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RESEARCH, PATENTING, AND TECHNOLOGICAL CHANGE[1]

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This paper develops a search-theoretic model of technological change that accounts for some puzzling trends in industrial research, patenting, and productivity growth. In the model, researchers sample from probability distributions of potential new production techniques. Past research generates a technological frontier representing the best techniques for producing each good in the economy. Technological breakthroughs, resulting in patents, become increasingly hard to find as the technological frontier advances. This explains why patenting has been roughly constant as research employment has risen sharply over the last forty years. Productivity is determined by the position of the technological frontier and hence by the stock of past research. If researchers sample from Pareto distributions, then productivity growth is proportional to the growth of the research stock. The Pareto specification accounts for why productivity growth has neither risen as research employment has grown nor fallen as patenting has failed to grow. The growth of research employment itself is driven, in equilibrium, by population growth. Calibrating the model's four parameters, the implied social return to research is over twenty percent.

Keywords: Innovation, patent productivity, research, technology.

1. INTRODUCTION

Numerous empirical studies have found a systematic positive relationship between total factor productivity and industrial research and between patented inventions and R&D across firms and industries.[3] These findings are not surprising since R&D, patenting, and productivity are all indicators of technological change. In contrast to the cross sectional evidence, however, the long-run time series behavior of the three indicators of technological change remains puzzling. Figure 1 shows the number of private-sector researchers in the United States, the number of U.S. patents for which they applied, and the number of patents they were granted.[4] The number of researchers has grown by nearly five percent per year since the early 1950's (see Table I) but the collective inventive

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[3] The literature on productivity and R&D is surveyed by BLS (1989) and Griliches (1979, 1992). Griliches (1990) surveys the literature that uses patent statistics. The best evidence that patents are indicators of inventive output comes from firm-level regressions of patents on R&D (Pakes and Griliches (1984) and Hall, Griliches, and Hausman (1986)).

[4] Private-sector researchers are the number of R&D scientists and engineers employed in industry, adjusted by the fraction of industry R&D that is company financed. A complete list of definitions and sources is in the Appendix.

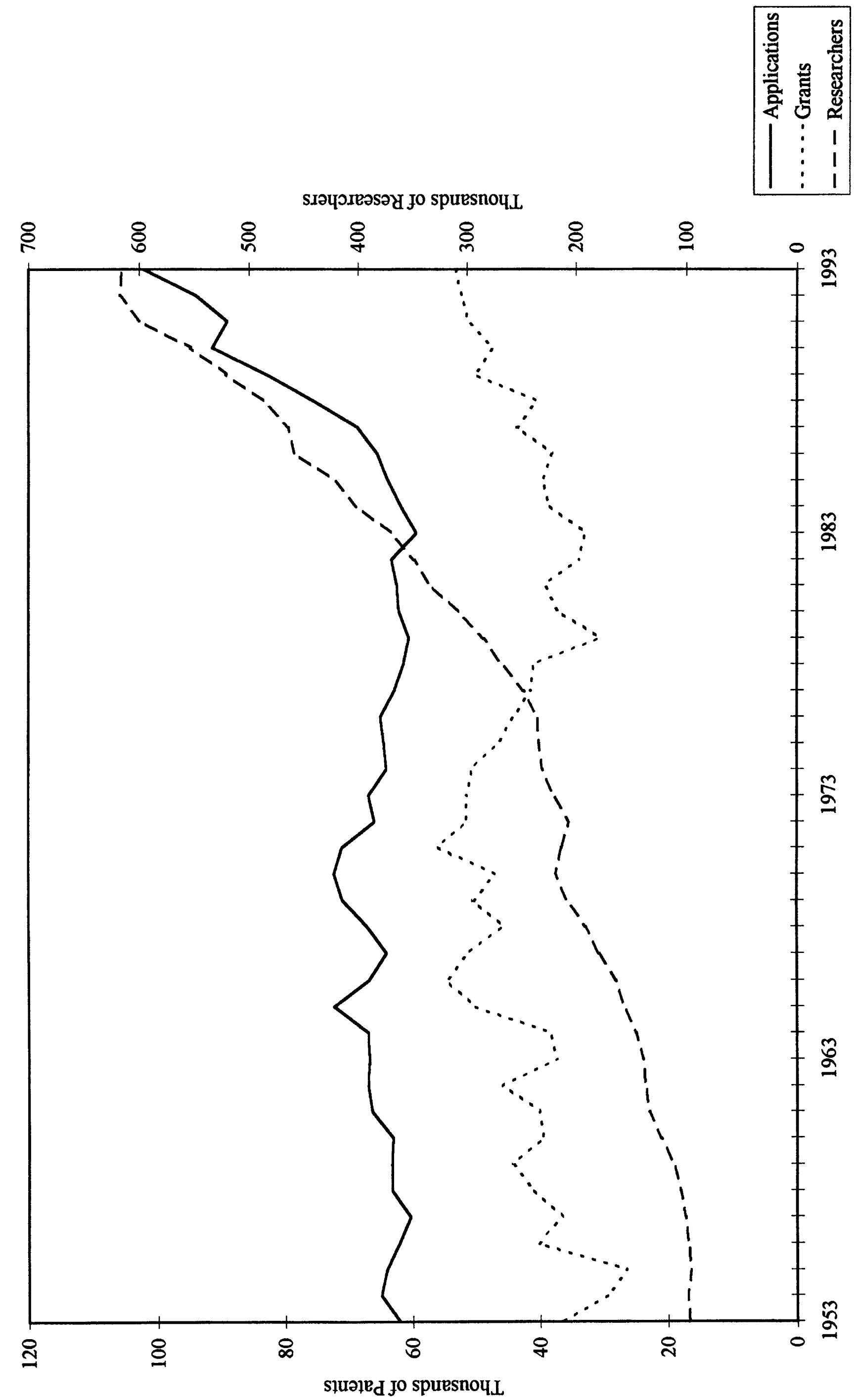

FIGURE 1.—Patents and researchers in the United States.

TABLE I

DATA FOR THE UNITED STATES

	Average Annual Growth Rates				
	1953–'63	1963–'73	1973–'83	1983–'93	1953–'93
Total Factor Productivity in Manufacturing	.016	.018	.004	.013	.012
Private-Sector R&D Scientists and Engineers	.036	.046	.052	.054	.047
Civilian Labor Force (16 or over)	.013	.022	.022	.014	.018
	Average Levels				
	1953–'63	1964–'73	1974–'83	1984–'93	1953–'93
Patents Granted to U.S. Residents (thousands)	37.9	49.7	39.8	45.4	43.1
Industry-Funded R&D Relative to Industry Compensation of Labor	.017	.021	.023	.031	.023

Sources: Total Factor Productivity is from BLS (1996). Private-Sector R&D S&E's is the number of R&D S&E's employed in industry multiplied by the fraction of R&D performed in industry that is industry funded, NSF (1987, 1995). The Civilian Labor Force Data are from CEA (1995). Patents Granted by the United States to U.S. Residents is from WIPO (1983), WIPO (various issues) and tabulations by the U.S. Patent Office. Total R&D Funded by Industry is from NSF (1995b) and Compensation is from BEA (1992) and BEA (1992b, 1994).

output of these researchers, as measured by patents, was roughly constant for three decades before applications rose sharply beginning in the mid-1980's. Table I also reports total factor productivity (TFP) growth by decade in the U.S. manufacturing sector. TFP growth has almost recovered from the slowdown during the 1970's but it has not increased with the secular rise in research employment nor has it responded to the recent jump in patent applications. A theory of invention and technological change should explain why research inputs have grown so rapidly, why patents per researcher have fallen, and why total factor productivity growth has not increased with the level of research.[5]

This paper develops a general equilibrium search-theoretic model of technological change to account for these puzzles. Researchers sample from probability distributions of potential new production techniques. These search distributions will shift if researchers obtain knowledge spillovers from past research efforts. Research output can be summarized by a technological frontier—an extreme value distribution—of the most efficient techniques for producing each good in

[5] The theory is not asked to account for the recent surge in U.S. patent applications since that seems to be a consequence of a major institutional change: patents have received stronger protection in the United States since the creation in 1982 of the Court of Appeals of the Federal Circuit (e.g. McConville (1994)). Nor is the theory asked to explain the rise of foreign patenting in the United States and the rise of U.S. patenting abroad. Although the vast majority of patents continue to be protected only in their domestic market, there is a long-term trend towards greater international protection of inventions. As mentioned in the conclusion, an increasing propensity to patent abroad is consistent with an implication of the theory developed here, that the value of patentable inventions is trending up.

the economy. A patent is treated as a research draw that advances the technological frontier. The aggregate rate of patenting depends on both current research (determining the rate of sampling from the search distributions) and the stock of past research (determining the probability that a given draw will advance the technological frontier). The technological frontier advances with accumulated research, generating a relationship between productivity—the mean of the technological frontier—and the research stock. Research effort itself is determined in equilibrium by the value of patents and the probability of discovering a patentable invention relative to the wage for producing output using existing techniques.

What restrictions to the model bring its predictions into line with the observed trends in research, patenting, and productivity growth? The observation that patenting has been constant as research effort has risen exponentially implies a restriction on how knowledge spillovers from past research improve the search distributions. If spillovers were too potent, the model would not account for the decline in patents per researcher. In addition, the fact that productivity has continued to grow implies a particular restriction on the form of the underlying search distributions, a restriction satisfied by the Pareto distribution. With the Pareto distribution, productivity growth is proportional to the growth of the research stock and to the level of patenting.[6] Lastly, the observation that research effort has grown suggests that the return to research has not fallen even as technological advances become more difficult. Research effort is sustained only if the labor force grows, which causes the value of patents to rise faster than the wage.

This interpretation of the facts synthesizes a number of arguments put forward in the literature. As far back as the 1930's writers have blamed the decline in patents per researcher on diminishing technological opportunities.[7] Further evidence for the diminishing technological opportunities hypothesis is provided by Evenson (1984) who finds that the decline in patents per researcher is a world-wide phenomenon.[8] The search model of invention, introduced by Evenson and Kislev (1976), formalizes the idea that technological improvements become increasingly difficult as the threshold for new discoveries rises.[9]

[6] The model does not consider the effect of research performed abroad on productivity in the United States. Eaton and Kortum (1997) extend the model to incorporate technology diffusion among a set of countries.

[7] Griliches (1990) reviews this early literature. Machlup (1962) compiles evidence on patents per researcher from 1870–1960 and shows that this ratio declined consistently after 1920.

[8] Domestic patent applications in France, Germany, and the United Kingdom continue to display little upward trend (WIPO (annual issues)). In the United States, Kortum (1993) finds that patenting relative to real R&D expenditures has fallen in all manufacturing industries, suggesting that the decline is not caused by shifts in industry composition. Researchers themselves do not report a decline in their propensity to patent inventions (Mansfield (1986)). Case studies of the textiles industry and the chemicals industry by Baily and Chakrabarti (1985) and of the pharmaceutical industry by Henderson and Cockburn (1996) suggest that inventions are becoming increasingly difficult to discover.

[9] Bental and Peled (1996) have integrated the search model of invention into a general equilibrium model of endogenous growth.

Diminishing opportunities need not imply stagnant productivity, however, as they may be offset by rising research effort. Furthermore, as noted by Griliches (1990), due to the "multiplicative aspect of their impact," a constant flow of patented inventions could, in principle, sustain positive productivity growth.[10] This idea of inventions as percentage improvements is formalized in the quality ladders model of Grossman and Helpman (1991). Jones (1995b), however, rejects the Grossman and Helpman model (in which inventions and productivity growth are proportional to research) since as research employment has grown rapidly in the United States, France, Germany, and Japan, TFP growth has not risen. The present paper combines Evenson and Kislev's search model (to explain patents and research) with Grossman and Helpman's quality ladders model (to explain productivity and patents). Given Pareto search distributions, the resulting productivity-research equation is identical to the specification used in studies that estimate the contribution of research to productivity growth (BLS (1989) and Griliches (1979)). When this equation is introduced into a general equilibrium model, TFP growth is ultimately tied to the growth of the labor force, as in Jones (1995a).[11]

Section 2 of the paper lays out the general equilibrium model of search. Section 3 restricts the general model based on evidence from the trends in aggregate research employment, patenting, and productivity. Section 4 concludes.

