (r+\alpha_\tau^T\tilde{x}_\tau)^\beta\} \\ &= (1/\beta)E\{\tilde{w}_{T-1}^\beta A_\beta(\alpha_T^T)\}, \end{aligned}$$

where

$$A_\beta(\alpha) \equiv E_{x_T}\{(r+\alpha\tilde{x}_T)^\beta\}. \tag{5}$$

Now, $(1/\beta)A_\beta(\alpha)$ is concave, since

$$A_\beta''(\alpha) = E\{(\beta-1)(r+\alpha\tilde{x})^{\beta-2}\tilde{x}^2\} < 0, \tag{6}$$

and has a unique maximum attained at α_β. (We will assume throughout that the maximum is attained in the interior of [0,1]; see the argument at the beginning of sect. 3.) Hence, by backwards induction

$$\alpha_T^T = \alpha_\beta,$$

and

$$\alpha_\tau^T = \alpha_\beta;\ \tau = 1, \ldots, T.$$

The argument for $U(w) = \log w$ is similar.

The simplicity and tractability of the power functions have long commended them for study. Furthermore, it can be shown that they have desirable axiomatic properties associated with stationarity as well. This suggests that a useful way to study the turnpike problem is to consider combinations of these 'good' functions.[3]

Section 3

Leaving the logarithmic case for last, we will begin by considering terminal utility functions of the form

$$U(w) = \sum_i a_i w^{\beta_i}, \tag{7}$$

where the β_i are ordered,

$$\beta_1 > \beta_2 > \ldots > \beta_n.$$

For (strict) concavity and monotonicity, we must have

$$\beta_1 < 1,$$

and monotonicity will require both

$$a_1 \gtrless 0 \quad \text{as} \quad \beta_1 \gtrless 0, \tag{8}$$

and

$$a_n \lesseqgtr 0 \quad \text{as} \quad \beta_n \lesseqgtr 0. \tag{9}$$

If (8) did not hold, $U(w)$ would decrease for large w, and if (9) were violated, it would decrease for w near 0. As a final point we will define $U(w)$ only on the positive orthant and assume that it is an improper function taking the value $-\infty$ on the negative orthant. This will immediately bound the portfolio weight, α, from above, if there is mass to negative values of $\tilde{x}$. By monotonicity and concavity it can be shown that if $\bar{x} > 0$, then α will always be positive so that without loss of generality, we will assume that α is restricted to the half-open unit interval, (0,1].[4]

Our study of the turnpikes associated with functions of the form of eq. (7) will

[3]A second class of functions for which the solution to (2) is well-known is the exponential or constant absolute risk aversion functions,

$$U(w) = -e^{-Aw}.$$

While the problem of this paper can be examined with such functions, the techniques are little different than the ones we employ and, as such, we will not treat this case explicitly. A forthcoming paper, Ross (1974), treats this case within the context of the general turnpike problem. See, also, the discussion of sect. 5 of the present paper.

[4]This is tantamount to assuming that for any positive ε, $\tilde{x}$ can attain a value less than $\varepsilon - r$ with positive probability.

depend on an understanding of the behavior of the functions $A_\beta(\alpha)$ introduced in sect. 2. From eq. (6) it is easy to see that

$$A_\beta(\alpha) \text{ is } \begin{cases} \text{concave for } 0 < \beta < 1, \\ \text{convex for } \beta < 0. \end{cases}$$

Depending on the values of β_1 and β_n, the work divides naturally into three cases and we will prove a preliminary lemma that covers each of these cases. The lemma tells us that the β_1 and β_n components dominate the intermediate values.

Lemma 1: If

$$\beta_1 > \beta_2 > \beta_3,$$

then $(\forall \alpha \in [0,1])$

$$A_{\beta_2}(\alpha) < A_{\beta_1}(\alpha) \vee A_{\beta_3}(\alpha).\text{[5]} \tag{10}$$

In particular, setting $\beta_3 = 0$ $(\exists \delta > 0)$ such that

$$A_{\beta_2}(\alpha) + \delta < A_{\beta_1}(\alpha) \vee 1, \tag{11}$$

and setting $\beta_1 = 0$ yields

$$A_{\beta_2}(\alpha) + \delta < A_{\beta_3}(\alpha) \vee 1. \tag{12}$$

Proof: For any $z > 0$, z^β is a strictly convex function of β. This can be verified by differentiation,

$$\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}\{z^\beta\} = z^\beta(\log z)^2 > 0. \tag{13}$$

Since $A_\beta(\alpha)$ is [by definition (5)] an average of such functions it, too, must be a strictly convex function of β. Proposition (10) is a consequence of this convexity.

Compactness now assures that $(\forall c < 1)$ (11) and (12) will hold on $[0, c]$, and this result extends directly for (11) to $[0, 1]$. A special argument is required for (12) since $B_2 < 0$ implies that $A_{\beta_2}(\alpha) \to \infty$ as $\alpha \to 1$. In this case, however, by Jensen's inequality

$$\begin{aligned} A_{\beta_3}(\alpha) &\equiv E\{(r+\alpha\tilde{x})^{\beta_3}\} \\ &= E\{[(r+\alpha\tilde{x})^{\beta_2}]^{\beta_3/\beta_2}\} \\ &> [A_{\beta_2}(\alpha)]^{\beta_3/\beta_2}, \end{aligned} \tag{14}$$

which guarantees (12) for $A_{\beta_2}(\alpha)$ bounded above unity. Q.E.D.

[5] We will employ the convenient notation

$$x \vee y \equiv \max\{x, y\},$$

and

$$x \wedge y \equiv \min\{x, y\}.$$

Now, consider solving the constant turnpike problem (3) for a utility function of the form

$$U(w) = a_1 w^{\beta_1} + a_2 w^{\beta_2}. \tag{15}$$

For a constant policy, α, we have

$$\begin{aligned} E\{U(\tilde{w}_T)\} &= a_1 E\{\tilde{w}_T^{\beta_1}\} + a_2 E\{\tilde{w}_T^{\beta_2}\}, \\ &= a_1 w^{\beta_1} E\{\prod_1^T (r+\alpha\tilde{x}_\tau)^{\beta_1}\} \\ &\quad + a_2 w^{\beta_2} E\{\prod_1^T (r+\alpha\tilde{x}_\tau)^{\beta_2}\}, \\ &= a_1 w^{\beta_1} \prod_1^T E\{(r+\alpha\tilde{x}_\tau)^{\beta_1}\} \\ &\quad + a_2 w^{\beta_2} \prod_1^T E\{(r+\alpha\tilde{x}_\tau)^{\beta_2}\}, \\ &\equiv a_1 w^{\beta_1} [A_{\beta_1}(\alpha)]^T \\ &\quad + a_2 w^{\beta_2} [A_{\beta_2}(\alpha)]^T. \end{aligned}$$

Differentiating with respect to α yields

$$\begin{aligned} \frac{\partial E\{U(\tilde{w}_\tau)\}}{\partial \alpha} &= T a_1 w^{\beta_1} [A_{\beta_1}(\alpha)]^{T-1} \frac{\partial A_{\beta_1}}{\partial \alpha} \\ &\quad + T a_2 w^{\beta_2} [A_{\beta_2}(\alpha)]^{T-1} \frac{\partial A_{\beta_2}}{\partial \alpha}, \end{aligned} \tag{16}$$

and the first order condition for a maximum is given by

$$a_1 w^{\beta_1} [A_{\beta_1}(\alpha^T)]^{T-1} \frac{\partial A_{\beta_1}}{\partial \alpha} + a_2 w^{\beta_2} [A_{\beta_2}(\alpha^T)]^{T-1} \frac{\partial A_{\beta_2}}{\partial \alpha} = 0.$$

From eq. (16) it is easy to see that if $A_{\beta_1}(\alpha) > A_{\beta_2}(\alpha)$, then for large T the behavior of $\partial E\{U(\tilde{w}_T)\}/\partial \alpha$ is governed by $\partial A_{\beta_1}/\partial \alpha$ and, conversely, if $A_{\beta_1}(\alpha) < A_{\beta_2}(\alpha)$, then the behavior is governed by $\partial A_{\beta_2}/\partial \alpha$ for large T. This observation will enable us to find the turnpikes for functions of the class (15).

