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Note

The relationship between quantum and classical correlation in games ☆

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ABSTRACT

Quantum games have been argued to differ from classical games by virtue of the quantum-mechanical phenomenon of entanglement. We formulate a baseline of classical correlation – which takes two forms according as signals added to the game are or are not required to be independent of chance moves in the underlying game. We show that independence is a necessary condition for the addition of quantum signals to have a different effect from the addition of classical signals.

1. Introduction

Quantum games, introduced by Meyer (1999) and Eisert et al. (1999), have gained attention in both the physics and game-theory communities for the possibility that they allow effects in games which are impossible in a classical setting.

Two streams of papers have emerged. In one stream, players receive signals from outside the game and can peg their choices of strategy in the underlying game on these signals. This is just as in classical correlation via coin or die tosses and the like, except that the signals the players receive may be "entangled" in an intrinsically quantum-mechanical fashion. Papers of this type include Huberman and Hogg (2003), Cleve et al. (2004), Iqbal and Weigert (2004), Dahl and Landsburg (2005), La Mura (2005), and Kargin (2008). In another stream of papers, new strategies which are quantum-mechanical in nature are added to the original game. (In technical terms, the new strategies are "superpositions" of the underlying classical strategies.) Papers of this type include Meyer (1999), Eisert et al. (1999), Eisert and Wilkens (2000), and Benjamin and Hayden (2001). In sum: in the first case, quantum signals are added to the game; in the second case, quantum strategies are added.[1]

But does quantum game theory really differ from classical game theory? Doubts were raised even at the start. Van Enk (2000) points out that the quantum strategy in Meyer (1999) can be mimicked classically (see also the reply by Meyer, 2000). Van Enk and Pike (2002) argue more generally that adding a quantum strategy to a game can be replicated by adding a suitable classical strategy. However, they hold out hope that adding quantum signals will bring in novel features. Levine (2005) makes essentially the same criticism of games with quantum strategies. Moreover, he argues that games with quantum signals can also be replicated classically, provided communication among the players is allowed. This is

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[1] Lambert-Mogiliansky et al. (in press) and Temzelides (2005) put quantum mechanics and decision theory together. They employ aspects of quantum mechanics – such as superposition and non-commutativity – to build non-classical decision theories.

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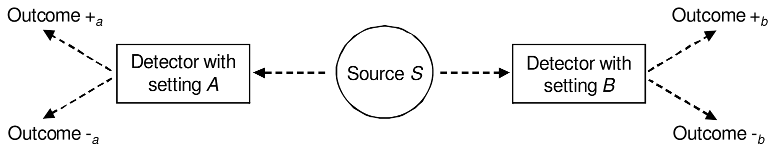

Fig. 1.

surely correct, but, as Levine himself allows, may give too much to the classical 'side,' since with quantum signals there is correlation but not actual communication among the players.

Here, we make a direct comparison between quantum and classical signals. To do so, we formulate a baseline analysis of classical signals. This leads to a key question:

Are classical signals which are external to the game independent or not independent of chance moves in the underlying game?

We will see that independence is necessary if adding quantum signals is to work differently from adding classical signals. We hope that identifying this condition will help clarify the issue of when and how quantum signals can change the analysis of games.

Section 2 explains some relevant aspects of the key phenomenon of entanglement – which is the distinctive form correlations can take in the quantum-mechanical realm. (The section can be understood without knowledge of physics.) Section 3 gives an example of how entanglement can be used in a game. Section 4 establishes the classical baseline. Section 5 compares the quantum and classical analyses, and Section 6 contains some additional comments.

2. Entanglement

The concept of entanglement in quantum mechanics (QM) was scrutinized in a famous paper by Einstein et al. (1935) (EPR).[2] Here is the idea, without any pretense of describing the physics properly.[3] Two particles in a special ("singlet") state are prepared and then separated, and their spins are measured via detectors with settings A and B. (The setting of a detector determines the direction in which a particle's spin is measured.) The outcome of each measurement is discrete (quantized) and can take one of two values: spin-up $(+)$ or spin-down $(-)$. See Fig. 1.