2. THE MODEL

The economy consists of a continuum of infinitely lived individuals $i \in [0, L(t)]$ at date t (the Appendix contains a list of symbols). Time is continuous and $L(t)$ is a nondecreasing sequence with $\int_{-\infty}^{t} L(s)\, ds$ bounded for finite t. Consumption goods come in a continuum of varieties $j \in [0, 1]$.

2.1. *Preferences*

Individual i's objective is to maximize the expectation of

$$(1) \qquad U_i(t) = \int_t^{\infty} e^{-\rho(s-t)} \exp\left[\int_0^1 \ln C_{ijs}\, dj\right] ds,$$

[10] Although he made extensive use of patent statistics, Schmookler (1966) discounted any relationship between patents and productivity. In his Ph.D. dissertation, Schmookler (1951) reports that domestic patent applications rose considerably from 1860 to 1930 but output per unit of input did not accelerate as he had hypothesized. Interestingly, the upward trend in patenting came to a halt: the five-year average of domestic patent applications centered on 1930 (reported in Schmookler (1954)) was not exceeded until 1988. Schmookler may have been less skeptical of the patent-productivity relationship had he seen the past sixty years of data.

[11] Nordhaus (1969) realized that population growth might be necessary to sustain technological change, a result which continues to be explored, e.g. Segerstrom (1995) and Young (1995). Young's model has the appealing feature that endogenous technological change is possible without population growth and yet a rising population does not lead to explosive growth. But, it does not account for the observed fall in patents per researcher.

where $\rho > 0$ is the discount rate and C_{ijs} is the individual's consumption of good j at date s. The individual takes prices of consumption goods P_{jt} as given. Utility maximization requires that expenditure on each variety be equal, $P_{js}C_{ijs} = X_i(s)$, where $X_i(s)$ denotes total expenditure by individual i at date s. The consumption index of individual i is therefore $\exp[\int_0^1 \ln C_{ijs}\, dj] = X_i(s)/P(s)$, where $P(s) = \exp[\int_0^1 \ln P_{js}\, dj]$ is the aggregate price index.

Taking the price index as numeraire, $U_i(t) = \int_t^\infty e^{-\rho(s-t)} X_i(s)\, ds$. Assuming individuals may borrow or lend at an interest rate of $r(t)$, equilibrium will require $r(t) = \rho$. In equilibrium, individuals are indifferent to the allocation of their expenditure over time.

2.2. *Techniques*

Over time researchers get ideas about new and potentially better techniques of production. Ideas arrive to an individual researcher as a Poisson process with a parameter normalized to unity. Each idea pertains to a technique for producing a single variety of good, drawn from the uniform density on $[0,1]$.[12]

Let $R(t) \leq L(t)$ be the aggregate measure of individuals engaged in research. The stock of all past research effort is $K(t) \equiv \int_{-\infty}^t R(s)\, ds$. Applying a standard result from the theory of search (Butters (1977), Peters (1991)) the number of techniques for producing good j discovered between time t and time $s > t$ has a Poisson distribution with parameter $K(s) - K(t)$.

Techniques vary according to how efficiently they transform labor services into output. If the efficiency of a technique for producing good j is q then, given a wage W, the good can be produced at a unit cost of W/q.[13] At the beginning of time any good can be produced at an efficiency level of $q_0 > 0$.

When a researcher gets an idea for a new technique, its efficiency is drawn from a probability distribution (the search distribution) representing technological opportunities. The efficiency of a new technique is independent of the variety of good it is used to produce. The search distribution is formulated as follows.

ASSUMPTION 2.1: *Let the random variable Q be the efficiency of a new technique. The search distribution from which Q is drawn, given a stock of past research K, is*

$$\Pr(Q \leq q; K) \equiv F(q; K) = \begin{cases} 1 - S(K)(1 - F(q)), & q \geq \bar{q}(K), \\ 0, & q < \bar{q}(K), \end{cases}$$

[12] Assuming that the variety cannot be chosen by the researcher avoids difficult strategic issues in research; see Reinganum (1989). Grossman and Helpman (1991) also model undirected research but in their model undirected research is not dominated by other strategies. In the present setup it would be dominated; hence undirected research must be taken as a feature of the search technology.

[13] Here, better techniques allow the same goods to be produced more cheaply. In Grossman and Helpman (1991) better techniques allow higher quality goods to be produced at the same cost. The two approaches are equivalent if output is measured in units of constant quality.

where $S(K)$ (the spillover function) is a nonnegative weakly increasing continuous function on $[0,\infty]$ satisfying $S(1)=1$ and $F(q)$ (the stationary search distribution) is a distribution function satisfying $F(q)=0$ on $(-\infty, q_0]$ and $F(q)=\int_{q0}^{q} f(x)\,dx$ on $[q_0,\infty)$, with a continuous density $f(q)$ on $[q_0,\infty)$. The lower support of the search distribution is $\bar{q}(K)=\min\{q \geq q_0 | 1-S(K)[1-F(q)] \geq 0\}$, hence it is weakly increasing in K (either $\bar{q}(K)=q_0$ or it satisfies $1-S(K)[1-F(\bar{q}(K))]=0$). The search distribution is continuous except for possibly a jump at q_0.

In the special case of $S(K)=1$ for all K the search distribution reduces to the stationary search distribution $F(q)$. Nelson (1982) calls this blind search and then goes on to consider how knowledge is used to focus search.[14] If the spillover function is increasing in K, then the efficiency of techniques drawn from the search distribution is stochastically increasing in K. As formalized in Assumption 2.1, the knowledge accumulated through past research efforts is available to everyone.

2.3. *Market Structure*

Suppose that m techniques have been discovered for producing good j. Denote their efficiency levels (including the initial technique) by q_l, $l=0,1,\ldots,m$, where the ordering is by date of discovery. The state of the art for producing good j is $z=\max\{q_0,q_1,\ldots,q_m\}$, i.e. the efficiency of the best technique yet discovered. If $q_m=z$, then the inventive step of the mth technique is $y=z/\max\{q_0,q_1,\ldots,q_{m-1}\}$. As in Kortum (1991), a technique is patentable if its inventive step exceeds unity.[15] The patent is infringed if anyone produces the jth good using a new technique of efficiency $q \in (z/y, z]$. Patent protection expires when a more efficient technique is discovered. When the patent expires, the technique is freely available to imitators but prior to that the patent holder can, at no cost, prevent others from infringing.[16]

The owner of an unexpired patent on a technique for producing good j sets a price so as to maximize profits and then hires workers at a wage of W to meet demand at that price. Suppose that the patented technique represents an inventive step y enabling good j to be produced at efficiency z. Competitors can

[14] "Let me formalize the idea that stronger knowledge means a better choice set actually explored by defining 'better' in terms of stochastic dominance" (Nelson (1982, p. 459)). Jovanovic and Rob (1990) and Jovanovic and Nyarko (1996) develop alternatives to blind search by casting the search problem in a Bayesian framework. Jaffe (1986) provides some empirical support for knowledge spillovers of the type incorporated into Assumption 2.1. He finds that a firm gets more patents per dollar of R&D if other firms are doing research in a related area of technology (i.e. if they are patenting in related patent classes).

[15] Under Section 102, a patent is barred for lack of novelty if there is enough in the prior art to enable someone skilled in the area to perform the process or produce the product described in the patent application. (Miller and Davis (1990, p. 46)).

[16] Patenting is assumed to be costless so that researchers patent all techniques that are patentable. Furthermore, patent examiners are assumed to grant patents only on patentable techniques.

imitate the previous state of the art technique and therefore can produce at a cost of $W/(z/y)$. The competitors take the price as given and produce only if it strictly exceeds their cost. Demand for good j is unit elastic, hence the patent holder maximizes profits by charging the highest price at which competitors do not enter. At time t, with a wage $W(t)$, this price is

$$P_{jt} = yW(t)/z. \tag{2}$$

2.4. *The Technological Frontier*

The market structure described above implies that only state of the art techniques will be used. The state of the art for producing a particular good is random, with a distribution function G_1 characterized in the following proposition.

PROPOSITION 2.1: *At time t, given the path of research up to that date, the distribution function G_1 of the state of the art for producing good j depends only on the stock of research, $K(t)$. Given $K(t) = K$,*

$$G_1(z;K) = \begin{cases} \exp\{-[1-F(z)]\Sigma(K)\}, & z \geq \bar{q}(K), \\ \exp\{-[1-F(z)]\Sigma(\bar{K}(z)) - [K - \bar{K}(z)]\}, & q_0 \leq z \leq \bar{q}(K), \\ 0, & z < q_0, \end{cases}$$

where $\Sigma(K) = \int_0^K S(x)\,dx$ and for $q_0 \leq z < \bar{q}(z)$, $\bar{K}(z) = \min\{\bar{K} \in [0, K] \mid 1 - S(\bar{K})[1-F(z)] \leq 0\}$. There is a mass point at the initial level of efficiency, $G_1(q_0; K) = \exp\{-\Sigma(K)\}$.

The proof is in the Appendix.

The technological frontier represents the state of the art across the entire spectrum of goods. This paper proceeds under the convention that, due to the independence of search across different goods, the distribution of the technological frontier is equal to the probability distribution of the state of the art for a specific good. Thus, $G_1(z; K(t))$ is the measure of goods produced at date t using a technique with efficiency less than or equal to z. This distribution summarizes the cumulative results of all past research effort. Aggregate productivity can be viewed as the mean of the distribution of the technological frontier. Proposition 2.1 implies that productivity, so defined, is completely determined by the stock of past research as in earlier work on research and productivity (Griliches (1979)).

The probability that an idea is patentable (given a stock of past research K), $p(K)$, is obtained by taking the probability that a new idea exceeds any efficiency level and integrating it over the distribution of the state of the art (from

Proposition 2.1). For $K \geq 1$,[17]

$$p(K) = \int_{q_0}^{\infty} (1 - F(z;K))\, dG_1(z;K) \tag{3}$$
$$= G_1(\bar{q}(K);K) + \int_{\bar{q}(K)}^{\infty} S(K)[1 - F(z)]\, dG_1(z;K)$$
$$= G_1(\bar{q}(K);K) + \int_0^1 x\Sigma(K)/S(K)\exp\{-x\Sigma(K)/S(K)\}\, dx$$
$$= \frac{S(K)}{\Sigma(K)}[1 - e^{-\Sigma(K)/S(K)}],$$

where the penultimate line changes the variable of integration from z to $x = S(K)[1 - F(z)]$. Notably, the probability that an idea will be patentable depends only on the spillover function $S(K)$, without regard to the form of the stationary search distribution $F(q)$.[18]

Several other functions, related to the distribution of useful techniques, are defined below. The probability that a patentable technique of efficiency q invented at time t will be used for less than x years is denoted $\Phi(q, K(t+x), K(t))$. Given a stock of past research K, the joint distribution of the inventive step of newly patented techniques and the efficiency of the techniques they supplant ($z' = z/y$) is denoted $H(z', y; K)$. Given a stock of past research K, the distribution of the inventive step of a technique in use is denoted $G_2(y;K)$. Expressions for each of these functions are derived in the Appendix.