Figs. 1 and 2 display two typical cases of A_{β_i} functions from which composite expected utility functions are formed. In Fig. 1 we have $1 > \beta_1 > \beta_2 > 0$ and the assumption is made here as it is throughout the paper that α_{β_1} and α_{β_2} are both internal to the unit interval. Consider a utility function of the form (15), where A_{β_1} and A_{β_2} are as in fig. 1 and where, for the moment, we assume that α^T does indeed converge to a turnpike α^*. Could α^* be outside of $[\alpha_{\beta_2}, \alpha_{\beta_1}]$? Clearly not, since, outside of this range, by eq. (16), both terms of $\partial E\{U(\tilde{w}_T)\}/\partial \alpha$ are positive for $\alpha < \alpha_{\beta_2}$ and negative for $\alpha > \alpha_{\beta_1}$. Could α^* be less than α_{β_1}?

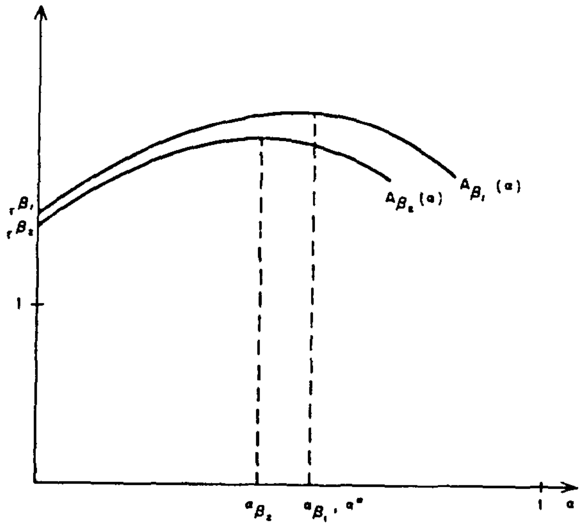

Fig. 1. One period expected utility as a function of the portfolio policy (α) for two utility functions with constant relative risk aversion coefficients ($1-\beta_1 < 1-\beta_2$) less than unity.

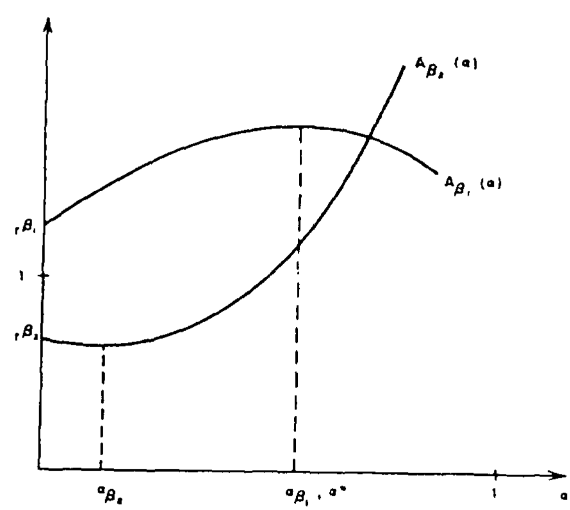

Fig. 2. One period expected utility as a function of the portfolio policy (α) for utility functions with constant relative risk aversion coefficients, $1-\beta_1 < 1$, and $1-\beta_2 > 1$.

Lemma 1 assures that $A_{\beta_1}(\alpha) > A_{\beta_2}(\alpha)$ for $\alpha \in [\alpha_{\beta_2}, \alpha_{\beta_1}]$ as in fig. 1. It follows that for large T, $\partial E\{U(\tilde{w}_T)\}/\partial\alpha$ has the sign of $\partial A_{\beta_1}/\partial\alpha$ which is positive everywhere except at α_{β_1}. If there is a turnpike, then it must be

$$\alpha^* = \alpha_{\beta_1}.$$

A similar line of reasoning applies for the situation of fig. 2 where $\beta_1 > 0 > \beta_2$. Recall that we will have $a_2 < 0$ so that our interest is in *minimizing* rather than *maximizing* the A_{β_2} contribution. Clearly then we must have $\alpha^* \leqslant \alpha_{\beta_1}$ since we will always set $\alpha^T \leqslant \alpha_{\beta_1}$, and, as before, we must in fact set

$$\alpha^* = \alpha_{\beta_1}.$$

The analysis is even more direct than the use of the first order condition would suggest. In either figs. 1 or 2, it can be seen that if $\alpha^* \neq \alpha_{\beta_1}$, then we can improve $E\{U(\tilde{w}_T)\}$ by moving closer to α_{β_1}. This will raise $A_{\beta_1}(\alpha)$ which can be thought of as the growth factor associated with the w^{β_1} portion of the utility function and, consequently, $E\{U(\tilde{w}_T)\}$ will asymptotically grow at a higher rate. This viewpoint is exploited in our first theorem.

Theorem 1: Let

$$U(w) = \sum_{i=1}^{n} a_i w^{\beta_i}$$

with $\beta_1 > 0$, $\beta_1 > \ldots > \beta_n$, and with

$$A^* \equiv A_{\beta_1}(\alpha_{\beta_1}) > A_{\beta_n}(\alpha_{\beta_1}), \tag{17}$$

if $\beta_n < 0$. It follows that

$$\alpha^T \to \alpha^* = \alpha_{\beta_1}, \tag{18}$$

uniformly on compact intervals of w not containing 0.

Before proving Theorem 1, we should note that assumption (17) simply assures that A_{β_1} and A_{β_n} are as in fig. 2 for $\beta_n < 0$. For $\beta_n > 0$, Lemma 1 proves (17).

Proof: Suppose, to the contrary, that $(\exists\varepsilon > 0)$ such that on a subsequence

$$|\alpha^T - \alpha_{\beta_1}| > \varepsilon. \tag{19}$$

By the strict concavity of A_{β_1}, $(\exists\delta > 0)$ such that $(\forall\alpha)$ $|\alpha - \alpha_{\beta_1}| > \varepsilon$ implies

$$A_{\beta_1}(\alpha) < A_{\beta_1}(\alpha_{\beta_1}) - \delta \equiv A^* - \delta.$$

In other words, if the policy is bounded away from α_{β_1}, then its growth factor is bounded below the maximum growth factor.

It follows that on the subsequence

$$\begin{aligned}E\{U(\tilde{w}_T)\} &= \sum_i a_i E\{\tilde{w}_T^{\beta_i}\} \\ &= \sum_i a_i w^{\beta_i} E\{\prod_1^T (r+\alpha^T\tilde{x}_\tau)^{\beta_i}\} \\ &= \sum_i a_i w^{\beta_i}[A_{\beta_i}(\alpha^T)]^T \\ &= a_1 w^{\beta_1}[A_{\beta_1}(\alpha^T)]^T \\ &\quad + \sum_{i\neq 1} a_i w^{\beta_i}[A_{\beta_i}(\alpha^T)]^T \\ &\leqslant a_1 w^{\beta_1}[A^*-\delta]^T + \sum_{i\neq 1} a_i w^{\beta_i}[A_{\beta_i}(\alpha^T)]^T.\end{aligned}$$

From Lemma 1, for $\beta_i > 0$, $(\exists\delta_i > 0)$ such that

$$A_{\beta_i}(\alpha^T) \leqslant A_{\beta_i}(\alpha_{\beta_i}) < A^*-\delta_i,$$

and for $\beta_i < 0$, $(\exists\theta_i > 0)$ such that

$$\begin{aligned}A_{\beta_i}(\alpha^T) &< [A_{\beta_n}(\alpha^T)-\theta_i] \vee [1-\theta_i] \\ &\leqslant [A_{\beta_n}(\alpha^T)-\theta_i] \vee [A^*-\theta_i],\end{aligned}$$

where we may take $\theta_i < \delta_i$.