QM allows the situation in which, for appropriate choice of A and B, the empirically measured probabilities q of the outcomes satisfy:

$$q(+_a, -_b|A, B) = q(-_a, +_b|A, B) = \frac{1}{2}.$$

That is, the spins of the two particles are perfectly anti-correlated. Does this situation require "spooky action at a distance" (in Einstein's famous phrase)? Does the act of measuring one particle somehow determine the spin of the other particle (which might be a long way away)? The answer, clearly, is no. The situation can easily be understood in terms of common-cause correlation. Formally, introduce an extra variable λ which can take either of two values λ^1 or λ^2. Build a probability measure p on the extended space that includes λ, as follows:

$$p(\lambda^1|A, B) = p(\lambda^2|A, B) = \frac{1}{2},$$
$$p(+_a, -_b|A, B, \lambda^1) = p(-_a, +_b|A, B, \lambda^2) = 1.$$

Then, p and q agree as far as the observed frequencies of outcomes are concerned. Moreover, the outcomes are now independent, conditional on the value of the extra variable λ. No spooky action at a distance is involved.[4]

This was exactly EPR's point. QM, as it stood, was "incomplete" (their term) and there were extra variables yet to be found and introduced into the theory.

But it turns out that QM is stranger than this. Bell (1964) produced a scenario in which the correlations cannot be explained via extra variables. Some kind of intrinsic entanglement between the particles does appear to be involved. Here, we sketch a variant of Bell's analysis, due to Hardy (1992, 1993). It captures what is essential for our purposes, while simplifying some calculations.

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[2] The term itself first appears in Schrödinger (1935). The version of EPR we are about to describe is due to Bohm (1951).

[3] Laloë (2001) is an accessible survey of the area (our Fig. 1 is similar to Fig. 1 there). A standard textbook on QM is Sakurai (1994).

[4] In correspondence with Schrödinger, von Neumann (1936) gave a nice example of common-cause correlation: "Let S_1 and S_2 be two boxes. One knows that 1 000 000 years ago either a white ball had been put into each or a black ball had been placed into each but one does not know which color the balls were. Subsequently one of the boxes (S_1) was buried on Earth, the other (S_2) on Sirius … . Now one digs S_1 on Earth out, opens it and sees: the ball is white. This action on Earth changes instantaneously the S_2 statistic on Sirius … ." See Section 6.4 for more on von Neumann and QM.

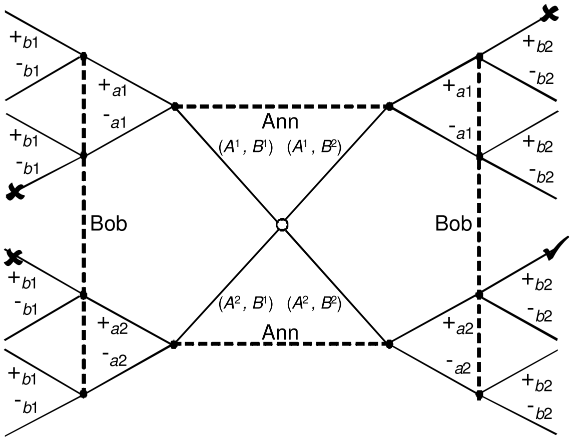

Fig. 2.

We again consider two particles, prepared in a certain way.[5] Ann can make one of two measurements A^1 or A^2 on her particle. Bob can make one of two measurements B^1 or B^2 on his particle. QM allows the following empirical probabilities of the outcomes:

$$q\left(-_a, -_b \middle| A^1, B^1\right) = 0, \tag{1}$$

$$q\left(+_a, +_b \middle| A^1, B^2\right) = 0, \tag{2}$$

$$q\left(+_a, +_b \middle| A^2, B^1\right) = 0, \tag{3}$$

$$q\left(+_a, +_b \middle| A^2, B^2\right) > 0. \tag{4}$$

Introducing an extra variable λ cannot explain the correlations this time. Here is the verbal argument. We see from (1) that there can be no non-null value of λ under which measurement A^1 yields outcome $-$ and measurement B^1 yields outcome $-$. That is, for all non-null λ, either: (i) A^1 yields outcome $+$, or (ii) B^1 yields outcome $+$ (or both). If (i), then, by (2), there can be no non-null value of λ under which measurement B^2 yields outcome $+$. If (ii), then, by (3), there can be no non-null value of λ under which measurement A^2 yields outcome $+$. In either case, there can be no non-null value of λ under which measurement A^2 yields outcome $+$ and measurement B^2 yields outcome $+$. This contradicts (4).