2.5. *Aggregrate Income*

Workers are paid a wage $W(t)$ while researchers are compensated by profits from any patented inventions they may have discovered. Aggregate income $X(t)$ is equal to the sum of aggregate wage income $W(t)[L(t) - R(t)]$ and aggregate profit income $\int_1^\infty \pi(y,t)\, dG_2(y; K(t))$. The profit from a patented technique with efficiency z and inventive step y is $\pi(y,t) = X(t)(1 - y^{-1})$, and hence aggregate income is

$$X(t) = W(t)[L(t) - R(t)] \Big/ \int_1^\infty y^{-1}\, dG_2(y; K(t)). \tag{4}$$

Using equation (2) and the fact that $P(t) = 1$, the production wage is

$$W(t) = \exp\left\{\int_{q_0}^{\infty} \ln(z)\, dG_1(z; K(t))\right\} \Big/ \exp\left\{\int_1^\infty \ln(y)\, dG_2(y; K(t))\right\}, \tag{5}$$

[17] The restriction to $K \geq 1$ implies $S(K)[1 - F(\bar{q}(K))] = 1$ and leads to the simple expression in equation (3). If $K \leq 1$, then $p(K) = (s(K)/\Sigma(K))[1 - e^{-\Sigma(K)}] + [1 - S(K)]e^{-\Sigma(K)}$. For $K = 1$ the equations are identical since $S(1) = 1$.

[18] Similarly, results in the theory of record breaking often do not depend on the functional form of the sampling distribution (Glick (1978)).

reflecting the (geometric) average efficiency level in the economy relative to the average markup.

2.6. *The Value of a Patent*

A patent has value because it gives the owner a claim to the future profits from using the patented technique. Let $V(t)$ be the expected value of a patent discovered at date t, taking account of the uncertainty about the patent's efficiency $z = z'y$, its inventive step y, and its profitable life,

$$V(t) = \int_t^{\infty}\int_{q_0}^{\infty}\int_1^{\infty} e^{-\rho(s-t)}[1 - \Phi(z'y, K(s), K(t))] \times \pi(y,s)\, dH(z', y; K(t))\, ds. \tag{6}$$

An individual choosing to engage in research expects a return of $p(K(t))V(t)$. This is the product of the rate of arrival of new ideas (unity), the probability that an idea arriving at date t is patentable, and the expected value of a patent discovered at date t.

2.7. *Equilbrium*

The return to research relative to the wage for production work, $E(t) \equiv p(K(t))V(t)/W(t)$, determines how individuals choose between research and production. The derivations above show how the return to research and the production wage depend on a path of the research stock. A path of the research stock, $\{K(s)\}$, is an equilibrium if it induces the level of research employment necessary to generate it, $\{R(s) = \dot{K}(s)\}$.

Definition: Given a stock of research $K(t)$ and a known path of the labor force $\{L(s) | s \geq t\}$, an *equilibrium* is a path of research $\{R(s) | s \geq t\}$ such that for all $s \geq t$: (i) the labor market allocation is optimal for each individual,

$$R(s) = \begin{cases} 0, & E(s) < 1, \\ \in [0, L(s)], & E(s) = 1, \\ L(s), & E(s) > 1, \end{cases}$$

where $E(s) \equiv p(K(s))V(s)/W(s)$, (ii) the wage satisfies equation (5), (iii) the expected value of a patent satisfies equation (6), and (iv) the probability that an idea is patentable satisfies equation (3).

Conditions are not provided for existence and uniqueness of the equilibrium at this level of generality. In what follows, an equilibrium path of research is constructed for a restricted version of the model.

3. IMPLICATIONS AND PLAUSIBLE RESTRICTIONS

This section examines, sequentially, the model's implications for patenting, productivity, and research. At each step the model is restricted further in light of the data trends noted in the introduction. Patenting is considered first because the model's implications for patenting (conditional on research) do not depend on the form of the stationary search distribution. Research is considered last because the model's implications for research are revealed only when the dynamic equilibrium problem is solved.

3.1. *Patenting*

The aggregate rate of patenting, I, is the product of the rate at which ideas are discovered and the fraction of those ideas that are patentable,

$$I(t) \equiv R(t)p(K(t)) = R(t)\frac{S(K(t))}{\Sigma(K(t))}[1 - e^{-\Sigma(K(t))/S(K(t))}], \tag{7}$$

for $K(t) \geq 1$. An implication is that the rate of patenting depends not only on current research but also on the amount of research done in the past.[19] The influence of past research depends on the spillover function but not on the stationary search distribution. Thus trends in patenting and research are informative about the spillover function. What form of the spillover function is consistent with the observation that patenting has remained roughly constant even as research has grown exponentially?

PROPOSITION 3.1: *If the rate of patenting approaches a constant I while research grows exponentially at rate g, then the spillover function must satisfy* $\lim_{K\mapsto\infty}\{S(K)/K^{I/g-1}\} = a$ *for some* $0 < a < \infty$.

The proof is in the Appendix.

To be consistent with the data trends, the spillover function must asymptote to a power function. The paper proceeds under the simplification that the spillover function is a power function even when the stock of research is finite.

ASSUMPTION 3.1: *The spillover function is* $S(K) = K^{\gamma}$, *where* $0 \leq \gamma < \infty$ *indexes the strength of spillovers.*

Given Assumption 3.1, $\Sigma(K) = K^{1+\gamma}/(1+\gamma)$ and if the research stock grows at rate g, the rate of patenting converges to $I = (1+\gamma)g$. If $\gamma = 0$ there are no

[19] Because the search process is sequential, equation (7) does not exhibit any congestion externalities from simultaneous research, as in Tandon (1983), Kortum (1993), and Stokey (1995). Decreasing returns to current research activity could also arise from heterogeneous research talent in the population, as in Eaton and Kortum (1997). Although potentially important empirically, decreasing returns are ignored here to maintain simplicity and to highlight the role of the search technology.

research spillovers, while if γ is large spillovers are potent. But, even with potent spillovers, patents per researcher will be strictly decreasing in the stock of research.

3.2. *Productivity*

Productivity is defined as the average efficiency of workers, i.e. the mean of the technological frontier,

$$A(t) \equiv A_{K(t)} \equiv \int_{q_0}^{\infty} z \, dG_1(z; K(t)),$$

where the notation A_K makes explicit the dependence of productivity on the stock of research. This relationship between productivity and research, unlike that between patents and research, is sensitive to the functional form of the stationary search distribution.

Three stationary search distributions are illustrative: (i) the Pareto, $F(q) = 1 - (q/q_0)^{-1/\lambda}$ for $\lambda > 0$, (ii) the exponential, $F(q) = 1 - e^{-(q-q_0)/\lambda}$ for $\lambda > 0$, and (iii) the uniform, $F(q) = (q - q_0)/(\lambda - q_0)$ for $\lambda > q_0$ and $q \leq \lambda$. The following results, given a stock of research $K(t) = K$, are derived in the Appendix. If the stationary search distribution is Pareto, then the distribution of the technological frontier is Fréchet (type 2 extreme value) and average efficiency is

$$A_K = c_1 K^{\lambda(1+\gamma)} + \epsilon(K). \tag{8}$$

If the stationary search distribution is exponential, then the distribution of the technological frontier is Gumbel (type 1 extreme value) and average efficiency is

$$A_K = c_0 + c_1 \ln K + \epsilon(K). \tag{9}$$

If the stationary search distribution is uniform, then the distribution of the technological frontier is Weibull (type 3 extreme value) and average efficiency is

$$A_K = c_0 - c_1 K^{-(1+\gamma)} + \epsilon(K). \tag{10}$$

The Appendix provides expressions for the constants ($c_0 > 0, c_1 > 0$) and proves that the approximation terms $\epsilon(K)$ are of smaller order than $e^{-K/(1+\gamma+b)}$ for $b > 0$.[20]

As the research stock gets large: (i) in the Pareto case the growth of productivity becomes proportional to the growth of the research stock, (ii) in the exponential case the increment to productivity becomes proportional to the growth of the research stock, and (iii) in the uniform case the level of productivity approaches an upper bound. None of these cases generate the endogenous growth equation, $A = e^{\lambda K}$, in which productivity growth is proportional to the level of research (Romer (1990), Grossman and Helpman (1991), and Aghion

[20] In other words, for any $b > 0$, $\lim_{K \to \infty} e^{K/(1+\gamma+b)} \epsilon(K) = 0$.

and Howitt (1992)). This is not troubling as Jones (1995b) showed that the endogenous growth specification does not come close to fitting the data. Only the Pareto case is consistent with the trend of constant productivity growth associated with constant growth in research and with the standard econometric equation used to quantify the impact of research on productivity growth (Griliches (1979)).

The following proposition shows how these results generalize to other stationary search distributions.

PROPOSITION 3.2: *As the stock of research K approaches infinity, the limiting form of the distribution of the technological frontier G_1 is either Fréchet, Gumbel, or Weibull (subject to normalizing sequences). In all three cases productivity satisfies* $\lim_{K\to\infty}\{((A_{k'K}/A_K)-1)/((A_{k'K}/A_K)-1)\}<\infty$ *for any* $0<k''<k'<\infty$. *If and only if the limiting form of G_1 is Fréchet does productivity satisfy*

$$\lim_{K\to\infty}(A_{kK}/A_K)=k^b$$

for some $b>0$ *and for all* $k>0$. *Stationary search distributions F leading to the Fréchet have unbounded upper support and satisfy* $\lim_{x\to\infty}\{(1-F(xa))/(1-F(x))\}=a^{-b}$, *for all* $a>0$ *and for some* $b>0$.

The proof is in the Appendix.

Proposition 3.2 is an application of the theory of extremal distributions as described succinctly in Billingsley (1986, pp. 197–199) and exhaustively in Galambos (1987).[21] Although this application is not the standard one, as the search distributions improve over time and the number of trials is random, the basic results from the theory of extremes go through. In particular, if there is a limiting form of the distribution of the technological frontier it will be either Fréchet, Gumbel, or Weibull. In each case, the productivity equation $A_K=e^{\lambda K}$ can be ruled out (given Assumption 3.1 on the spillover function).

The central result of Proposition 3.2 is to characterize the family of stationary search distributions that can produce the observed trend in productivity given the trend in research. This family consists of exactly those stationary search distributions that are in the domain of attraction of the Fréchet extreme value distribution. It includes the Pareto and Cauchy distributions among others. Any stationary search distribution in the Fréchet's domain of attraction leads (asymptotically) to the empirically plausible power function relationship between productivity and the stock of research. Since the Pareto is the easiest to work with, it is assumed throughout the remainder of the paper.

ASSUMPTION 3.2: *The stationary search distribution is Pareto,* $F(q)=1-q^{-1/\lambda}$ *with* $0<\lambda<1$. *The initial level of efficiency is* $q_0=1$.

[21] This theory was also applied by Muth (1986) in his model of the learning curve.

The mean of the Pareto distribution is $1/(1-\lambda)$.[22]

Given the assumptions above, productivity growth $\dot{A}(t)/A(t)$ is approximately $\lambda(1+\gamma)\dot{K}(t)/K(t)$. The elasticity of productivity with respect to the stock of research, $\lambda(1+\gamma)$, is large if the Pareto search distribution is rich (as measured by λ) or if research spillovers are potent (as measured by γ).[23] Combining the patent equation and the productivity equation, productivity growth is approximately $\lambda I(t)$. It can be shown that λ is the average size of a patent if size is measured as the logarithm of the inventive step.[24] Productivity growth is simply the average patent size multiplied by the rate at which patentable inventions are discovered.[25] More potent research spillovers do not affect the average size of patents but do increase the rate at which they are discovered. A richer stationary search distribution increases the average size of patents but does not affect the rate at which they are discovered, given research effort.

The discussion above equates productivity with average efficiency, yet a more typical measure of productivity is output per worker, $X(t)/[L(t)-R(t)]$. It turns out that the two measures move in parallel as the stock of research approaches infinity. To show this requires an expression for the limiting distribution of the inventive step of inventions in use.

PROPOSITION 3.3: *The limiting distribution of the inventive step of inventions in use (as the stock of research $K\to\infty$) is $G_2(y;K)\xrightarrow{d} G_2(y)=1-a(y)y^{-1/\lambda}$, for $y\in[1,\infty)$ where $a(y)\equiv \ln(y^{1/\lambda})/(1-y^{-1/\lambda})>1$ for all $y>1$ and $a(1)=1$. Output per worker relative to average efficiency,*

$$b(K(t))\equiv \frac{X(t)}{L(t)-R(t)}\bigg/A(t),$$

converges to a constant, $\lim_{K\to\infty} b(K)=b<\infty$.