Hence, along the subsequence,

$$\begin{aligned}\limsup \frac{E\{U(\tilde{w}_T)\}}{[A^*]^T} &\leqslant \limsup a_1 w^{\beta_1}\left[\frac{A^*-\delta}{A^*}\right]^T \\ &\quad + a_n w^{\beta_n}\left[\frac{A_{\beta_n}(\alpha^T)}{A^*}\right]^T \\ &\quad + \sum_{i\neq 1,n} |a_i| w^{\beta_i}\left[\frac{[A_{\beta_n}(\alpha^T)-\theta_i] \vee [A^*-\theta_i]}{A^*}\right]^T, \\ &= \limsup a_n w^{\beta_n}\left[\frac{A_{\beta_n}(\alpha^T)}{A^*}\right]^T \\ &\quad + \sum_{i\neq 1,n} |a_i| w^{\beta_i}\left[\frac{A_{\beta_n}(\alpha^T)-\theta_i}{A^*}\right]^T, \\ &= \limsup \left[\frac{A_{\beta_n}(\alpha^T)}{A^*}\right]^T \Bigg\{a_n w^{\beta_n} \\ &\quad + \sum_{i\neq 1,n} |a_i| w^{\beta_i}\left[\frac{A_{\beta_n}(\alpha^T)-\theta_i}{A_{\beta_n}(\alpha^T)}\right]^T\Bigg\}, \\ &= \limsup a_n w^{\beta_n}\left[\frac{A_{\beta_n}(\alpha^T)}{A^*}\right]^T, \\ &\leqslant 0.\end{aligned} \tag{20}$$

Asymptotically, then, on the subsequence $E\{U(w_T)\}$ does not grow as rapidly as $[A^*]^T$. On the other hand, if we follow a policy of setting

$$\alpha^T = \alpha_{\beta_1},$$

we have

$$\frac{\sum_i a_i E\{\tilde{w}_T{}^{\beta_i}\}}{[A_{\beta_1}(\alpha_{\beta_1})]^T} = \frac{1}{[A_{\beta_1}(\alpha_{\beta_1})]^T}\left\{a_1 w^{\beta_1}[A_{\beta_1}(\alpha_{\beta_1})]^T + \sum_{i\neq 1} a_i w^{\beta_i}[A_{\beta_i}(\alpha_{\beta_1})]^T\right\},$$

$$= a_1 w^{\beta_1} + \sum_{i\neq 1} a_i w^{\beta_i}\left[\frac{A_{\beta_i}(\alpha_{\beta_1})}{A_{\beta_1}(\alpha_{\beta_1})}\right]^T, \tag{21}$$

$$\rightarrow a_1 w^{\beta_1}.$$

Thus (21) violates optimality and we must have

$$\alpha^T \rightarrow \alpha_{\beta_1}$$

pointwise. To prove uniform convergence we could simply take a subsequence of w as well and the proof would be unaltered with (21) exceeding

$$\liminf a_1 w^{\beta_1} > 0.$$

Q.E.D.

It can be seen from the proof of Theorem 1 why convergence is not uniform on all of R^+. As $w \rightarrow 0$, the relative weight $w^{\beta_n}/w^{\beta_1} \rightarrow \infty$ and the α^T policies can be kept arbitrarily close to α_{β_n} by choosing a rapidly enough falling w sequence.

Theorem 1, however, far from finishes the story. Fig. 3 depicts a case where $\beta_1 > 0 > \beta_2$ and

$$A_{\beta_2}(\alpha_{\beta_1}) > A_{\beta_1}(\alpha_{\beta_1}).$$

Clearly now the turnpike can no longer be at α_{β_1} since that would imply that $E\{U(\tilde{w}_T)\} \rightarrow -\infty$! Where then?

It is tempting to guess that α^* is the point where the vertical distance $A_{\beta_1}(\alpha) - A_{\beta_2}(\alpha)$ is maximized, but a little thought reveals that this cannot be. Raising α closer to the α value where A_{β_1} and A_{β_2} cross clearly raises the major growth factor $A_{\beta_1}(\alpha)$ and causes $E\{U(\tilde{w}_T)\}$ to grow at an asymptotically faster rate. In general, then, we can prove the following theorem.

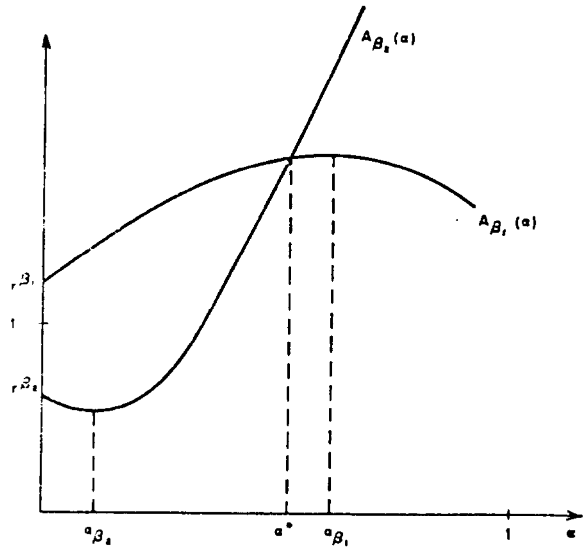

Fig. 3. One period expected utility as a function of the portfolio policy (α) for two utility functions. This figure differs from fig. 2 in that it displays a situation where at the optimal policy, α_{β_1}, for the utility function with a coefficient of relative risk aversion less than unity the expected value for the more risk averse function is dominant.

Theorem 2: Let

$$U(w) = \sum_{i=1}^{n} a_i w^{\beta_i}$$

with $\beta_1 > 0$, $\beta_1 > \ldots > \beta_n$, $\beta_n < 0$ and with

$$A_{\beta_n}(\alpha_{\beta_1}) > A_{\beta_1}(\alpha_{\beta_1}). \tag{22}$$

It follows that

$$\alpha^T \rightarrow \alpha^*,$$

where $A^* \equiv A_{\beta_n}(\alpha^*) = A_{\beta_1}(\alpha^*)$, uniformly on compact intervals bounded away from the origin. (Notice that α^* is unique.)

Proof: We first show that $(\forall \alpha \in [0, 1])$

$$A_{\beta_i}(\alpha) \leqq A_{\beta_1}(\alpha) \vee A_{\beta_n}(\alpha); \tag{23}$$

with a uniform strict bound if $i \neq 1, n$.

By (22) we have

$$A_{\beta_1}(\alpha) \vee A_{\beta_n}(\alpha) > A_{\beta_1}(0) = r^{\beta_1} > 1,$$

and from Lemma 1, (23) must hold for all $\beta_i \gtrless 0$.

Suppose, now, contrary to the theorem, that $(\exists \varepsilon > 0)$ such that on a subsequence of optimal constant policies

$$|\alpha^T - \alpha^*| > \varepsilon. \tag{24}$$

By the strict concavity of A_{β_1} and the strict convexity of A_{β_n} there exists $\delta > 0$ and $\hat{A} \equiv A_{\beta_1}(\hat{\alpha}) < A^*$ such that $(\forall\alpha)|\alpha - \alpha^*| > \varepsilon$ implies either

$$A_{\beta_1}(\alpha) < \hat{A} - \delta,$$

or

$$A_{\beta_1}(\alpha) < A_{\beta_n}(\alpha) - \delta, \tag{25}$$

depending on whether $\alpha \lesseqgtr \alpha^*$.