A formal proof is not much harder (see Proposition 4.2). The important step is to identify the precise conditions under which the argument holds. In the following sections, as we build a bridge to game theory, we will describe these conditions.

3. Quantum signals in games

Fig. 2 is an example of how a game with quantum signals works. The game tree is classical. Nature goes first and chooses between four moves: (A^1, B^1), (A^1, B^2), (A^2, B^1), and (A^2, B^2). Ann observes the first component of Nature's move. If she observes A^1, she chooses between the moves $+_{a1}$ and $-_{a1}$. If she observes A^2, she chooses between the moves $+_{a2}$ and $-_{a2}$. Bob observes the second component of Nature's move, but not Ann's move (he moves simultaneously with her). Like Ann, Bob has two moves at each of his two information sets. The labelling of the moves is deliberately chosen, as we are about to see. There will be no need to specify payoffs – or, therefore, to talk about what the players and the observer know about payoffs. (We will explain the crosses and check mark in the figure.)

Now consider an outside observer of the game (or a third player). Let this observer have a probability assessment q on paths of play, satisfying:

$$q\left(-_{a1}, -_{b1} \middle| A^1, B^1\right) = 0, \tag{5}$$

$$q\left(+_{a1}, +_{b2} \middle| A^1, B^2\right) = 0, \tag{6}$$

$$q\left(+_{a2}, +_{b1} \middle| A^2, B^1\right) = 0, \tag{7}$$

$$q\left(+_{a2}, +_{b2} \middle| A^2, B^2\right) > 0, \tag{8}$$

where each $q(A^i, B^j) > 0$. In Fig. 2, the paths that must get probability 0 under q are marked with a $\times$, the path that must get positive probability with a $\checkmark$.

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5 See Hardy (1992, 1993) for the actual physical set-up.

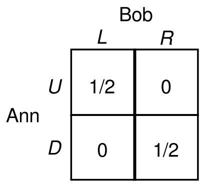

Fig. 3.

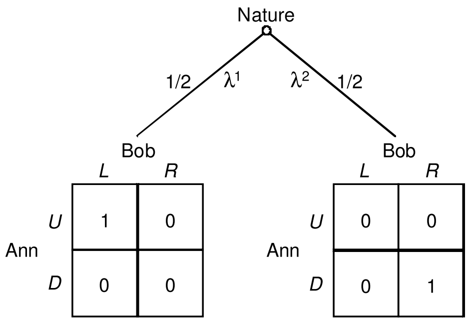

Fig. 4.

Obviously, the system (5)–(8) is patterned after (1)–(4) from the previous section. Indeed, let us see how the assessment q of (5)–(8) can arise if an observer thinks that Ann and Bob make their choices in accordance with a quantum device. Two particles are prepared in an entangled state à la Hardy, as in the previous section. Then the game is played. If Nature moves so that Ann observes A^1, Ann makes measurement A^1 on her particle. If the outcome of this measurement is spin-up (resp. spin-down), Ann makes the move $+_{a1}$ (resp. $-_{a1}$). If Ann observes A^2, she makes measurement A^2 on her particle, and, if the outcome is spin-up (resp. spin-down), she makes the move $+_{a2}$ (resp. $-_{a2}$). Likewise with Bob. By design, the observer's assessment q will then satisfy (5)–(8).

Of course, the key question is whether the assessment q can be justified via classical signals. If no, then we will have identified – in what would seem to be a very simple way – how adding quantum signals can go beyond classical analysis of games.[6] If yes, then we will have lent support to the view that quantum game theory does not add to classical game theory.

4. Classical signals in games

To answer the question, we need to set up the theory of classical correlation and see what it allows in the game of Fig. 2. By this we mean that we allow ourselves to add classical signals such as arise from coin or dice tosses to the underlying game, as in Aumann (1974, 1987). (Formally, we add payoff-irrelevant moves by Nature.) Players can then peg their choices on the realizations of these signals.