[22] Bental and Peled (1996) use the Pareto distribution to obtain results on endogeneous growth in their search-theoretic model.

[23] The productivity growth equation can be rearranged to yield the form used by Jones (1995a), $\dot{A}(t)\approx\lambda(1+\gamma)R(t)A(t)^{[\lambda(1+\gamma)-1]/[\lambda(1+\gamma)]}$. The exponent on the level of productivity is negative if the degree of spillovers γ is small and if λ is not too big.

[24] From the derivation of $H(z',y;K)$ in the Appendix, the distribution of the inventive step for new inventions $H_2(y|z';K)$ is Pareto if the stationary search distribution is Pareto. Adapting an argument in Billingsley (1986, p. 191) $H_2(y|z';K)$ is Pareto only if the stationary search distribution is Pareto. In the Pareto case $H_2(y|z';K)=H_2(y)=1-y^{-1/\lambda}$ and hence the logarithm of the inventive step is exponentially distributed with mean λ. The return on a patent is $\pi=(1-y^{-1})X(t)$ and hence the coefficient of variation of the distribution of returns is $(1+2\lambda)^{-1/2}\leq 1$. Empirical evidence suggests greater variation in returns to patents. Estimates of a patent renewal model by Schankerman and Pakes (1986) imply a coefficient of variation of about 3 for the returns to patent protection in three European countries. Scherer (1996), extending a note in Scherer (1965), calculates a coefficient of variation of nearly 5 for the distribution of royalties on Harvard University's patents (sometimes royalties on related patents were combined). Using data from Grabowski and Vernon (1990) on the value of new pharmaceutical entities, Scherer calculates a coefficient of variation somewhat below 2 (I am indebted to Professor Scherer for sharing these statistics, which do not appear in his paper).

[25] Kortum and Lach (1995) find evidence for such a relationship between productivity and patenting at the industry level.

The proof is in the Appendix.

The limiting distribution $G_2(y)$ is of independent interest because it represents the distribution of price markups. A corollary of Proposition 3.3 is that the limiting distribution of price markups is stochastically increasing in the richness of the stationary search distribution, λ.[26]

3.3. *Research*

Given exponential growth of the research stock, the assumptions above ensure that patents are discovered at a constant rate and productivity grows at a constant rate. To ensure that exponential growth of the research stock is an equilibrium requires an assumption about the path of the labor force.

If the research stock grows exponentially, the number of researchers must eventually grow exponentially and so too must the labor force, since $\dot{K}(t) = R(t) \leq L(t)$. The following assumption is therefore maintained throughout the rest of the paper

ASSUMPTION 3.3: *The labor force grows at a constant rate, $L(t+s)/L(t) = e^{ns}$, for some $n > 0$ and all $s \geq 0$.*

Note that $L(t)$ should be interpreted as effective units of labor in which case its growth could in part be due to gains in education or other factors improving the efficiency of labor in both production work and research.

Assumption 3.3 guarantees enough *potential* researchers to generate exponential growth of the research stock, but the *actual* number is determined in equilibrium. According to the definition of equilibrium, there will be no researchers at date t unless the return to doing research is at least as great as the wage, $E(t) \geq 1$. The equation for $E(t)$ is greatly simplified by letting both the stock of research and the labor force at date t pass to infinity holding fixed the ratio $k(t) \equiv K(t)/L(t)$. Under such limiting conditions (which will be maintained in what follows), and given an arbitrary future path of research intensity $\{\alpha(\cdot)\} \equiv \{R(t+s)/L(t+s) \mid s \geq 0\}$, the Appendix shows that $E(t)$ can be expressed as $E(k(t); \{\alpha(\cdot)\})$.

The research stock normalized by the labor force evolves according to the differential equation $\dot{k}(t) = \alpha(t) - nk(t)$, where $\alpha(t) \equiv R(t)/L(t)$. An initial condition $k(t) = k^*$ and a level of research intensity $\alpha(t) = \alpha^* = nk^*$, if they satisfy $E(k^*; \{\alpha^*\}) = 1$, will support an equilibrium in which both research intensity and k are constant.[27]

[26] The distribution of price markups stochastically dominates the distribution of markups on new patentable inventions, $H_2(y)$, because inventions with larger inventive steps tend to be used for longer before becoming obsolete.

[27] Although the following proposition provides sufficient conditions for local stability, it does not specify how the equilibrium with constant research intensity is reached from an arbitrary initial condition $k(t)$.

PROPOSITION 3.4: *Given an arbitrarily large research stock and labor force at date t and assuming $\rho > \lambda n(1+\gamma) - n\gamma$, there exists an initial condition $k(t) = k^*$ and corresponding $\alpha^* = nk^*$ such that $\{k(t+s) = k^*,\ \alpha(t+s) = \alpha^* | s \geq 0\}$ is an equilibrium. The level of research intensity is*

$$\alpha^* = \alpha^*(r^*, \lambda) \equiv \frac{1}{1 + \dfrac{\theta(\lambda)(1 + r^* - \lambda)}{1 - \phi(r^*, \lambda)}},$$

where $\theta(\lambda) \equiv \int_1^\infty y^{-1}\, dG_2(y)$, $\phi(r^*, \lambda) \equiv (1 + r^* - \lambda)\int_0^\infty w^{r^*} e^w \Gamma(\lambda - r^*, w)\Gamma(1 - \lambda, w)\, dw$, $r^* \equiv ((\rho/n) - 1)/(1+\gamma)$, *and* $\Gamma(a, w) \equiv \int_w^\infty e^{-s} s^{a-1}\, ds$ *is the incomplete gamma function. Equilibrium research intensity is decreasing in* r^* *and increasing in* λ. *The equilibrium is locally stable (it will be approached from any* $k(t)$ *in a neighborhood of* k^**) if*

$$(1+\gamma)^{-1} > \frac{\lambda}{\theta(\lambda)(1+\lambda)\alpha^*(r^*, \lambda)} - r^*.$$

For other regions of the parameter space a sufficient condition for local stability is provided in the Appendix.

The proof is in the Appendix.

Since $r^* \equiv ((\rho/n) - 1)/(1+\gamma)$, Proposition 3.4 implies that equilibrium research intensity decreases in the discount rate and increases in the rate of labor force growth and in the degree of spillovers. If $r^* = \lambda$, research intensity is equal to the profit share of income, $\alpha^* = 1 - \theta(\lambda)$.[28] In general, the equilibrium level of research intensity must be computed numerically; see Table II.

In the equilibrium characterized by Proposition 3.4, research intensity is constant, the research stock grows at an exponential rate n, productivity and the wage grow at rate $n\lambda(1+\gamma)$, and aggregate income grows at rate $n[\lambda(1+\gamma) + 1]$. The expected value of new patentable inventions grows at the same rate as aggregate income, providing an incentive for continued research even as the opportunity cost of doing research rises (due to wage growth) and the chance of discovering a patentable invention falls (due to the growth of the research stock). Interestingly, the rise in the expected value of new patents is not experienced by any existing cohort of patents. A patent faces a hazard of being surpassed that increases over its life at an exponential rate of $n(1+\gamma)$. In the aggregate, this increasing hazard is exactly offset by the inflow of new patents

[28] This simplification occurs because if $\rho = n[\lambda(1+\gamma) + 1]$ and the stock of research grows at the rate n then discounting and market-growth exactly offset each other in the equation for the return to doing research relative to the wage.

TABLE II

SIMULATION RESULTS

λ	$\int_1^\infty y\,dG_2(y)$	$1-\theta(\lambda)$	$r^* = .03$ $\alpha^*(r^*,\lambda)$	$r^* = .10$ $\alpha^*(r^*,\lambda)$	$r^* = .30$ $\alpha^*(r^*,\lambda)$	$r^* = 1.0$ $\alpha^*(r^*,\lambda)$
.01	1.017	.016	.016	.015	.012	.007
.02	1.034	.032	.032	.029	.023	.013
.03	1.052	.047	.047	.043	.034	.020
.10	1.192	.143	.157	.143	.115	.067
.30	1.850	.340	.440	.410	.340	.208

Notes: The numerical calculations are performed using Mathematica. The column labeled $\int_1^\infty y\,dG_2(y)$ shows the average level of price markups, the column labeled $1-\theta(\lambda)$ shows the profit share of income, and the remaining columns show the equilibrium level of research intensity for different values of $r^* = ((\rho/n)-1)/(1+\gamma)$. The rows correspond to different values of λ.

with, on average, lower hazard rates so that the aggregate hazard $\eta = (1+\gamma)n$ is constant.[29]

The private rate of return to research in the equilibrium characterized by Proposition 3.4 is simply the discount rate, ρ. The social rate of return to research is the implicit discount rate such that a social planner chooses the equilibrium level of research intensity. The social planner's problem, solved in the Appendix, implies that the equilibrium social rate of return to research is equal to productivity growth relative to research intensity, $\lambda(1+\gamma)n/\alpha^*(r^*,\lambda)$.[30] The social rate of return to research is high if the spillover parameter γ is large. For example, the social rate of return exceeds the private rate of return for all of the cases covered in Table II if $\gamma > 3/2$.[31]

3.4. *Quantitative Implications*

The restricted model exhibits a steady state in which: (i) research grows at a constant rate, (ii) productivity grows at a constant rate, (iii) patenting is constant, and (iv) research intensity is constant. Predictions (i)–(iii) are roughly consistent with the experience of the United States over the past forty years, as

[29] A patented invention with efficiency z at time t, faces a hazard rate of $\eta(z,t) = R(t)K(t)^\gamma z^{-1/\lambda} = (1+\gamma)(\alpha(t)/k(t))z^*$, where $z^* = (1+\gamma)^{-1}K(t)^{1+\gamma}z^{-1/\lambda}$. The Appendix (in deriving an expression for $E(k(t); \{\alpha(\cdot)\}))$ shows that the random variable associated with z^* has a distribution $H_1^*(z^*)$ which is exponential with a mean of one. In the equilibrium described by Proposition 3.4, $(\alpha(t)/k(t)) = n$ and hence $\eta(z,t) = \eta(z^*) \equiv (1+\gamma)nz^*$. The aggregate hazard is therefore $\eta = \int_0^\infty \eta(z^*)\,dH_1^*(z^*) = (1+\gamma)n$.

[30] A similar result is invoked in empirical studies that infer the social return to research from the relation between productivity growth and research intensity (Griliches and Lichtenberg (1984)). Jones and Williams (1995) investigate this result in a model with diminishing returns to current research.

[31] The transversality condition for the social planner's problem requires $r^* > \lambda$, so that the comparison between the social and private rate of return can be made in only fourteen of the cases in Table II.

can be seen in Table I.[32] Prediction (iv) is clearly rejected, as can be seen from the rise in research intensity (in the bottom row of Table I).[33] Rather than trying to account for rising research intensity, the goal is to find a vector of parameters that generates research intensity close to the level observed recently. The model can then be assessed based on its quantitative implications for price markups, the rate at which patents lose value, and the social rate of return to research.

The following parameter vector is explored (the discount rate, labor force growth, the parameter of the stationary search distribution, and the spillover parameter, respectively):

$$\rho = 0.07, \quad n = 0.03, \quad \lambda = 0.03, \quad \gamma = 9.$$

The discount rate is chosen to match the real return on the stock market. The parameter n is chosen as a compromise between labor force growth of almost two percent and research growth (which should equal labor force growth in a steady state) of almost five percent (the data are presented in Table I). The parameters λ and γ yield productivity growth, $\lambda(1+\gamma)n$, of nearly one percent, a bit less than total factor productivity growth from 1953–1993.[34] The implied elasticity of productivity with respect to the stock of research $\lambda(1+\gamma)$ is 0.3, the same value proposed by Griliches (1992) as being representative of estimates of the combined direct and indirect impacts of research on productivity.[35] Given the elasticity of productivity with respect to the research stock, research intensity is smaller if research spillovers are greater, i.e. if γ is larger and λ is smaller. The values that were chosen imply research intensity of about four percent, somewhat above the observed ratio of industry funded research to the compensation of labor over the last decade.