It follows that

$$\begin{aligned}
E\{U(\tilde{w}_T)\} &= \sum_i a_i E\{\tilde{w}_T{}^{\beta_i}\}, \\
&= a_1 w^{\beta_1}[A_{\beta_1}(\alpha^T)]^T + a_n w^{\beta_n}[A_{\beta_n}(\alpha^T)]^T \\
&\quad + \sum_{i \neq 1,n} a_i w^{\beta_i}[A_{\beta_i}(\alpha^T)]^T, \\
&< \{a_1 w^{\beta_1}[\hat{A} - \delta]^T + \sum_{i \neq 1} a_i w^{\beta_i}[A_{\beta_i}(\alpha^T)]^T\} \\
&\quad \vee \{a_1 w^{\beta_1}[A_{\beta_n}(\alpha^T) - \delta]^T + a_n w^{\beta_n}[A_{\beta_n}(\alpha^T)]^T \\
&\quad + \sum_{i \neq 1,n} a_i w^{\beta_i}[A_{\beta_i}(\alpha^T)]^T\}.
\end{aligned}$$

Applying (23) and (25) we have

$$\begin{aligned}
\frac{E\{U(\tilde{w}_T)\}}{[\hat{A}]^T} &< \left\{a_1 w^{\beta_1}\left[\frac{\hat{A} - \delta}{\hat{A}}\right]^T + \sum_{i \neq 1} a_i w^{\beta_i}\left[\frac{A_{\beta_i}(\alpha^T)}{\hat{A}}\right]^T\right\} \\
&\quad \vee \left\{a_1 w^{\beta_1}\left[\frac{A_{\beta_n}(\alpha^T) - \delta}{\hat{A}}\right]^T + a_n w^{\beta_n}\left[\frac{A_{\beta_n}(\alpha^T)}{\hat{A}}\right]^T\right. \\
&\quad \left. + \sum_{i \neq 1,n} a_i w^{\beta_i}\left[\frac{A_{\beta_i}(\alpha^T)}{\hat{A}}\right]^T\right\}, \\
&\to 0.
\end{aligned}$$

Our task now is to display a superior alternate policy sequence $\langle \alpha^T \rangle$, but letting $\alpha^T = \hat{\alpha}$ yields

$$\begin{aligned}
\frac{E\{U(\tilde{w}_T)\}}{[\hat{A}]^T} &= a_1 w^{\beta_1}\left[\frac{\hat{A}}{\hat{A}}\right]^T + \sum_{i \neq 1} a_i w^{\beta_i}\left[\frac{A_{\beta_i}(\hat{\alpha})}{\hat{A}}\right]^T, \\
&\to a_1 w^{\beta_1} > 0.
\end{aligned}$$

This contradicts the optimality of the original policy and we must have

$$\alpha^T \to \alpha^*$$

pointwise. Uniform convergence is proved as in Theorem 1. Q.E.D.

The case depicted in fig. 3 is important for several reasons. To begin with it disproves an appealing conjecture that grows out of some results of Mossin (1968). Mossin shows that for utility functions with linear risk tolerance,

$$-U'/U'' = aw+b,$$

the optimal general turnpike solution α_1^T converges to the turnpike associated with the constant relative risk aversion function

$$-U'/U'' = aw,$$

for $a > 0$. This is simply the power function and log class we have been analyzing. One might conjecture, then, that the same result would hold for

$$-U'/U'' = aw+f(w), \tag{26}$$

where in the sup norm,

$$||f(w)|| < b < \infty,$$

on the positive orthant.[6] That this is not the case can be seen by constructing a counterexample of the form (15) with $\beta_1 > 0 > \beta_2$. It is possible to choose values of β_1 and β_2 such that (26) is satisfied but, nevertheless, fig. 3 obtains. The constant a is now $(1-\beta_1)$, the coefficient of relative risk aversion associated with w^{β_1}. By focusing on this term, however, we are concentrating only on the behavior of $U(w)$ for large w and ignoring the losses associated with w^{β_2} for w small. Under the postulated conditions these losses swamp the gains for the policy α_{β_1}.

Second, fig. 3 illustrates a lack of closure for the class of optimal constant policies with α^* as the turnpike and $|a_n| > a_1$. In this case, although the optimal policies approach α^*, it would be folly to actually get on the turnpike since

$$a_1[A_{\beta_1}(\alpha^*)]^T + a_n[A_{\beta_n}(\alpha^*)]^T = (a_1+a_n)[A_{\beta_1}(\alpha^*)]^T,$$
$$\to -\infty!$$

In other words, it is best to converge to a path whose utility goes to $-\infty$. By contrast, in Theorem 1 a straight-down-the-turnpike constant α^* policy does as well in an asymptotic utility sense as the optimal policy. This is *not*, however, a valid argument for following such a policy – see Goldman (1974).

[6]Leland (1972) has dealt with this class, but he imposed an additional assumption whose effect on limiting the class of terminal utility functions is difficult to assess. It should also be stressed that the counterexample is valid for the constant turnpike problem, but not for the general problem.

The final case we consider, were all $\beta_i < 0$, is depicted in fig. 4. Once again we require $a_1, a_n < 0$ for monotonicity, and now our objective is to cut losses. The appropriate turnpike policy is that one which asymptotically minimizes the maximum A_{β_i} loss. Formally, we have the following theorem.

Theorem 3: Let

$$U(w) = \sum_{i=1}^{n} a_i w^{\beta_i},$$

with $0 > \beta_1 > \ldots > \beta_n$. It follows that

$$\alpha^T \to \alpha^*,$$

where α^* is set so that $(\forall \alpha)$

$$A^* \equiv \sup_i A_{\beta_i}(\alpha^*) \leqslant \sup A_{\beta_i}(\alpha).$$

Convergence is uniform on compact intervals bounded away from the origin.

Proof: By Lemma 1, we have that for all $i \neq 1, n$ and all $\alpha \in [0, 1]$

$$A_{\beta_i}(\alpha) < A_{\beta_1}(\alpha) \vee A_{\beta_n}(\alpha),$$

and we can equivalently define α^* by

$$A_{\beta_1}(\alpha^*) \vee A_{\beta_n}(\alpha^*) \leqslant A_{\beta_1}(\alpha) \vee A_{\beta_n}(\alpha).$$

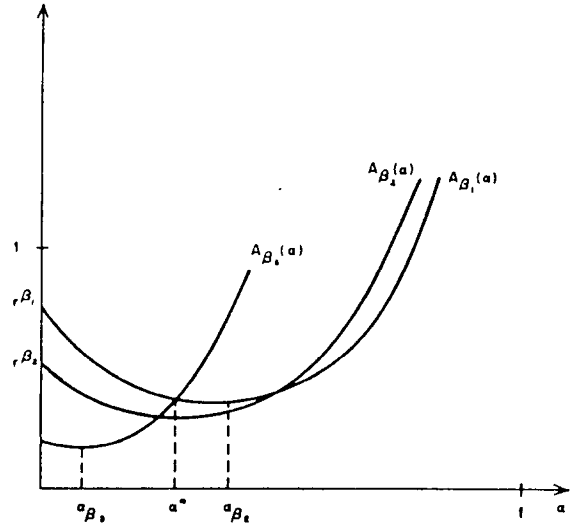

Fig. 4. One period expected utility as a function of the portfolio policy for three utility functions with constant relative risk aversion coefficients greater than unity.