The basic idea is well known. Fig. 3 is a typical example of classical correlation, where the observer has an assessment which assigns probability $\frac{1}{2}$ to the event that Ann and Bob choose U and L, and probability $\frac{1}{2}$ to the event that they choose D and R. Fig. 4 is obtained from Fig. 3 by adding a signal, which is λ^1 or λ^2 with equal probability. The observer thinks that: (i) if the signal is λ^1, then Ann chooses U and Bob chooses L; and (ii) if the signal is λ^2, then Ann chooses D and Bob chooses R. The two matrices give the observer's (degenerate) conditional probabilities.[7] The correlation in the original assessment is explained via the introduction of the signal.

Here is the general treatment. We start with the case where the underlying game does not contain any moves by Nature. (So, this covers Fig. 3 but not yet Fig. 2.) Consider a game in strategic form (or a simultaneous-move tree). Let X_a (resp. X_b) be the set of moves for Ann (resp. Bob). The observer has a probability assessment q on $X_a \times X_b$. We now build an extended game with classical signals. Formally, in the extended game, Nature goes first and makes a payoff-irrelevant move which consists of choosing a point (λ_a, λ_b) from some finite product space $\Lambda_a \times \Lambda_b$. Ann (resp. Bob) observes the component λ_a (resp. λ_b).[8] The players then make choices as in the underlying game. Let p be the observer's probability measure on the extended game $X_a \times X_b \times \Lambda_a \times \Lambda_b$. We want p to be consistent with q, i.e. $\mathrm{marg}_{X_a \times X_b}\, p = q$. But we also want p to respect

[6] But not quite the simplest way. See Section 6.2.

[7] Note the identification of moves in the two subtrees in Fig. 4. This is the basis for saying that the assessment in Fig. 4 agrees with that in Fig. 3.

[8] Finiteness of Λ_a and Λ_b is, of course, a restriction, but allows us to avoid all measure-theoretic issues. On the other hand, the product structure is without loss of generality. If, instead, Ann (resp. Bob) has a partition $\mathcal{H}_a$ (resp. $\mathcal{H}_b$) of a space Λ, just take Λ_a (resp. Λ_b) to be the quotient space $\{h_a\colon\ h_a \in \mathcal{H}_a\}$ (resp. $\{h_b\colon\ h_b \in \mathcal{H}_b\}$) and move the probabilities over in the obvious way.

the idea that the players choose their moves independently – i.e., that they do not choose together. To achieve this, we adapt some conditions from Brandenburger and Friedenberg (2008)[9]:

Conditional Independence. *The observer assesses Ann's and Bob's moves as independent, conditional on the signals. Formally, whenever* $p(\lambda_a, \lambda_b) > 0$,

$$p(x_a, x_b|\lambda_a, \lambda_b) = p(x_a|\lambda_a, \lambda_b) \times p(x_b|\lambda_a, \lambda_b).$$

Sufficiency. *If the observer knows Ann's signal, and comes to learn Bob's signal, this won't change the observer's assessment of Ann's move. Likewise with Ann and Bob interchanged. Formally, whenever* $p(\lambda_a, \lambda_b) > 0$,

$$p(x_a|\lambda_a, \lambda_b) = p(x_a|\lambda_a),$$
$$p(x_b|\lambda_a, \lambda_b) = p(x_b|\lambda_b).$$

Conditional Independence says that correlation in the observer's assessment of Ann's and Bob's moves comes from uncertainty about the signals they observe. If the observer knew these signals, he would have an independent assessment. Sufficiency says that Bob's signal should provide information about Ann's move only to the extent it provides information about her signal. This is because Ann knows only her own signal, not Bob's. (Likewise with Ann and Bob interchanged.) Here is an easy consequence of the two conditions. (See Brandenburger and Friedenberg, 2008, Proposition 9.1 for a much more general result.)

Proposition 4.1. *Under Conditional Independence and Sufficiency, if* $p(\lambda_a, \lambda_b) = p(\lambda_a) \times p(\lambda_b)$ *for all* λ_a, λ_b, *then* $p(x_a, x_b) = p(x_a) \times p(x_b)$ *for all* x_a, x_b.