The small value of λ implies modest price markups of only 5 percent (see the second column of Table II). Five percent markups are on the low side of most estimates, although they are consistent with findings of Basu and Fernald (1997). The aggregate hazard rate for patent returns is thirty percent. Evidence from patent renewal data suggests that the actual hazard rate is closer to ten

[32] Total factor productivity growth in manufacturing is used because output is relatively well measured in the manufacturing sector and because labor productivity growth due to pure capital deepening should not be attributed to technological change.

[33] Subtracting line 3 from line 2 in Table I reveals an even more dramatic rise in research intensity. But, research intensity should be interpreted as the fraction of human capital devoted to research, rather than simply the fraction of researchers in the labor force. To capture this, in line 5, industry financed R&D expenditure is divided by the total compensation of employees in private industry.

[34] Since total factor productivity treats capital as an input, $L(t)$ in the model might be interpreted as an aggregate of labor and capital that, in a steady state, grows faster than labor itself. This provides some justification for choosing n to be much larger than labor force growth.

[35] The value of 0.3 is at least three times the typical econometric estimate of the direct effect of R&D on productivity.

percent.[36] Given $\lambda(1+\gamma)=.3$, there is a tension between the larger value of the spillover parameter γ that would match the level of research intensity and the smaller value of γ that would match a hazard rate consistent with the renewal data. The social rate of return to research implied by the parameters is 22 percent, three times the private rate of return.[37] A social planner would raise research intensity to 13 percent.

4. CONCLUSION

This paper develops a search theory of invention to explain the long-run behavior of research employment, patenting, and TFP growth. According to this theory, patents per researcher decline over time as technological breakthroughs become increasingly hard to come by. In fact the number of researchers must rise exponentially to generate a constant flow of new patented inventions. The growth in research effort produces constant productivity growth if the size distribution of inventions is stationary (where size is measured by percent efficiency gain). The size distribution will in fact be stationary if the search distribution of potential new techniques is Pareto. The growth in research employment itself is fueled by an increase in the value of patented inventions relative to wages, which is in turn sustained by growth in the labor force.

A key implication of the theory is that the value of patented inventions rises over time causing researchers to expend ever greater resources to discover them. There are two pieces of corroborating evidence in patent statistics. First, Schankerman and Pakes (1986) found that in the United Kingdom, France, and Germany the age at which patents were allowed to lapse (because the inventor failed to pay a renewal fee) tended to rise over time. From 1965–1975 patents per researcher fell sharply in these countries, but the decline was offset by a rise in the average value of a patent (as estimated from the renewal statistics). Second, there is a long-run trend for inventions to be patented more internationally. In the 1950's, the ratio of U.S. inventions seeking patent protection in the United Kingdom to U.S. inventions seeking patent protection in the United States was about ten percent. By the 1970's this ratio had climbed to almost twenty percent and by the early 1990's it was 25 percent. The increase has been even more dramatic for U.S. inventions seeking patent protection in Germany

[36] In many countries inventors must pay annual patent renewal fees to maintain their patent rights. Failing to renew may indicate that a patent has been surpassed by a better patent or that the inventive step is so small that the patent is not worth renewing. In the United States, twelve-year renewal fees were collected for the first time in 1994. Of patents granted in 1982, thirty-six percent paid the fee, indicating an annual hazard of eight percent (Brown (1995)). Lanjouw (1993) finds hazard rates of about ten percent in Germany, while Pakes (1986) reports similar hazard rates for France and the United Kingdom.

[37] In their case studies of individual innovations, Mansfield et al. (1977) find that the median social rate of return is about twice the median private rate of return.

and Japan.[38] Although costs of patenting are not part of the model, the trend towards broader patent protection is indicative of patentable inventions becoming more valuable.

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APPENDIX A: DATA

The patent data in Figure 1 are patent applications in the United States by U.S. inventors and patents issued (granted) by the U.S. Patent Office to U.S. Inventors (from WIPO (1983), WIPO (annual issues), and tabulations by the U.S. Patent Office). The research data are R&D scientists and engineers employed in U.S. industry adjusted by the fraction of R&D performed in industry that is industry funded, NSF (1987, 1995). The productivity data in Table I are from BLS (1996). Multifactor productivity in manufacturing is the growth of the net output of the manufacturing sector less the contribution of the growth in hours worked, capital input, purchased services, and materials purchased from outside the manufacturing sector (Guillickson (1992)). The civilian labor force data are from CEA (1995). Total R&D funded by industry is from NSF (1995b) and Industry Compensation is from BEA (1992) and BEA (1992b, 1994).

APPENDIX B: SYMBOLS

$L(t)$	Labor force.
$R(t)$	Researchers.
$K(t)$	Stock of past research effort ($\dot{K}(t)=R(t)$).
$k(t)$	Research stock relative to the labor force.
$P(t)$	Aggregate price index (numeraire).
$W(t)$	Wage for production workers.
$X(t)$	Aggregate income.
$I(t)$	Rate of patenting.
$A(t)$	Average efficiency (productivity).
$E(t)$	Return to research relative to the wage.
q	Index for the efficiency of a technique.
$F(q;K)$	Search distribution.
$F(q)$	Stationary search distribution.
$S(K)$	Spillover function.
$\Sigma(K)$	Accumulated spillover function $\int_0^K s(x)\,dx$.
z	Index for the state of the art.
y	Index for the inventive step.
$V(t)$	Expected value of a patentable invention.
$\pi(y,t)$	Profits on a technique with inventive step y.
$G_1(z;K)$	Distribution of the state of the art.
$G_2(y;K)$	Distribution of the inventive step (for inventions in use).
$p(K)$	Probability that an idea is patentable.
$\Phi(q,K(t+x),K(t))$	Distribution of patent lives.
$H(z';y;K)$	Joint distribution of the efficiency surpassed and the inventive step of new patentable inventions.

[38] These figures are from WIPO (various issues) and Federico (1964).

ρ	Discount rate.
n	Population growth.
γ	Spillover parameter.
η	Hazard rate for patents.
λ	Parameter of the stationary search distribution.
$\theta(\lambda)$	Mean inverse inventive step.
r^*	$=((\rho/n)-1)/(1+\gamma)$.
$\alpha(\cdot)$	Path of research intensity, $R(t)/L(t)$.
$\alpha^*(r^*, \lambda)$	Equilibrium research intensity.

APPENDIX C: PROOFS

Proof of Proposition 2.1

Due to Poisson arrivals, $e^{-[K(s+ds)-K(s)]}$ is the probability that no new technique for producing good j is discovered between time s and $s+ds$. For ds arbitrarily small, $e^{-R(s)[1-F(z;K(s))]\,ds}$ is the probability that no technique more efficient than z is discovered. Thus,

$$G_1(z; K(s+ds)) = G_1(z; K(s))e^{-R(s)[1-F(z;K(s))]\,ds},$$

where $G_1(z; K(t))$ is the probability that the state of the art for producing good j at date t has an efficiency no greater than z. Letting $J(s) \equiv \ln G_1(z; K(s))$ it follows that $\dot{J}(s) = -R(s)[1-F(z; K(s))]$. The initial level of efficiency is q_0, hence $\lim_{s\to-\infty} J(s) = 0$. Integrating this differential equation yields $J(t) = -\int_{-\infty}^{t} R(s)[1-F(z; K(s))]\,ds = -\int_0^{K(t)}[1-F(z;x)]\,dx$. To simplify the expression for $J(t) = \ln G_1(z; K(t))$, two cases must be considered: (i) if $z \geq \bar{q}(K(t))$ then, from Assumption 2.1,

$$\ln G_1(z; K(t)) = -[1-F(z)]\int_0^{K(t)} S(x)\,dx,$$

and (ii) if $z \in [q_0, \bar{q}(K(t))]$ then, from Assumption 2.1,

$$\ln G_1(z; K(t)) = -[1-F(z)]\int_0^{\bar{K}(z)} S(x)\,dx - \int_{\bar{K}(z)}^{K(t)} dx,$$

where, in the latter case, $\bar{K}(z) = \min\{\bar{K} \in [0, K(t)] \mid 1 - S(\bar{K})[1-F(z)] \leq 0\}$. The result follows by exponentiating these equations and letting $\Sigma(K) = \int_0^K S(x)\,dx$. The derivation above uses the fact that if $z \in [q_0, \bar{q}(K(t))]$ then either $\bar{K}(z) = 0$ or else if $0 \leq K < \bar{K}(z)$ then $1 - S(K)[1-F(z)] > 0$. The two expressions for $\ln G_1(z; K(t))$ are equal at $z = \bar{q}(K(t))$ because either $\bar{K}(\bar{q}(K(t))) = K(t)$ or else if $\bar{K}(\bar{q}(K(t))) < K < K(t)$ then $1 - S(K)[1-F(\bar{q}(K(t)))] = 0$. *Q.E.D.*

An Expression for $\Phi(q, K(t+x), K(t))$

A patent having efficiency q faces a hazard rate of $\eta(q,s) \equiv R(s)[1-F(q; K(s))]$ of being surpassed at time s. It follows that a patent being used at time t has a probability $\exp\{-\int_t^{t+x}\eta(q,s)\,ds\} = \exp\{-\int_{K(t)}^{K(t+x)}[1-F(q;K)]\,dK\}$ of surviving through time $t+x$. Thus,

$$\Phi(q, K(t+x), K(t)) = 1 - \exp\left\{-\int_{K(t)}^{K(t+x)}[1-F(q;K)]\,dK\right\}.$$

Given $K(t)$, the probability of a patent surviving from date t to date $t+x$ is a decreasing function of $K(t+x)$.

An Expression for $H(z', y; K)$

Let the random variable Y be the inventive step and let Z' be the efficiency of the technique it supplants. Their joint density, $h(z', y; K)$ can be factored as $h_1(z'; K)h_2(y|z'; K)$. The distribution functions associated with these densities can be derived, beginning with the second. The conditional distribution of the inventive step is

$$H_2(y|z'; K) \equiv \Pr(Y \leq y|z'; K) = \frac{[1 - F(z'; K)] - [1 - F(z'y; K)]}{1 - F(z'; K)}$$

$$= \frac{F(z'y) - F(z')}{1 - F(z')},$$

for $z' \geq \bar{q}(K)$. For $z' < \bar{q}(K)$, any idea will surpass the state of the art, thus $H_2(y|z'; K) = F(z'y; K)$ in that case. The distribution of the efficiency surpassed is

$$H_1(z'; K) \equiv \Pr(Z' \leq z'; K) = p(K)^{-1}\left\{\int_{q_0}^{z'} [1 - F(x; K)]\, dG_1(x; K)\right\}.$$

Note that this distribution will have a mass point at the initial level of efficiency: $H_1(q_0; K) = p(K)^{-1}G_1(q_0; K)$.