Suppose, as before, that $(\exists \varepsilon > 0)$ such that on a subsequence

$$|\alpha^T - \alpha^*| > \varepsilon.$$

By the strict convexity of A_{β_i}, $(\exists \delta > 0)$ such that if

$$|\alpha - \alpha^*| > \varepsilon,$$

then

$$\bar{A}(\alpha) \equiv A_{\beta_1}(\alpha) \vee A_{\beta_n}(\alpha) > A^* + \delta.$$

By (14), from Lemma 1 we have that

$$\begin{aligned}
\frac{E\{U(\tilde{w}_T)\}}{[\bar{A}(\alpha^T)]^T} &= \sum_i a_i w^{\beta_i}\left[\frac{A_{\beta_i}(\alpha^T)}{\bar{A}(\alpha^T)}\right]^T, \\
&< [a_1 w^{\beta_1} \vee a_n w^{\beta_n}] \\
&\quad + \sum_{i \neq 1,n} a_i w^{\beta_i}\left[\frac{A_{\beta_i}(\alpha^T)}{\bar{A}(\alpha^T)}\right]^T, \\
&\to a_1 w^{\beta_1} \vee a_n w^{\beta_n}, \\
&< 0.
\end{aligned}$$

On an alternative path where we keep α^T a constant at α^*, however,

$$\begin{aligned}
\frac{E\{U(\tilde{w}_T)\}}{[\bar{A}(\alpha^T)]^T} &= \sum_i a_i w^{\beta_i}\left[\frac{A_{\beta_i}(\alpha^*)}{\bar{A}(\alpha^T)}\right]^T, \\
&> (a_1 w^{\beta_1} + a_n w^{\beta_n})\left[\frac{A^*}{A^*+\delta}\right]^T \\
&\quad - \sum_{i \neq 1,n} |a_i| w^{\beta_i}\left[\frac{A_{\beta_i}(\alpha^*)}{\bar{A}(\alpha^T)}\right]^T, \\
&\geqslant (a_1 w^{\beta_1} + a_n w^{\beta_n})\left[\frac{A^*}{A^*+\delta}\right]^T \\
&\quad - \sum_{i \neq 1,n} |a_i| w^{\beta_i}\left[\frac{A^*}{A^*+\delta}\right]^T, \\
&\to 0,
\end{aligned}$$

contradicting optimality. This result implies pointwise convergence and the argument of Theorem 1 establishes uniform convergence. *Q.E.D.*

The results obtained above seem very restrictive but fortunately they can be easily and substantially generalized. There are several routes to such generalizations. One method is to pass directly from finite sums to infinite sums and then to their closure, the integral form. When we do this we obtain functions of the form

$$U(w) = \int_{-\infty}^{\infty} w^{\beta} \, \mathrm{d}F_{\beta}, \tag{27}$$

where $U(\cdot)$ is now the Mellin–Stieltjes transform of a function $F(\cdot)$ (of bounded variation on compact sets) which assigns zero mass above $b \leq 1$ and below some $a > -\infty$.[7] Little is known about the class of utility functions admitting of this representation, and which we will call Mellin functions, but it is clearly extensive.

There are essentially two ways to establish the turnpike results for the Mellin class. We can verify the stability of the convergence to the turnpike as we converge to integrals, or we can work directly with (27) and this latter approach is the easiest. Analogous to the discrete form, the mass function, $F(\cdot)$, cannot assign negative mass in a neighborhood of its highest power b, if $b > 0$, and if $b < 0$, it cannot assign positive mass in a neighborhood of b. Similarly, it cannot assign negative mass in a neighborhood of the lowest power a, if $a > 0$, nor positive mass if $a < 0$. We can now prove that any policy sequence that fails to converge to α^* will be dominated exactly as in the previous theorems.

The fundamental constant turnpike theorem for Mellin utility functions

Let $U(\cdot)$ be a Mellin function representable as

$$U(w) = \int_a^b w^{\beta} \, \mathrm{d}F_{\beta},$$

where

$$-\infty < a \leq b \leq 1.$$

and define

$$\alpha^* \equiv \begin{cases} \text{(i) } \alpha_b \text{ if } b > 0 \text{ and } A_b(\alpha_b) \geq A_a(\alpha_a), \\ \text{(ii) the crossing point where } A_b(\alpha^*) = A_a(\alpha^*) \\ \quad \text{if } b > 0 \text{ and } A_b(\alpha_b) < A_a(\alpha_a), \\ \text{(iii) otherwise the point where} \\ \quad A_a(\alpha^*) \vee A_b(\alpha^*) \leq A_a(\alpha) \vee A_b(\alpha). \end{cases} \tag{28}$$

It follows that $\alpha^T \rightarrow \alpha^*$ uniformly on compact sets bounded from the origin.

Proof. The proof of the theorem is greatly simplified at little cost, if we slightly strengthen the requirement on $F(\cdot)$ so that $\mathrm{d}F$ assigns positive mass in a neighborhood, $[b-\Delta_b, b]$ of b and negative mass on $[a, a+\Delta_a]$ (Δ_a or Δ_b could

[7]See Widder (1941) for a detailed study of this transform. Unfortunately, neither the truncated form nor the relationship between the inversion problem and the class of concave functions appear to have been well studied in the mathematical literature.

be zero). Since the modifications of the proofs of Theorems 1, 2 and 3 are all very similar, we will only do the first case.

Suppose, then, that the case (i) conditions are satisfied (see the statement of Theorem 1) and assume, contrary to the theorem, that (24) holds on a subsequence. Using the strict concavity of $A_\beta(\alpha)$ (in a neighborhood of b) and the continuity of $A_\beta(\alpha_\beta)$ in β, ($\exists\delta > 0$ and $\sigma \in [0, \Delta_b]$) such that

$$\begin{aligned} E\{U(\tilde{w}_T)\} &= \textstyle\int_a^b w^\beta[A_\beta(\alpha^T)]^T \,\mathrm{d}F_\beta \\ &< \textstyle\int_{(b-\sigma)}^b w^\beta[A^*-\delta]^T \,\mathrm{d}F_\beta \\ &\quad + \textstyle\int_a^{(b-\sigma)^-} w^\beta[A_\beta(\alpha^T)]^T \,\mathrm{d}F_\beta . \end{aligned}$$

Choose $\beta' \in [b-\sigma, b]$ such that

$$B^* \equiv A_{\beta'}(\alpha_{\beta'}) \in (A^*-\delta, A^*].$$

As in Theorem 1, we have

$$\begin{aligned} \limsup \frac{E\{U(\tilde{w}_T)\}}{[B^*]^T} &< \limsup \left\{ \int_{(b-\sigma)}^b w^\beta \left[\frac{A^*-\delta}{B^*}\right]^T \mathrm{d}F_\beta \right. \\ &\quad + \int_a^{(a+\Delta_a)} w^\beta \left[\frac{A_\beta(\alpha^T)}{B^*}\right]^T dF_\beta \\ &\quad \left. + \int_{(a+\Delta_a)^+}^{(b-\sigma)^-} w^\beta \left[\frac{A_{a+\Delta_a}(\alpha^T) - 0_\beta}{B^*}\right]^T \mathrm{d}|F_\beta| \right\}, \\ &< \limsup \left[\frac{A_{a+\Delta_a}(\alpha^T)}{B^*}\right]^T \\ &\qquad \int_a^{(a+\Delta_a)} w^\beta \left[\frac{A_\beta(\alpha^T)}{A_{a+\Delta_a}(\alpha^T)}\right]^T \mathrm{d}F_\beta , \\ &< 0, \end{aligned}$$

where we have used the convergence theorem for Lebesgue–Stieltjes integrals.

Now, consider the alternative policy, $\alpha^T = \alpha_{\beta'}$;

$$\begin{aligned} \frac{E\{U(\tilde{w}_T)\}}{[B^*]^T} &= [B^*]^{-T} \textstyle\int_a^b w^\beta[A_\beta(\alpha_{\beta'})]^T \,\mathrm{d}F_\beta , \\ &\to \int_{\beta'}^b w^\beta \left[\frac{A_\beta(\alpha_{\beta'})}{B^*}\right]^T \mathrm{d}F_\beta , \\ &> \textstyle\int_{\beta'}^b w^\beta \,\mathrm{d}F_\beta , \\ &> 0, \end{aligned}$$

by $\beta' \in (b-\Delta_b, b]$.