Proof. We have

$$\begin{aligned} p(x_a, x_b) &= \sum_{\{(\lambda_a,\lambda_b):\ p(\lambda_a,\lambda_b)>0\}} p(x_a, x_b|\lambda_a, \lambda_b) p(\lambda_a, \lambda_b) = \sum_{\{(\lambda_a,\lambda_b):\ p(\lambda_a,\lambda_b)>0\}} \big[p(x_a|\lambda_a) \times p(x_b|\lambda_b)\big] p(\lambda_a, \lambda_b) \\ &= \sum_{\{\lambda_a:\ p(\lambda_a)>0\}} p(x_a|\lambda_a) p(\lambda_a) \times \sum_{\{\lambda_b:\ p(\lambda_b)>0\}} p(x_b|\lambda_b) p(\lambda_b) = p(x_a) \times p(x_b), \end{aligned}$$

as required. □

In words, under Conditional Independence and Sufficiency, if the observer assesses Ann's and Bob's signals as independent, then he assesses their moves as independent. Taking the contrapositive, we see that Conditional Independence and Sufficiency guarantee that correlation in moves implies ('comes from') correlation in signals. This is the notion of classical correlation we wanted to formalize. (The two conditions are tight – see Brandenburger and Friedenberg, 2008, Appendix A.)

It is easy to check that the assessment in Fig. 4 satisfies Conditional Independence and Sufficiency – so, this was indeed an example of explaining correlation as now formalized.[10] But can such a maneuver always be carried out? If the underlying game doesn't contain any moves by Nature, the answer is yes. The idea of the construction is well known. Set $\Lambda_a = X_a$ and $\Lambda_b = X_b$, and build p on the diagonal by setting

$$p(x_a, x_b, \lambda_a, \lambda_b) = \begin{cases} q(x_a, x_b) & \text{if } \lambda_a = x_a \text{ and } \lambda_b = x_b, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\text{marg}_{X_a \times X_b}\, p = q$, and Conditional Independence and Sufficiency both hold.

But, what if the underlying game already contains moves by Nature – as in Fig. 2? (These moves might be payoff-relevant, unlike the (λ_a, λ_b)-moves which we add to the game.) To investigate, suppose that, in the underlying game, Nature goes first and chooses a point (μ_a, μ_b) from some finite product space $M_a \times M_b$. Ann (resp. Bob) observes the component μ_a (resp. μ_b). Ann (resp. Bob) then makes a choice $x_a \in X_a$ (resp. $x_b \in X_b$). Thus, in Fig. 2, $X_a = \{+_{a1}, -_{a1}, +_{a2}, -_{a2}\}$, $X_b = \{+_{b1}, -_{b1}, +_{b2}, -_{b2}\}$, $M_a = \{A^1, A^2\}$, and $M_b = \{B^1, B^2\}$.

Note carefully that we use μ rather than λ to distinguish moves by Nature in the given game (possibly payoff-relevant) from payoff-irrelevant moves by Nature (signals) in an extended game. Also, not all tuples of moves will be legal, given the tree. (For example, in Fig. 2, after Nature moves (A^1, B^1), Ann cannot move $+_{a2}$.) For our purposes, it will suffice to handle this in a simple way by supposing that the probability measures q and p put weight 0 on illegal tuples.

Here are the extensions of Conditional Independence and Sufficiency from before (we won't give them new names):

[9] The focus of Brandenburger and Friedenberg (2008) is on developing a concept of "intrinsic" correlation in games. This is correlation that comes not from outside signals but from the players' own beliefs about the game. The conditions formulated in Brandenburger and Friedenberg (2008) are easily adapted to the current "extrinsic" setting (with signals), as we now do.

[10] Strictly speaking, if the signals λ^1 and λ^2 are put into an (obvious) product format.