An Expression for $G_2(y; K)$

Let $G(z, y; K(t))$ be the measure of goods which are produced at date t with efficiency less than or equal to z using a technique whose inventive step is less than or equal to y. An expression for this joint distribution can be obtained by integrating over all past cohorts $(s < t)$ of patentable inventions, taking account of their qualities, their inventive steps, and the likelihood that they will still be in use by date t:

$$G(z, y; K(t)) = \int_{-\infty}^{t} p(K(s))R(s)$$

$$\times\left\{\int_1^y \int_{q_0}^{z/y'} [1 - \Phi(z'y', K(t), K(s))]\, dH(z', y'; K(s))\right\} ds.$$

Changing the variable of integration to $x = K(s)$, setting $K = K(t)$, and letting z go to infinity,

$$G_2(y; K) = \int_0^K p(x)\left\{\int_1^y \int_{q_0}^{\infty} [1 - \Phi(z'y', K, x)]\, dH(z', y'; x)\right\} dx.$$

Proof of Proposition 3.1

For $K(t) \geq 1$, the rate of patenting is given by

$$I(t) = R(t)\frac{S(K(t))}{\Sigma(K(t))}[1 - e^{-\Sigma(K(t))/S(K(t))}].$$

If the rate of patenting approaches a constant I even as research increases, then $\lim_{t\to\infty} S(K(t))/\Sigma(K(t)) = 0$. If research grows exponentially at rate g, then $\lim_{t\to\infty} R(t)/(gK(t)) = 1$. Therefore, if research grows exponentially at rate g, the rate of patenting approaches a constant I if and only if

$$\lim_{K\to\infty} \frac{gKS(K)}{\Sigma(K)} = I. \tag{11}$$

If (11) held identically for all $K \geq 1$, then the spillover function would be $S(K) = K^{(I/g)-1}$. But, since (11) is only an asymptotic condition, more analysis is required.

Suppose that there exists a number b such that $\lim_{K \to \infty} K^{-b}S(K) = a$ for some $0 < a < \infty$. It follows that for any $\epsilon > 0$ there exists a number $\bar{x} < \infty$ such that for all $x \geq \bar{x}$, $(a - \epsilon)x^b \leq S(x) \leq (a + \epsilon)x^b$. Define $\underline{S} \equiv \min_{x \in [0, \bar{x}]}\{S(x)\} \geq 0$ and $\bar{S} \equiv \max_{x \in [0, \bar{x}]}\{S(x)\} < \infty$. For any $K > \bar{x}$,

$$\Sigma(K) \equiv \int_0^K S(x)\, dx \leq \bar{x}\bar{S} + \frac{a + \epsilon}{b + 1}(K^{b+1} - \bar{x}^{b+1})$$

and $\Sigma(K) \geq \bar{x}\underline{S} - ((a - \epsilon)/(b + 1))(K^{b+1} - \bar{x}^{b+1})$. Since ϵ can be arbitrarily small, given that K can be arbitrarily large, it follows that $\lim_{K \to \infty} K^{-(b+1)}\Sigma(K)/(b + 1) = a$. Plugging these results into equation (11),

$$I = \lim_{K \to \infty} \frac{gKS(K)}{\Sigma(K)} = \lim_{K \to \infty} \frac{g(b + 1)K^{-b}S(K)}{K^{-(b+1)}\Sigma(K)/(b + 1)} = g(b + 1).$$

The equation above holds if and only if $b = (I/g) - 1$. *Q.E.D.*

Derivation of Equations (8), (9), *and* (10)

Substituting in for $F(q)$ either a Pareto distribution, an exponential distribution, or a uniform distribution, $G_1(z; K)$ becomes a Fréchet (type 2 extreme value) distribution, a Gumbel (type 1 extreme value) distribution, or a Weibull (type 3 extreme value) distribution respectively, for $z \geq \bar{q}(K)$. The approximation terms in equations (8), (9), and (10) arise because $G_1(z; K)$ does not equal any of these extreme value distributions for $a < z < \bar{q}(K)$, where $a = 0$ for the Fréchet distribution and $a = -\infty$ for the other two. Leaving aside this issue, the properties of the three extreme value distributions (in particular their means) are provided in Castillo (1988, Ch. 5). Applying the results there: (i) in equation (8) $c_1 = q_0\Gamma(1 - \lambda)/(1 + \gamma)^{\lambda}$, where $\Gamma(1 - \lambda) \equiv \int_0^{\infty} x^{-\lambda}e^{-x}\, dx$; (ii) in equation (9) $c_0 = q_0 + \lambda(\psi - \ln(1 + \gamma))$ and $c_1 = \lambda(1 + \gamma)$, where $\psi \approx .57772$ is Euler's constant; and (iii) in equation (10) $c_0 = \lambda$ and $c_1 = \lambda(1 - q_0/\gamma)(1 + \gamma)$.

To derive the terms of small order, $\epsilon(K)$, define $G_1^*(z; K) \equiv e^{-(1-F(z))(K^{1+\gamma}/(1+\gamma))}$ for $z \geq a$, zero otherwise. Let $A_K^* \equiv \int_a^{\infty} z\, dG_1^*(z; K)$ and $A_K \equiv \int_{q_0}^{\infty} z\, dG_1(z; K)$ (it is expressions for A_K^* that Castillo provides). The difference in means (Billingsley (1986, problem 21.9)) is

$$\begin{aligned} A_K - A_K^* &= \int_{-\infty}^{\infty} [G_1^*(z; K) - G_1(z; K)]\, dz \\ &= \int_a^0 G_1^*(z; K)\, dz + \int_0^{\bar{q}(K)} G_1^*(z; K)\, dz - \int_{q_0}^{\bar{q}(K)} G_1(z; K)\, dz. \end{aligned}$$

The first term on the right equals zero for the Fréchet distribution (since $a = 0$), it equals $e^{-q_0/\lambda}(1 + \gamma)K^{-(1+\gamma)}e^{-(e^{q_0/\lambda}K^{1+\gamma})/(1+\gamma)}$ for the Gumbel distribution, and for the Weibull distribution it equals $(\lambda - q_0)(1 + \gamma)K^{-(1+\gamma)}e^{-\lambda K^{1+\gamma}/((\lambda - q_0)(1+\gamma))}$. The second and third terms are each bounded above by $\bar{q}(K)G_1^*(\bar{q}(K); K) = \bar{q}(K)e^{-K/(1+\gamma)}$, for $K > 1$. Given $K > 1$, $\bar{q}(K) = q_0K^{\lambda\gamma}$ if $F(z)$ is Pareto, $\bar{q}(K) = q_0 + \lambda\gamma \ln K$ if $F(z)$ is exponential, and $\bar{q}(K) = \lambda - (\lambda - q_0)K^{-\gamma}$ if $F(z)$ is uniform. Thus, the difference between A_K and A_K^* is of smaller order than $e^{-K/(1+\gamma+b)}$, for $b > 0$.

Proof of Proposition 3.2

It is convenient to work with the following approximation to the distribution of the technological frontier, $G_1^*(z; K) \equiv e^{-(1-F(z))K^{1+\gamma}/(1+\gamma)}$ for $z \geq q_0$ (zero otherwise). From Proposition 2.1 and applying Assumption 3.1, $G_1(z; K) = G_1^*(z; K)$ for $z \geq \bar{q}(K)$. Expressing productivity as $A_{K(t)} = \int_{q_0}^{\infty} z\, dG_1(z; K(t))$ and defining $A_K^* = \int_{q_0}^{\infty} z\, dG_1^*(z; K)$, it follows that $A_K - A_K^* = \int_{q_o}^{\bar{q}(K)}[G_1^*(z; K) - G_1(z; K)]\, dz$ and $A_K^* \geq \int_{\bar{q}(K)}^{\infty} \bar{q}(K)\, dG_1(z; K)$. Using these results and given $K \geq 1$,

$$\frac{|A_K - A_K^*|}{A_K^*} \leq \frac{\bar{q}(K)G_1(\bar{q}(K); K)}{A_K^*} \leq \frac{G_1(\bar{q}(K); K)}{1 - G_1(\bar{q}(K); K)} = \frac{e^{-K/(1+\gamma)}}{1 - e^{-K/(1+\gamma)}}.$$

Thus, $\lim_{K\to\infty}((A_K - A_K^*)/A_K^*) = 0$ and hence results on the behavior of productivity as K gets large can be derived from G_1^*.

Note that $G_1^*(z;K) = F^*(z)^N$, where $N = K^{1+\gamma}/(1+\gamma)$ and $F^*(z) = e^{-(1-F(z))}$ for $z \geq q_0$ ($F^*(z) = 0$ otherwise). Thus G_1^* can be interpreted as the distribution of the maximum of N independent trials from the distribution F^*. The theory of extremal distributions derives asymptotic properties of distributions arising in this way (Galambos (1987), Billingsley (1986, pp. 197–199)). Given regularity conditions on F^* there exist sequences $a_N > 0$ and b_N such that $[F^*(a_N x + b_N)]^N \xrightarrow{d} G^*(x)$. The extremal distribution takes one of only three forms (Billingsley (1986, Theorem 14.3)):

$$G^*(x) = e^{-x^{-d_0}}, \quad x \geq 0 \text{ (zero otherwise), or}$$
$$G^*(x) = e^{-e^{-x}}, \quad \text{or}$$
$$G^*(x) = e^{-(-x)^{d_0}}, \quad x \leq 0 \text{ (one otherwise),}$$

where $d_0 > 0$ is a constant. These are just the standard forms of the Fréchet, Gumbel, and Weibull distributions, respectively. This proves the first statement in Proposition 3.2.

Proving the next two statements requires results about the mean A_K^* of G_1^*. It is related to the mean d_1 of the appropriate distribution G^* by

$$\lim_{K\to\infty} \frac{A_K^* - b_{N(K)}}{a_{N(K)}} = d_1,$$

where $N(K) \equiv K^{1+\gamma}/(1+\gamma)$. To derive the asymptotic behavior of the normalizing sequences, note that for $m > 0$, $[F^*(a_N x = B_N)]^{mN} \xrightarrow{d} [G^*(x)]^m$ and $[F^*(a_{mN} x + b_{mN})]^{mN} \xrightarrow{d} G^*(x)$. Thus by Theorem 14.2 in Billingsley (1986), $[G^*(x)]^m = G^*(a_m^* x + b_m^*)$ where $a_m^* = \lim_{N\to\infty}(a_N/a_{mN})$ and $b_m^* = \lim_{N\to\infty}(b_N - b_{mN})/a_{mN}$. Solving the equation $m \ln G^*(x) = \ln G^*(a_m^* x + b_m^*)$: (i) if $G^*(x)$ is Fréchet then $a_m^* = m^{-1/d_0}$ and $b_m^* = 0$, (ii) if $G^*(x)$ is Gumbel, then $a_m^* = 1$ and $b_m^* = -\ln m$, and (iii) if $G^*(x)$ is Weibull then $a_m^* = m^{1/d_0}$ and $b_m^* = 0$. Choose $c' > 0$ and $c'' > 0$ and define $m' \equiv N(c'K)/N(K) = c'^{1+\gamma}$ and $m'' \equiv N(c''K)/N(K) = c''^{1+\gamma}$. Then

$$\lim_{K\to\infty} \frac{\dfrac{A_{c'K}}{A_K} - 1}{\dfrac{A_{c''K}}{A_K} - 1} = \lim_{K\to\infty} \frac{\dfrac{A_{c'K}^* - b_{N(K)}}{a_{N(K)}} - \dfrac{A_K^* - b_{N(K)}}{a_{N(K)}}}{\dfrac{A_{c''K}^* - b_{N(K)}}{a_{N(K)}} - \dfrac{A_K^* - b_{N(K)}}{a_{N(K)}}}$$
$$= \frac{(a_{m'}^*)^{-1}[d_1 - b_{m'}^*] - d_1}{(a_{m''}^*)^{-1}[d_1 - b_{m''}^*] - d_1} < \infty.$$

This proves the second statement in the proposition.

Solving for

$$\lim_{K\to\infty} \frac{\dfrac{A_{c'K}}{A_K} - 1}{\dfrac{A_{c''K}}{A_K} - 1}$$

in each case, the Fréchet yields $((c')^{(1+\gamma)/d_0} - 1)/((c'')^{(1+\gamma)/d_0} - 1)$, the Gumbel yields $\ln(c')/\ln(c'')$, and the Weibull yields $(1 - (c')^{-(1+\gamma)/d_0})/(1 - (c'')^{-(1+\gamma)/d_0})$. Note that the Fréchet case is necessary for productivity to satisfy $\lim_{K\to\infty}(A_{cK}/A_K) = c^b$. To prove that it is also sufficient, set $b_N = 0$ in the Fréchet, following Galambos (1987, Theorem 2.1.1). Thus,

$$\lim_{K\to\infty} \frac{A_{c'K}}{A_K} = \lim_{K\to\infty} \frac{\dfrac{a_{N(c'K)}}{a_{N(K)}} A_{c'K}^*/a_{N(c'K)}}{A_K^*/a_{N(K)}} = (c')^{(1+\gamma)/d_0},$$

which is the property that was sought.