The contradiction establishes pointwise convergence and uniform convergence is done as in Theorem 1. Theorems 2 and 3 are similar extensions. *Q.E.D.*

Notice that, as before, the key to the proof techniques is the use of a dominating alternative path that lies close to the turnpike.

A further generalization can be obtained by considering appropriate perturbations of the Mellin utility functions. For example, suppose under the conditions of Theorem 1, we consider

$$U(w) = \sum_{i=1}^{n} a_i w^{\beta_i} + f(w), \tag{29}$$

where ($\exists$, m, k_1, k_n and $K > 0$) such that

$$[k_1 - a_1]w^{\beta_1} - mw^{\beta_n} - K < f(w)$$
$$< mw^{\beta_1} - (a_n + k_n)w^{\beta_n} + K. \tag{30}$$

(Notice that we might as well assume that $k_1 < a_1$.) These constraints on $f(w)$ insure both that it will not 'undo' the influence of w^{β_1} or w^{β_n} by simply cancelling them and that $f(w)$ will not 'overpower' them on their respective asymptotically dominant domains.[8]

We can now prove an extended version of Theorem 1.

Theorem 1*. *Assume that* $U(w)$ *is given by* (29) *and satisfies the conditions of Theorem* 1 *and* $f(w)$ *satisfies* (30). *It follows that*

$$\alpha^T \to \alpha^*$$

uniformly on compact sets bounded away from the origin.

Proof. The proof is nearly identical to that of Theorem 1. By (30), if (19) is satisfied, then

$$E\{f(\tilde{w}_T)\} \leqq E\{m\tilde{w}_T^{\beta_1} - (a_n + k_n)\tilde{w}_T^{\beta_n}\} + K,$$
$$\leqq mw^{\beta_1}[A^* - \delta]^T - (a_n + k_n)[A_{\beta_n}(\alpha^T)]^T + K,$$

and (20) will still be satisfied.

Similarly, along the α^* turnpike we will still have

$$\limsup \frac{E\{U(\tilde{w}_T)\}}{[A^*]^T} > 0,$$

[8] While conditions (30) are sufficient, they may not be necessary since it is possible to find functions $f(w)$ such that, for example,

$$f(w)/w^{\beta_1} \to \infty \quad \text{as} \quad w \to \infty,$$

but

$$f(w)/w^{\beta_1+\delta} \to 0 \quad \text{as} \quad w \to \infty,$$

for any $\delta > 0$. One example is

$$f(w) = w^{\beta_1} \log w.$$

since

$$a_1 w^{\beta_1} + \frac{E\{f(\tilde{w}_T)\}}{[A^*]^T} \geqq a_1 w^{\beta_1} + (-m w^{\beta_n})\left[\frac{A_{\beta_n}(\alpha^*)}{A^*}\right]^T + (k_1 - a_1) w^{\beta_1}\left[\frac{A^*}{A^*}\right]^T - \frac{K}{[A^*]^T},$$

$$\to k_1 w^{\beta_1} > 0.$$

Q.E.D.

Extended versions of Theorems 2 and 3 can also be proved and, in summary, it is possible to extend the fundamental theorem. Since the techniques are straightforward at this stage we will simply state the appropriate conditions and the extended theorem without proof.

The fundamental constant turnpike theory for extended Mellin utility functions

Let $U(\cdot)$ be an extended Mellin utility function

$$U(w) = \int_a^b w^\beta \, \mathrm{d}F_\beta + f(w), \tag{31}$$

where

$$-\infty < a \leqq b \leqq 1,$$

and define the turnpike policy, α^*, in the three regimes as in (28). Furthermore, in cases (i) and (ii), $f(w)$ must satisfy (30) with $\beta_n = a$ and $\beta_1 = b$, and in case (iii), we require

$$-m(w^a + w^b) \leqq f(w) \leqq -(a_1 + k_1)w^b - (a_n + k_n)w^a,$$

where a_n and a_1 are now interpreted as the mass assigned by F_β at a and b respectively, and where we may set k_n or k_1 at zero, if F_β assigns mass of the same sign as a^+ or b^- on deleted neighborhoods of a or b respectively.[9]

Proof. See the proof of Theorem 1*. *Q.E.D.*

[9]The additional generality of the perturbed form comes from the fact that not all admissible perturbations will possess Mellin–Stieltjes transforms. A slightly greater increase in generality can be had by considering the class of utility functions for which $(\exists b)$

$$U(w)/w^b \to k, \tag{f1}$$

as $w \to \infty$ with similar conditions as $w \to 0$. These conditions and their relation to the general turnpike problem are discussed in Ross (1974). The proofs of the turnpike theorems for utility satisfying (f1) are still straightforward extensions of the arguments of the text with condition (f1) guaranteeing that no portfolio policy can asymptotically dominate the turnpike and assuring the asymptotic growth of the turnpike policy. These results permit us to directly treat utility functions of the form $(1/\beta)(w+c)^\beta$ where $\beta < 0$, but despite the increased generality, the constructive approach of the text was considered of greater interest in the present context. Alternatively, much of the analysis can be carried out directly on the class of functions of the form $(w+c_\beta)^\beta$ developing sums, integral forms and perturbed forms from this base, as in the text.

The reader can easily verify, for example, that functions of the form $(1/\beta)(w+c)^\beta$, $\beta > 0$, where c is a constant, belong to the extended Mellin family and therefore have turnpikes α_β. [This last result was obtained by Mossin (1968) for the general turnpike problem.] In sect. 5, we will discuss the generality of these results.

We will conclude this section with a turnpike theorem for the Bernoulli logarithmic case. The special role assumed by the logarithmic utility function has warranted that it be examined separately [although it has a representation in the topological closure of (31)]. Intuitively, if not mathematically, it can be treated as the power function with the lowest $\beta > 0$ and the largest $\beta < 0$. It is not surprising, then, that the only new wrinkle it presents in the turnpike theory is that it dominates suboptimal policies in an additive fashion rather than multiplicatively. Define the additive growth factor in the logarithmic case as

$$L(\alpha) \equiv E\{\log(r+\alpha\tilde{x})\},$$

and assume its maximum is attained at α_L;

$$L^* \equiv L(\alpha_L) \geqq L(\alpha).$$

Theorem 4. *If*

$$U(w) = \log w + f(w),$$

where for some concave $H(\cdot)$ and some convex $G(\cdot)$,

$$f(w) \in (G(w), H(w)), \tag{32}$$

and for any $a \geqq 0$

$$\frac{G(a^T)}{T}, \frac{H(a^T)}{T} \to 0, \tag{33}$$

then α_L has the turnpike property.

Proof. Since the proof is similar to our previous ones we will only outline it. From (33), it follows that $H(\cdot)$ is increasing. To see this note that by concavity, if $H'(a) < 0$ for some a, then we can take $a > 1$, and

$$\frac{H(a^T)}{T} < \frac{H(a)+H'(a)[a^T-a]}{T}$$

$$\to -\infty,$$

violating (33). Similarly, $G(\cdot)$ must be a decreasing function.