Conditional Independence. *Whenever* $p(\mu_a, \mu_b, \lambda_a, \lambda_b) > 0$,

$$p(x_a, x_b|\mu_a, \mu_b, \lambda_a, \lambda_b) = p(x_a|\mu_a, \mu_b, \lambda_a, \lambda_b) \times p(x_b|\mu_a, \mu_b, \lambda_a, \lambda_b). \tag{9}$$

Sufficiency. *Whenever* $p(\mu_a, \mu_b, \lambda_a, \lambda_b) > 0$,

$$\begin{aligned} p(x_a|\mu_a, \mu_b, \lambda_a, \lambda_b) &= p(x_a|\mu_a, \lambda_a), \\ p(x_b|\mu_a, \mu_b, \lambda_a, \lambda_b) &= p(x_b|\mu_b, \lambda_b). \end{aligned} \tag{10}$$

The rationale for these conditions is as before, with the difference that now we take account of the fact that Ann observes both μ_a (from the given game) and λ_a (from the extended game), and likewise for Bob.

Now that the underlying game includes moves by Nature, there is a third condition we can consider. We can ask that these moves be independent of any signals we add to the game. In the next section, we indicate when this condition might – or might not – be appropriate. First, we establish what it implies.

λ-Independence. *The observer assesses moves by Nature in the underlying game and moves by Nature in the extended game as independent. Formally*,

$$p(\mu_a, \mu_b, \lambda_a, \lambda_b) = p(\mu_a, \mu_b) \times p(\lambda_a, \lambda_b). \tag{11}$$

Proposition 4.2. *Consider the game of Fig. 2 and the assessment q given by (5)–(8). There do not exist (finite) spaces Λ_a and Λ_b, and a probability measure p on $X_a \times X_b \times M_a \times M_b \times \Lambda_a \times \Lambda_b$, such that p agrees with q on $X_a \times X_b \times M_a \times M_b$ and satisfies Conditional Independence, Sufficiency, and λ-Independence.*

Proof. From (5) and agreement, we get

$$\begin{aligned} 0 &= \sum_{\{(\lambda_a,\lambda_b):\ p(A^1,B^1,\lambda_a,\lambda_b)>0\}} p\big(-_{a1}, -_{b1}\big|A^1, B^1, \lambda_a, \lambda_b\big) p\big(\lambda_a, \lambda_b\big|A^1, B^1\big) \\ &= \sum_{\{(\lambda_a,\lambda_b):\ p(\lambda_a,\lambda_b)>0\}} \big[p\big(-_{a1}\big|A^1, \lambda_a\big) \times p\big(-_{b1}\big|B^1, \lambda_b\big)\big] p(\lambda_a, \lambda_b), \end{aligned}$$

where the second line uses λ-Independence, $p(A^1, B^1) = q(A^1, B^1) > 0$, Conditional Independence, and Sufficiency. Letting $M = \{(\lambda_a, \lambda_b):\ p(\lambda_a, \lambda_b) > 0\}$, we get that for all $(\lambda_a, \lambda_b) \in M$,

$$p\big(-_{a1}\big|A^1, \lambda_a\big) \times p\big(-_{b1}\big|B^1, \lambda_b\big) = 0. \tag{12}$$

Similar arguments using (6) and (7) show that for all $(\lambda_a, \lambda_b) \in M$,

$$p\big(+_{a1}\big|A^1, \lambda_a\big) \times p\big(+_{b2}\big|B^2, \lambda_b\big) = 0, \tag{13}$$

$$p\big(+_{a2}\big|A^2, \lambda_a\big) \times p\big(+_{b1}\big|B^1, \lambda_b\big) = 0. \tag{14}$$

Fix any $(\lambda_a, \lambda_b) \in M$ and consider (12). Case (i): $p(-_{a1}|A^1, \lambda_a) = 0$. Then $p(+_{a1}|A^1, \lambda_a) = 1$ (by the rules of probability), so, by (13), $p(+_{b2}|B^2, \lambda_b) = 0$. Case (ii): $p(-_{b1}|B^1, \lambda_b) = 0$. Then $p(+_{b1}|B^1, \lambda_b) = 1$, so, by (14), $p(+_{a2}|A^2, \lambda_a) = 0$. In either case, we get, using agreement, λ-Independence, Conditional Independence, and Sufficiency,

$$q\big(+_{a2}, +_{b2}\big|A^2, B^2\big) = \sum_{\{(\lambda_a,\lambda_b):\ p(\lambda_a,\lambda_b)>0\}} \big[p\big(+_{a2}\big|A^2, \lambda_a\big) \times p\big(+_{b2}\big|B^2, \lambda_b\big)\big] p(\lambda_a, \lambda_b) = 0,$$

contradicting (8). □

5. Comparison

We can now answer the question: Under what conditions is adding quantum signals different from adding classical signals to the game in Fig. 2?