Finally, turn to the domain of attraction of the Fréchet distribution. The extremal distribution takes this form (Galambos (1987, Theorem 2.4.3)) if and only if $F^*(x)$ has unbounded upper support and, for all $a>0$, there exists a $b>0$ such that $\lim_{x\to\infty}(1-F^*(xa))/(1-F^*(x))=a^{-b}$. By l'Hôpital's rule, the stationary search distribution $F(x)$ can be used in place of $F^*(x)=e^{-(1-F(x))}$ to check this condition. *Q.E.D.*

Proof of Proposition 3.3

An expression for the distribution of the markup, $G_2(y;K)$ for $y\geq 1$, was established in the Appendix above. Since it could be represented as an integral, it has a density of the continuous type,

$$g_2(y;K)=\int_0^K\left\{\int_{q_0}^{\infty}p(x)[1-\Phi(z'y,K,x)]\,dH(z',y;x)\right\}dx.$$

A good starting point is to establish an inequality relating $g_2(y;K)$ and the limiting density $g_2(y)$. Ignoring the values of z' in the interval $[q_0,\bar{q}(K))$ (and applying Assumptions 3.1 and 3.2) leads to simple expressions for the objects in the integrand:

$$p(x)\,dH(z',y;x)=(1+\gamma)^{-1}\lambda^{-2}x^{1+2\gamma}z'^{-(2+\lambda)/\lambda}y^{-(1+\lambda)/\lambda}e^{-(1+\gamma)^{-1}z'^{-1/\lambda}x^{1+\gamma}}\,dz'$$

and $1-\Phi(z'y,K,x)=e^{-(1+\gamma)^{-1}(z'y)^{-1/\lambda}(K^{1+\gamma}-x^{1+\gamma})}$.

Replacing q_0 with $\bar{q}(K)=K^{\lambda\gamma}$ in the second integral, substituting in the expressions for the integrand, defining $a(x,y,K)\equiv(1+\gamma)^{-1}[(1-y^{-1/\gamma})x^{1+\gamma}+y^{-1/\lambda}K^{1+\gamma}]$, and changing the variable of integration from z' to $w=z'^{-1/\gamma}a(x,y,K)$,

$$g_2(y;K)\geq\int_0^K\frac{x^{1+2\gamma}y^{-(1+\lambda)/\lambda}}{\lambda(1+\gamma)a(x,y,K)^2}[1-(K^{-\gamma}a(x,y,K)+1)e^{-K^{-\gamma}a(x,y,K)}]\,dx.$$

Since $K^{-\gamma}a(x,y,K)\geq(1+\gamma)^{-1}y^{-1/\lambda}K$ for $x\in[0,K]$,

$$g_2(y;K)\geq[1-c(y,K)]\int_0^K\frac{x^{1+2\gamma}y^{-(1+\lambda)/\lambda}}{\lambda(1+\gamma)a(x,y,K)^2}\,dx,$$

where $c(y,K)\equiv[(1+\gamma)^{-1}y^{-1/\lambda}K+1]e^{-(1+\gamma)^{-1}y^{-1/\lambda}K}$.

Changing the variable of integration from x to $w=x^{1+\gamma}$ and applying the result that $\int w(b_0w+b_1)^{-2}\,dw=b_0^{-2}[\ln|b_0w+b_1|+b_1/(b_0w+b_1)]+b_2$ (for constants b_0, b_1, and b_2), $g_2(y;K)\geq[1-c(y,K)]g_2(y)$, where $g_2(y)\equiv\lambda^{-1}y^{-(1+\lambda)/\lambda}(1-y^{-1/\lambda})^{-2}[y^{-1/\lambda}-\ln y^{-1/\gamma}-1]$. Note that $g_2(y)$ is a density function on $y\geq 1$, with a cumulative distribution function $G_2(y)=1-\lambda^{-1}(y^{1/\lambda}-1)^{-1}\ln y$.

It is thus established that $g_2(y;K)\geq g_2(y)-c(y,K)g_2(y)$, where g_2 is a probability density function of the continuous type for $y\geq 1$. Note that $c(y,K)$ is bounded above by 1, increasing in y, decreasing in K, that $\lim_{y\to\infty}c(y,K)=1$ for $K<\infty$, and that $\lim_{K\to\infty}c(y,K)=0$ for $y<\infty$. For any number $1<y^*<\infty$,

$$\int_1^{\infty}c(y',K)g_2(y')\,dy'\leq c(y^*,K)\int_1^{y^*}g_2(y')\,dy'+\left[\lim_{y\to\infty}c(y,K)\right]\int_{y^*}^{\infty}g_2(y')\,dy'.$$

For any $\epsilon>0$ choose y^* so that $1-G_2(y^*)\leq\epsilon/4$ and choose K_ϵ so that $c(y^*,K)\leq\epsilon/4$ for $K\geq K_\epsilon$. It follows that for all $K\geq K_\epsilon$,

$$\int_1^{y}c(y',K)g_2(y')\,dy'\leq\int_1^{\infty}c(y',K)g_2(y')\,dy'\leq\epsilon/2.$$

For any $\epsilon > 0$, $K \geq K_\epsilon$, and $y \geq 1$,

$$|G_2(y;K) - G_2(y)| \leq \left| \int_1^y g_2(y';K)\,dy' - \left[\int_1^y g_2(y')\,dy' - \int_1^y c(y',K) g_2(y')\,dy' \right] \right| + \epsilon/2$$
$$\leq \left| \int_1^\infty g_2(y';K)\,dy' - \int_1^\infty g_2(y')\,dy' + \int_1^\infty c(y',K) g_2(y')\,dy' \right| + \epsilon/2$$
$$\leq |1 - 1 + \epsilon/2| + \epsilon/2 = \epsilon.$$

Thus, letting K tend to infinity, $G_2(y;K) \xrightarrow{d} G_2(y)$.

From equations (4) and (5), the ratio of output per worker to average efficiency is,

$$\text{(12)} \qquad \frac{X(t)/A(t)}{L(t) - R(t)} = \frac{\exp\left\{ \int_{q_0}^\infty \ln(z)\,dG_1(z;K(t)) \right\} \Big/ \int_{q_0}^\infty z\,dG_1(z;K(t))}{\exp\left\{ \int_1^\infty \ln(y)\,dG_2(y;K(t)) \right\} \int_1^\infty y^{-1}\,dG_2(y;K(t))} = b(K(t)).$$

The second part of Proposition 3.3 is that $\lim_{K \to \infty} b(K) = b$ for $0 < b < \infty$.

The numerator of equation (12) is the ratio of the geometric mean to the arithmetic mean efficiency of techniques in use, where the underlying distribution of efficiencies is Fréchet. Letting $K(t) = K$, the arithmetic mean of the Fréchet is given by equation (8) as $(\Gamma(1-\lambda)/(1+\gamma)^\lambda) K^{\lambda(1+\gamma)} + \epsilon(K)$. It is known that if a random variable Z is distributed Fréchet than ln Z is distributed Gumbel. Therefore the geometric mean of the Fréchet is simply the exponential of the arithmetic mean of the Gumbel which, from substitution of equation (9), is

$$\exp\{\lambda\psi - \lambda \ln(1+\gamma) + \lambda(1+\gamma)\ln K + \epsilon(K)\} = \frac{e^{\lambda\psi}}{(1+\gamma)^\lambda} K^{\lambda(1+\gamma)} e^{\epsilon(K)}.$$

The numerator of equation (12) therefore converges to $e^{\lambda\psi}/\Gamma(1-\lambda)$. By Proposition 3.3 the denominator of (12) converges to $\exp\{\int_1^\infty \ln(y)\,dG_2(y)\} \int_1^\infty y^{-1}\,dG_2(y) > 0$. Thus, the ratio of output per worker to average efficiency converges in the manner proposed. *Q.E.D.*

An Expression for $E(k(t);\{\alpha(\cdot)\})$

The expected return to research relative to the wage is a formidable object,

$$\text{(13)} \qquad E(t) = \int_0^\infty \int_1^\infty \int_1^\infty e^{-\rho s} \frac{(1+\gamma)(1 - e^{-K(t)/(1+\gamma)})}{\int_1^\infty y^{-1}\,dG_2(y;K(t+s))} \frac{K(t+s)}{K(t)}$$
$$\times \left[\frac{L(t+s)}{K(t+s)} - \frac{R(t+s)}{K(t+s)} \right] \frac{W(t+s)}{W(t)} (1 - y^{-1})$$
$$\times [1 - \Phi(z'y, K(t+s), K(t))]\,dH(z',y;K(t))\,ds.$$

The goal is to express $E(t)$, given a path of future research intensity, as a function of $k(t) = K(t)/L(t)$, with both $K(t)$ and $L(t)$ passing to infinity.

Equation (13) is particularly difficult to work with because Φ and dH depend on the stock of research. This difficulty is overcome by working with a transformation of the state of the art z that depends on the level of the research stock. A convenient transformation, given a stock of research $K(t) = K$, is $z^* \equiv (1+\gamma)^{-1} K^{1+\gamma} z^{-1/\lambda}$. Denoting random variables by capital letters, noting that $Z = Z'Y$, and letting H^* be the joint distribution of Z^* and Y,

$$H^*(z^*, y; K) \equiv \Pr(Z^* \leq z^*, Y \leq y; K) = \int_1^y \int_{x^{-1}(1+\gamma)^{-\lambda} K^{\lambda(1+\gamma)} z^{*-\lambda}}^\infty dH(z', x; K).$$

From the derivation of an expression for $H(z', y; K)$, given Assumptions 3.1 and 3.2,

$$dH(z', x; K) = \frac{K^{2(1+\gamma)} z'^{-(2+\lambda)/\lambda} x^{-(1+\lambda)/\lambda} e^{-(1+\gamma)^{-1} z'^{-1/\lambda} K^{1+\gamma}}}{(1 - e^{-K/(1+\gamma)})(1+\gamma)^2 \lambda^2} \, dz' \, dx,$$

for $z' \geq \bar{q}(K) = K^{\lambda\gamma}$. Thus taking $z^* \leq y^{-1/\lambda}(1+\gamma)^{-1}K$ (so that $z' \geq \bar{q}(K)$), changing the variable of integration from z' to $(1+\gamma)^{-1}K^{1+\gamma}z'^{-1/\lambda}$, and solving,

$$H^*(z^*, y; K) = \int_1^y (1 - e^{-K/(1+\gamma)})[1 - e^{-x^{1/\lambda} z^*}(x^{1/\lambda} z^* + 1)] \lambda^{-1} x^{-(1+\lambda)/\lambda} \, dx.$$

This integral simplifies to

$$\frac{1 - e^{-z^*} - y^{-1/\lambda} e^{-z^* y^{1/\lambda}}}{1 - e^{-K/(1+\gamma)}}.$$

Taking the limit as $K \to \infty$,

$$H^*(z^*, y; K) \xrightarrow{d} H^*(z^*, y) \equiv 1 - e^{-z^*} - y^{-1/\lambda} e^{-z^* y^{1/\lambda}},$$

for any $0 < z^* < \infty$ and $1 < y < \infty$. The marginal distribution of the transformed state of the art $H_1^*(z^*)$ is exponential, the marginal distribution of the inventive step $H_2^*(y)$ is Pareto, and the joint density is $\lambda^{-1} y^{\lambda^{-1}-1} z^* e^{-z^* y^{1/\lambda}} \equiv h^*(z^*, y)$.