Now, if α_L does not have the turnpike property, then (24) holds on a subsequence and on this subsequence $(\exists\delta > 0)$ such that

$$\begin{aligned}
\frac{E\{U(\tilde{w}_T)\}}{T} &= \frac{1}{T}[\log w + TL(\alpha^T) + E\{f(\tilde{w}_T)\}], \\
&\leqq \frac{1}{T}[\log w + TL(\alpha^T) + E\{H(\tilde{w}_T)\}], \\
&\leqq \frac{1}{T}[\log w + TL(\alpha^T) + H(E\{\tilde{w}_T\})], \\
&= \frac{1}{T}\log w + L(\alpha^T) + \frac{H(w(r+\alpha^T\bar{x})^T)}{T}, \\
&\leqq \frac{1}{T}\log w + L(\alpha^T) + \frac{H(a^T)}{T}, \\
&\to L(\alpha^T), \\
&< L^* - \delta,
\end{aligned}$$

where

$$a \equiv [r+\bar{x}] \vee [w(r+\bar{x})].$$

On the alternative path where $\alpha^T = \alpha_L$ we have

$$\begin{aligned}
\frac{E\{U(\tilde{w}_T)\}}{T} &= \frac{\log w}{T} + L^* + \frac{E\{f(\tilde{w}_T)\}}{T}, \\
&\geqq \frac{\log w}{T} + L^* + \frac{G[E\{\tilde{w}_T\}]}{T}, \\
&\geqq \frac{\log w}{T} + L^* + \frac{G(a^T)}{T}, \\
&\to L^*.
\end{aligned}$$

This result violates optimality and, since uniform convergence is as in the proof of Theorem 1, α_L possesses the turnpike property. *Q.E.D.*

The bounds on $f(w)$ are interesting in their own right because they suggest a direction of generalization of our earlier results. For large $w(w \gg 1)$, $H(\cdot)$ might be of the form $(1/\beta)(\log w)^\beta$, $\beta < 1$ and

$$\frac{1}{\beta}\frac{[\log(a^T)]^\beta}{T} = \frac{1}{\beta}(\log a)^\beta \frac{T^\beta}{T} \to 0.$$

By compounding the power functions and the logarithm in this fashion, further turnpike results might emerge.

Section 4

We have already remarked on the similarity between the turnpike literature of growth theory and the results developed above.[10] The intent of this section is to draw attention to this analogy and to illustrate it. Its full exploration, however, is really the subject of another paper. For the sake of a concrete example consider a 2-form terminal utility function,

$$U(w) = a_1 w^{\beta_1} + a_2 w^{\beta_2};\ 1 > \beta_1 > \beta_2 > 0. \tag{34}$$

If we allow only constant policies $\langle \alpha_1, \ldots, \alpha_T \rangle$, where α_τ is a predetermined constant (not functionally dependent on $w_{\tau-1}$) then the portfolio turnpike problem is

$$\max_{\langle \alpha_1, \ldots, \alpha_n \rangle} \{E\{U(\tilde{w}_T)\} = a_1 w^{\beta_1} \prod_1^T A_{\beta_1}(\alpha_\tau) + a_2 w^{\beta_2} \prod_1^T A_{\beta_2}(\alpha_\tau)\}.$$

(As we have already seen, the solution to this problem sets $\alpha_\tau = \alpha^T$, for each τ).

To establish the analogy with growth turnpike theory we only have to regard $E\{\tilde{w}_T^{\beta_1}\}$ and $E\{\tilde{w}_T^{\beta_2}\}$ as two different goods. The terminal utility function tells us that they are priced at a_1 and a_2 respectively, i.e., the terminal utility valuation is simply a linear price valuation. If $\beta_i < 0$, then we might set some $a_i < 0$ indicating that the associated power function is a 'bad' rather than a 'good'.

The technology of our problem can be described in a fashion quite similar to that of an ordinary production set. At the beginning of a period inputs $\langle x_1, x_2 \rangle$ go into the production process where these represent w^{β_1} and w^{β_2} respectively.[11]

[10] For an excellent bibliography on traditional growth theory the reader is referred to Burmeister and Dobell (1970).

[11] An alternative specification of the technology is to adopt a kinked production scheme which more closely mimics the inputs-as-wealth analogy. Given inputs $\langle x_1, x_2 \rangle$ we would find the maximum w subject to $(\forall i)$

$$w^{\beta_i} \leqq x_i.$$

Output is then defined as

$$y_i \leqq w^{\beta_i} A_{\beta_i}(\alpha) \leqq x_i A_{\beta_i}(\alpha).$$

This approach has some appealing features and, in particular, the production envelope Y (defined below) is now convex. The price of this convexity, however, is to sacrifice constant returns to scale. To see this let w and $\langle x \rangle$ be such that

$$w^{\beta_i} = x_i.$$

Now a change of scale, by $\lambda > 0$, to λx will require the existence of $\hat{w}$ such that $(\forall i)$

$$\lambda w^{\beta_i} \leqq \hat{w}^{\beta_i} \leqq \lambda w_i,$$

or $(\forall i)$

$$\hat{w}^{\beta_i} = \lambda x_i,$$

which cannot in general be satisfied.

Thus, the production set will not be shifted in scale by a scale change in inputs. The analysis now becomes significantly complicated and it seemed preferable to pursue the analogy in the text.

Output, $\langle y_1, y_2 \rangle$, is then defined as the expected next period values of the β_i-power function, and admitting free disposal we have

$$y_i \leqq x_i A_{\beta_i}(\alpha),$$

where α is the particular portfolio policy chosen for the period. The policy α is the analogue of a variable that indexes the activity set in ordinary production theory. Formally, we can define the production technology set as

$$S \equiv \{\langle x, y \rangle | (\exists \alpha)(\mathrm{A}_i) y_i \leqq x_i A_{\beta_i}(\alpha)\}.$$

If a particular good is to be valued as a bad, then free disposal will be turned around to allow

$$y_i \geqq x_i A_{\beta_i}(\alpha).$$

In this sense, unlike ordinary production theory, the valuation will influence the definition of the production possibility set. To avoid the tedium of obvious qualifications we will not consider this case. Assume, then, that a_i, $\beta_i > 0$.

We can show that S has many of the properties of an ordinary production set. First, S exhibits constant returns to scale. Clearly, if $\langle x, y \rangle \in S$, then $(\forall \lambda \geqq 0)$ $\langle \lambda x, \lambda y \rangle \in S$. Second, it satisfies the no Land of Cockaigne assumption; if inputs $x = 0$, then $y \leqq 0$. Third, the continuity of $A_{\beta_i}(\alpha)$ implies that S is closed. It should also be noted that these last two properties assure that S_x is bounded for fixed x. The final critical neoclassical property is that of convexity. Notice that since S admits of constant returns, it cannot be strictly convex.

It is easily shown that the analogue of the classical production possibility set,

$$S_x \equiv \{y | \langle x, y \rangle \in S\},$$

is convex. To see this let y^1, $y^2 \in S_x$. Taking a convex combination we have that $(\forall i)$

$$\begin{aligned} y_i &\equiv \lambda y_i^1 + (1-\lambda) y_i^2, \\ &\leqq \lambda x_i A_i(\alpha^1) + (1-\lambda) x_i A_i(\alpha^2), \\ &= x_i[\lambda A_i(\alpha^1) + (1-\lambda) A_i(\alpha^2)], \\ &\leqq x_i A_i(\alpha), \end{aligned}$$

where $\alpha \equiv \lambda \alpha^1 + (1-\lambda)\alpha^2$ and we have made use of the concavity of $A_i(\cdot)$. The convexity of S_x, however, is only necessary and not sufficient for S to be convex. In fact, we can show the rather surprising result that in an appropriate projective space, the complement of S rather than S is convex!