First, our analysis of classical correlation. This involved adding signals λ_a, λ_b to the game, subject to satisfying Conditional Independence (9) and Sufficiency (10). (Recall: These conditions are to ensure that correlations respect the assumption that players choose independently.) In this case, adding quantum signals cannot differ from adding classical signals. We can immediately reproduce any assessment q, simply by extending the 'diagonal' argument from the previous section to games with moves by Nature. Just define signal spaces $\Lambda_a = X_a \times M_a$ and $\Lambda_b = X_b \times M_b$ and again build p on the diagonal. Of course, this construction will apply, in particular, to the assessment q of (5)–(8) for Fig. 2. So, although in Section 3 we got this q via entanglement à la Hardy (1992, 1993), this was not essential.

Now, in addition, ask for λ-Independence (11). Then, Proposition 4.2 shows that the q of (5)–(8) cannot be obtained via classical signals. This time, the use of entanglement allows an assessment which is otherwise impossible. For a scenario where λ-Independence might be appropriate, suppose that Nature's move (μ_a, μ_b) in the underlying game corresponds to private coin tosses – one by Ann and one by Bob. (Ann's component μ_a takes the value Heads or Tails depending to how her coin lands. Likewise with Bob's component μ_b.) By contrast, a move (λ_a, λ_b) by Nature which we add to the game is some kind of signal from the external environment. (To suggest the idea, such signals are often called "sunspots.") In this case, requiring independence of the two kinds of moves seems reasonable. If so, classical correlation cannot produce the q of (5)–(8). We explained in Section 3 that quantum entanglement can.

But, we don't mean to push this (or another) scenario. We aren't 'taking sides.' Our goal is only to pinpoint when the quantum and the classical can or cannot differ. The key, we have seen, is whether or not we impose λ-Independence on the classical analysis of correlation. This is our resolution of the differing opinions in the literature on whether or not adding quantum signals can differ from adding classical signals.

6. Discussion

Here, we offer some additional formal remarks and also a historical note.

6.1. No-go theorems

Proposition 4.2 is, in effect, a presentation of the famous Bell's Theorem (1964) of QM – in the variant due to Hardy (1992, 1993). Bell asked whether all correlations in QM can be explained via the introduction of extra – or what in QM are called "hidden"-variables. He imposed some restrictions on the exercise. To see what these are, reinterpret the variables in Section 4. Let μ_a (resp. μ_b) now be the measurement which Ann (resp. Bob) makes on her (resp. his) particle. Let X_a (resp. X_b) now be the outcome of Ann's (resp. Bob's) measurement. The extra (hidden) variables λ_a (resp. λ_b) now describe how the particles respond to measurements.

In QM, condition (9) is called Outcome Independence (Jarrett, 1984; Shimony, 1986). It says that conditional on the values of the measurements undertaken and the hidden variables, the outcomes of the measurements are independent. Condition (10) is called Parameter Independence (Jarrett, 1984; Shimony, 1986) and says that, given Ann's choice of measurement and the value of the a-hidden variable, knowledge of Bob's choice of measurement and the value of the b-hidden variable doesn't change (probabilistically) the outcome of Ann's measurement. Likewise with Ann and Bob interchanged. Bell required:

Locality. *Whenever* $p(\mu_a, \mu_b, \lambda_a, \lambda_b) > 0$,

$$p(x_a, x_b | \mu_a, \mu_b, \lambda_a, \lambda_b) = p(x_a | \mu_a, \lambda_a) \times p(x_b | \mu_b, \lambda_b). \tag{15}$$

This is easily seen (Jarrett, 1984) to be equivalent to the conjunction of (9) and (10). Bell also imposed (11), which, in the QM setting, says that the process determining the values of the hidden variables is independent of what measurements are conducted. Alternatively put, Ann and Bob are 'free to choose' their measurements. The term λ-Independence for this condition is found in Dickson (2005, p. 140). (We deliberately borrowed the term in Section 4.)