Three other expressions in equation (13) simplify if both $K(t)$ and $K(t+s)$ tend to infinity with their ratio fixed, i.e., fixing $c(s) \equiv K(t+s)/K(t)$ as $K(t) = K \to \infty$. First,

$$1 - \Phi(z, c(s)K, K) = e^{-(1+\gamma)^{-1} z^{-1/\lambda}[(c(s)K)^{1+\gamma} - K^{1+\gamma}]} = e^{-z^*[c(s)^{1+\gamma} - 1]},$$

for $z \geq \bar{q}(c(s)K) = (c(s)K)^{\lambda\gamma}$. The latter condition holds with a probability approaching one since, as K approaches infinity, $\Pr(Z \geq (c(s)K)^{\lambda\gamma}) = \Pr(Z^* \leq (1+\gamma)^{-1}Kc(s)^{-\gamma}) \to H_1^*((1+\gamma)^{-1}Kc(s)^{-\gamma}) = 1 - e^{-(1+\gamma)^{-1}Kc(s)^{-\gamma}} \to 1$. Second, applying Proposition 3.3 implies $\lim_{K\to\infty}[\int_1^\infty y^{-1} \, dG_2(y; K)] = \int_1^\infty y^{-1} \, dG_2(y) \equiv \theta(\lambda)$. Third, from equation (4), equation (8), and Proposition 3.3 (again holding $K(t+s)/K(t) \equiv c(s)$ fixed) $\lim_{K\to\infty}[W(t+s)/W(t)] = c(s)^{\lambda(1+\gamma)}$.

Changing the variable of integration in equation (13) from z to z^* (as defined above), using the simplifications above, and taking the limit as $K(t) = K \to \infty$,

$$(14) \qquad E(k(t); \{\alpha(\cdot)\}) = \int_0^\infty \int_1^\infty \int_0^\infty e^{-\rho s} \frac{(1+\gamma)}{\theta(\lambda)} e^{n[1+\lambda(1+\gamma)](s + \ln k(t+s) - \ln k(t))}$$
$$\times \frac{1 - \alpha(t+s)}{k(t+s)} (1 - y^{-1}) e^{-z^*[e^{n(1+\gamma)(s + \ln k(t+s) - \ln k(t))} - 1]} h^*(z^*, y) \, dz^* \, dy \, ds.$$

Proof of Proposition 3.4

The proposed equilibrium $\{k(t+s) = k^*, \alpha(t+s) = \alpha^* | s \geq 0\}$ must satisfy $\alpha^* = nk^*$ and $E(k^*; \{\alpha^*\}) = 1$. Combining these conditions with equation (14),

$$1 = \int_0^\infty \int_1^\infty \int_0^\infty e^{-\rho s} \frac{(1+\gamma)}{\theta(\lambda)} e^{n[1+\lambda(1+\gamma)]s} \frac{1 - \alpha^*}{\alpha^*/n} (1 - y^{-1})$$
$$\times e^{-z^*[e^{n(1+\gamma)s} - 1]} h^*(z^*, y) \, dz^* \, dy \, ds,$$

where $\alpha^* \equiv \alpha(k^*)$. Changing the variable of integration from s to $x = e^{n(1+\gamma)s}$,

$$1 = \frac{1 - \alpha^*}{\alpha^* \theta(\lambda)} \int_1^\infty \int_0^\infty (1 - y^{-1}) e^{z^*} (z^*)^{r^* - \lambda} \Gamma(\lambda - r^*, z^*) h^*(z^*, y) \, dz^* \, dy,$$

where $r^* = ((\rho/n) - 1)/(1+\gamma)$ and $\Gamma(a,w) \equiv \int_w^\infty e^{-x}x^{a-1}\,dx$ is the incomplete gamma function. Integrating over y,

$$1 = \frac{1-\alpha^*}{\alpha^*\theta(\lambda)} \int_0^\infty e^{z^*}(z^*)^{r^*-\lambda}\Gamma(\lambda - r^*, z^*)[e^{-z^*} - (z^*)^\lambda \Gamma(1-\lambda, z^*)]\,dz^*. \tag{15}$$

Applying the identity, $\int_0^\infty w^{a-1}\Gamma(b,w)\,dw = \Gamma(a+b)/a$,

$$1 = \frac{(1-\alpha^*)(1-\phi(r^*,\lambda))}{\alpha^*\theta(\lambda)(1+r^*-\lambda)},$$

where $\phi(r^*,\lambda) \equiv (1+r^*-\lambda)\int_0^\infty w^{r^*}e^w\Gamma(\lambda-r^*,w)\Gamma(1-\lambda,w)\,dw$. Solving for $\alpha^* = \alpha^*(r^*,\lambda)$ yields the expression given in Proposition 3.4.

Holding α^* fixed, the right side of equation (15) is decreasing in r^* since $(z^*)^{r^*-\lambda}\Gamma(\lambda-r^*,z^*) = \int_{z^*}^\infty e^{-x}x^{-1}(x/z^*)^{\lambda-r^*}\,dx$ is decreasing r^* for any $z^*>0$. Similarly, $(z^*)^{r^*-\lambda}\Gamma(\lambda-r^*,z^*)$ is increasing in λ as is $-(z^*)^\lambda\Gamma(1-\lambda,z^*)$. Furthermore, since the distribution of the markup is stochastically increasing in λ, it must be that $1/\theta(\lambda)$ is increasing in λ. Thus, α^* is decreasing in r^* and increasing in λ.

To check the local stability of the equilibrium, suppose that for any $s \geq 0$: $\alpha(t+s) = a_1 > \alpha^*$ if $k(t+s) < k^*$, $\alpha(t+s) = \alpha^*$ if $k(t+s) = k^*$, and $\alpha(t+s) = a_2 < \alpha^*$ if $k(t+s) > k^*$. Given $k(t)$ and this path of research intensity, $k(t+s) = [k(t) - a_i/n]e^{-ns} + a_i/n$ for $s \leq s_i^*$ and $k(t+s) = k^*$ for $s \geq s_i^*$, where $s_i^* = \ln\{(a_i/n - k(t))/(a_i/n - k^*)\}/n$ and $i=1$ if $k(t) < k^*$, $i=2$ if $k(t) > k^*$. The proposed path of research intensity $\{\alpha(\cdot)\}$ is an equilibrium (given $k(t)$ in a neighborhood of k^*) if $\partial E(k(t);\{\alpha(\cdot)\})/\partial k(t)|_{k(t)=k^*} \leq 0$. If the inequality is strict, then $a_1 = 1$ and $a_2 = 0$.

Plugging the proposed paths for research intensity and the research stock relative to the labor force into equation 14, it can be shown that

$$\left.\frac{\partial E(k(t);\{\alpha(\cdot)\})}{\partial k(t)}\right|_{k(t)=k^*} = \left[\frac{n(1+\gamma)\lambda}{\theta(\lambda)(1+\lambda)\alpha^*} - \rho\right]\Big/\alpha^*.$$

The derivative is weakly negative if $\rho \geq (n(1+\gamma)\lambda)/(\theta(\lambda)(1+\lambda)\alpha^*)$, which is equivalent to the sufficient condition for local stability given in Proposition 3.4.

Otherwise, either the equilibrium is not locally stable or else $k(t)$ approaches k^* only asymptotically. Taking up the latter possibility, suppose $\alpha(t+s) = \alpha^* - b[k(t+s) - k^*]$ for $b \geq -n$ and all $s \geq 0$. The solution to the resulting differential equation is $k(s+t) = k^* + e^{-(b+n)s}[k(t) - k^*]$, given $k(t)$. The proposed path of research intensity $\{\alpha(\cdot)\}$ is an equilibrium locally (and hence the equilibrium in Proposition 3.4 is locally stable) if, starting from $k(t)$ in a neighborhood of k^*, $\partial E(k(t);\{\alpha(\cdot)\})/\partial k(t)|_{k(t)=k^*} = 0$.

Plugging the proposed paths for research intensity and the research stock relative to the labor force into equation 14, it can be shown that the derivative $\partial E(k(t);\{\alpha(\cdot)\})/\partial k(t)|_{k(t)=k^*}$ is a continuous function of b. If $b = -n$ the derivative is negative and as b approaches infinity the derivative converges to a term with the same sign as

$$\begin{aligned} B(\lambda, r^*, \gamma) \equiv{}& \frac{1+\gamma}{1+r^*-\lambda} - (1+\gamma)\int_0^\infty x^{r^*}e^x\Gamma(\lambda - r^* + 1, x)\Gamma(1+\lambda, x)\,dx \\ &- \frac{1+\lambda(1+\gamma)}{1+r^*-\lambda} + (1+\lambda(1+\gamma))\int_0^\infty x^{r^*}e^x\Gamma(\lambda - r^*, x)\Gamma(1-\lambda, x)\,dx. \end{aligned}$$

If $B > 0$, then there must exist some intermediate value of $b \in (-n, \infty)$ such that $\partial E(k(t);\{\alpha(\cdot)\})/\partial k(t)|_{k(t)=k^*} = 0$. Thus another sufficient condition for local stability is that the parameters satisfy $B(\lambda, r^*, \gamma) > 0$ (a condition which can be checked numerically). *Q.E.D.*

The Social Planner's Problem

Consider a social planner with an objective function

$$U(t) = \int_0^{\infty} e^{-\tilde{\rho} s} X(t+s)\, ds,$$

which represents the aggregate of individuals' instantaneous utilities but discounted at rate $\tilde{\rho}$ which may differ from ρ. The social planner determines only the fraction of labor services allocated to research, leaving other production and allocation decisions to the market. The social planner's objective function is simplified by representing it in terms of $k = K/L$ and $\alpha = R/L$ while letting $K(t)$ and $L(t)$ tend to infinity. Applying equations (4) and (5) and Proposition 3.3 (while dropping an irrelevant constant),

$$U(t) = \int_0^{\infty} e^{-[\tilde{\rho} - (1+\lambda(1+\gamma))n]s} k(t+s)^{\lambda(1+\gamma)} [1 - \alpha(t+s)]\, ds.$$

The social planner controls $\alpha(t+s)$, which determines the evolution of the state variable, $\dot{k}(t+s) = \alpha(t+s) - nk(t+s)$. The current value Hamiltonian is

$$H(t+s) = k(t+s)^{\lambda(1+\gamma)}[1 - \alpha(t+s)] + \Lambda(t+s)[\alpha(t+s) - nk(t+s)],$$

where Λ is the shadow price of k.

For all $s \geq 0$, the solution must satisfy

$$\frac{\partial H(t+s)}{\partial k(t+s)} = [\tilde{\rho} - (1+\lambda(1+\gamma))n]\Lambda(t+s) - \dot{\Lambda}(t+s),$$

$$\lim_{s \to \infty} e^{-[\tilde{\rho} - (1+\lambda(1+\gamma))n]s} \Lambda(t+s) k(t+s) = 0,$$

$$\alpha(t+s) = 0 \quad \text{if} \quad \frac{\partial H(t+s)}{\partial \alpha(t+s)} < 0, \quad \text{and} \quad \alpha(t+s) = 1 \quad \text{if} \quad \frac{\partial H(t+s)}{\partial \alpha(t+s)} > 0.$$

The transversality condition requires the parameter restriction, $\tilde{\rho} > (1 + \lambda(1+\gamma))n$. The solution is $\alpha(t+s) = 1$ for $k(t+s) < \tilde{k}$, $\alpha(t+s) = \tilde{\alpha}$ for $k(t+s) = \tilde{k}$, and $\alpha(t+s) = 0$ for $k(t+s) > \tilde{k}$ for all $s \geq 0$, where $\tilde{k} = \lambda(1+\gamma)/\tilde{\rho}$ and $\tilde{\alpha} = n\lambda(1+\gamma)/\tilde{\rho}$. After a period of transition (if $k(t) \neq \tilde{k}$) the solution displays constant research intensity of $\tilde{\alpha}$. The social return to research in the market equilibrium is the value of $\tilde{\rho}$ such that $\tilde{\alpha} = \alpha^*(r^*, \lambda)$. *Q.E.D.*

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