We will illustrate this point with the 2-form. Since S admits of constant returns to scale it is easiest to normalize and we will do so by setting the input sum,

$$x_1 + x_2 = 1.$$

Figure 5 illustrates two ways to conceive of S under this restriction. The set of possible outputs, Y, is defined as

$$Y \equiv \bigcup_{x} S_x, \tag{35}$$

where the union is taken over normalized input pairs. Thus, Y is bounded by the outer envelope of the individual convex production possibility sets. The union of convex sets, however, is not necessarily (or usually in a category sense) convex and Y is no exception. An equivalent definition of Y is to take the union over policies and from (35)

$$\begin{aligned} Y &= \bigcup_{x} S_x, \\ &= \bigcup_{x} \{y | \langle x, y \rangle \in S\}, \\ &= \bigcup_{x} \{y | (\exists \alpha)(\forall i) y_i \leqq x_i A_{\beta_i}(\alpha)\}, \\ &= \bigcup_{x} \bigcup_{\alpha} \{y | (\forall i) y_i \leqq x_i A_{\beta_i}(\alpha)\}, \\ &= \bigcup_{\alpha} [\bigcup_{x} \{y | (\forall i) y_i \leqq x_i A_{\beta_i}(\alpha)\}]. \end{aligned}$$

The set in square brackets, though, is simply the set of feasible outputs for a fixed portfolio policy, α. As is shown in fig. 5, this set is simply bounded by a line much like an ordinary budget or cost set. Since $A_{\beta_1}(\alpha)$ and $A_{\beta_2}(\alpha)$ are monotone in different directions in the relevant undominated range between α_{β_2} and α_{β_1}, the budget sets do not dominate and the outer envelope will be concave rather than convex.[12]

This considerably complicates matters if we wish to invoke Radner's (1961) beautiful lemma and the resulting turnpike theorem to verify that $\alpha^T \to \alpha^*$ where α^* is the maximal steady state growth rate, α_{β_1}, in the case considered above. (The argument becomes more difficult still with 'bads' because of the possible non-existence of a turnpike in the usual sense due to lack of closure as in the case of Theorem 2.) There are, however, two further reasons why we chose not to attack the problem through this analogy.

First, such an approach would not have made use of two special features of our problem. On the one hand, we can adopt a constant policy for each horizon and this simplifies our problem. In the economy-wide problem, resources cannot be assumed to be costlessly shiftable in each period and we must drop the price taking assumption we have employed. The analogue of our results would, thus, not use this property, since it is lacking in ordinary turnpike theory. Second, and more importantly, our technology is completely decomposable and this feature

[12] The argument is reminiscent of the analysis of the factor price frontier. See Burmeister and Dobell (1970).

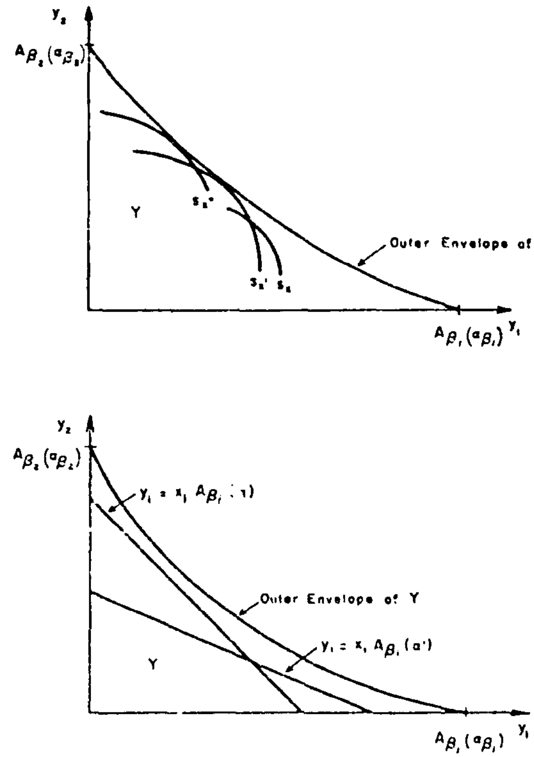

Fig. 5. Two alternative techniques for constructing analogs to the production possibility set of neoclassical theory by taking the envelop of attainable expected utilities. The top graph holds inputs constant and varies the portfolio policy (α) and the bottom figure holds the policy fixed and varies the inputs.

allows us to study the structure of the problem in detail. The β_1-good or sector in the above case grows more rapidly than any other and, asymptotically, the other sectors become negligible. [Warning! Caution must be used with notions of convergence in utility space, see Goldman (1974).] This is a characteristic feature of a decomposable system and we have exploited it in all of our earlier results.

The second reason for not relying too heavily on the analogy is that it does not generalize very easily. The techniques and results of sect. 3 are used extensively in a forthcoming paper, Ross (1974), to analyze the general turnpike problem, but it is difficult to extend the traditional growth turnpike theory to the general portfolio turnpike problem. In general, the optimal policy in period τ, α_τ, will be functional in the random wealth level, $\tilde{w}_{\tau-1}$, inherited from the previous period, i.e.,

$$\alpha_\tau = \alpha_\tau(\tilde{w}_{\tau-1}).$$

The state variables which define the current position must therefore include the past realization of $\tilde{w}_{t-1}$, and, conversely, it would be inadequate merely to store $\langle A_{\beta_i}(\alpha)\rangle$ as we were able to do in the above. As the horizon increases, though, the number of feasible terminal wealth realizations will become infinite (even if $\tilde{w}_t$ is finitely discrete) and the commodity space of our analogue must be infinite dimensional. Radner's elegant development, however, breaks down in infinite dimensional spaces.

To make all this explicit consider our concrete 2-form function (34). In the general T-period turnpike problem, there will be an optimum initial portfolio policy α_0 for given w. This will lead to a random return

$$\tilde{w}_1 = w[r+\alpha_0\tilde{x}].$$

The second period optimal policy will now depend on the realized $\tilde{w}_1$ value, and not simply on the *ex ante* $A_{\beta_1}(\alpha_0)$ and $A_{\beta_2}(\alpha_0)$ expected outputs. If $T = 2$, then our final outputs will be

$$w^{\beta_i}E\{(r+\alpha_0\tilde{x})^{\beta_i}(r+\alpha_1(\tilde{w}_1)\tilde{x}_1)^{\beta_i}\},$$

rather than

$$w^{\beta}E\{(r+\alpha_0\tilde{x})^{\beta_i}\}E\{(r+\alpha_1\tilde{x})^{\beta_i}\}.$$ [13]

Section 5

There seem to be two broad avenues of generalization that are of interest.[14] First, within the confines of the model as it is currently stated, we would like a more complete understanding of the class of terminal utility valuations that admit turnpike results. We have introduced and examined one class, the extended Mellin family (including the log function), in some detail in this paper and in Ross (1974) it is shown that an analogous Laplace class of the form

$$\int_a^\infty e^{-w\beta}\, dF_\beta;\ a > 0,$$

can be analyzed in a similar fashion. These two classes are quite broad, but at present it is not known whether they are necessary as well as sufficient for a turnpike theory. [A negative conjecture on this issue is spelled out in Ross (1974).]

Second, as is shown in Ross (1974), the general turnpike problem introduced earlier can be studied within the same framework as the current paper. In particular, by drawing on the methods developed here, we can show that in

[13] It is possible, though, to incorporate some limited probability state dependence. For example, if there is a finite number of states in each period, then α_t could be permitted to depend on any finite number of realized past states or, equivalently, *rates* of return.

[14] It should be clear that nothing we have done so far depended critically on the assumption of a single risky asset and the extension to many risky assets can, generally speaking, be accomplished by simply reinterpreting α to be a portfolio of risky assets and $\tilde{x}$ to be a vector of random returns premiums.

many of our cases the turnpikes for the constant policies and the true optimal sequence are the same. The induced dependence of the optimal policy on the past realized wealth level (see sect. 1) does not prevent such an analysis. A much more serious complication is introduced if the exogenously given rates of return are permitted to be stochastically dependent. Nevertheless, we might hope that our present findings will serve as a guide to the solution of turnpike problems with stochastic interdependence.

In general, the findings of this paper indicate that the study of the dynamic properties of stochastic portfolio problems can be greatly facilitated by an appropriate treatment of the valuation of uncertain payoffs. By generalizing from the stationary cases which admit of closed form analysis we have been able to derive solutions for a large class of valuation criteria. Furthermore, the techniques employed have stressed overtaking principles that promise to be robust across many different problems.

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