Bell's Theorem says that there are quantum-mechanical probabilities for which no hidden-variable analysis satisfying (11) and (15) is possible. It is "no-go" for a hidden-variable approach to QM. The spooky action at a distance cannot be removed. The proof is exactly that of Proposition 4.2 (again, in Hardy's (1992, 1993) variant). Of course, this correspondence is precisely our point: If a hidden-variable analysis of QM was always possible, then all correlations would be classical, subject to the addition of extra variables. An implication would be that the effect of adding quantum signals to a game could not be different from that of adding classical signals.

But, we have to be careful. The λ-Independence condition is hard to argue against in the physics setting. To do so would seem to be a denial of the 'free-will' of the experimenters (Ann and Bob). In the game setting, however, λ-Independence is certainly not inevitable. In the previous section, we painted a scenario where imposing it seemed reasonable. But there are surely scenarios where we wouldn't insist on it. In such cases, "no-go" in game theory would break down and adding quantum signals would not differ from adding classical signals.

6.2. Kochen–Specker

We have used Bell's Theorem because it is the best known no-go theorem of QM. But there is a stronger no-go theorem – i.e., one using fewer conditions. The Kochen–Specker Theorem (1967) involves a quantum-mechanical set-up for which no hidden-variable analysis is possible even asking for just (10) and (11).[11] A game corresponding to Kochen–Specker – as the game of Fig. 2 corresponds to Bell – is easily constructed. (We omit the details.)

[11] Some more detail for the interested reader: The spin of a certain particle is measured in various directions. The spin in any direction can take values $+1$, 0, or -1. The set-up is such that if the spin is measured in each of three orthogonal directions, and the squares of the three spins are calculated, we will always get two 1's and one 0. But, it is impossible to assign to each point on a sphere a 1 or a 0 so that every set of three orthogonal points has two 1's and one 0. (Kochen–Specker actually gave an argument involving only finitely many different directions of measurement.) The conclusion is that the outcome of a measurement of (the square of) spin in one direction depends on which other directions of measurement are also chosen. But, under (10) and (11), such dependence – called Contextuality in QM – is impossible.

What about examining just (9) and (11)? In this case, any probabilities – including, therefore, quantum-mechanical probabilities – can be reproduced via hidden variables. (See Brandenburger and Yanofsky, 2008, Theorem 3.2 for a simple proof.) We wouldn't be able to drive a wedge between adding quantum rather than classical signals if we imposed only these two conditions on our analysis. But, since we see both Conditional Independence (9) and Sufficiency (10) as basic to classical analysis of correlation, we don't emphasize this point.

6.3. *A classical question*

Back to classical correlation. We said that λ-Independence is not inevitable in the game setting. Still, we think it would be interesting to characterize what it yields. More precisely, we could require Conditional Independence, Sufficiency, λ-Independence, and an optimality condition – such as the requirement that each player is rational, thinks each other player is rational, etc. A proper formulation and analysis of this question appears to be open.

To put this in a broader context, Aumann-style correlation (Aumann, 1974, 1987) is very 'permissive' in that no restrictions are placed on what can constitute a signal. A theory of restricted signals seems worth investigating. No doubt, there are many choices here, but the λ-Independence route seems an interesting one.

6.4. *Historical note*

There is, of course, a historical connection between game theory and QM, via the towering figure of von Neumann.

Von Neumann initiated the hidden-variable program in QM. In von Neumann (1932, pp. ix–x in the 1955 translation), he wrote: "[T]here will be a detailed discussion of the problem as to whether it is possible to trace the statistical character of quantum mechanics to an ambiguity (i.e., incompleteness) in our description of nature." Von Neumann also initiated the study of correlation (von Neumann, 1928; von Neumann and Morgenstern, 1944) in game theory. He defined a cooperative game by starting with a non-cooperative game and allowing correlated behavior among the players. (Formally, the characteristic function for a subset A of players is the maximin payoff to A in the associated zero-sum game between A and not-A.) However, there does not appear to be any evidence that he saw a formal connection between these two enterprises.[12]

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[12] Jean de Valpine brought to my attention that there are several references – but only of a rather general kind – to QM in von Neumann and Morgenstern (1944). See von Neumann and Morgenstern (1944, pp. 3, 33, 148, 401).

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