Comment on "Galileo's discovery of scaling laws," by Mark A. Peterson [Am. J. Phys. 70 (6), 575–580 (2002)]–Galileo and the existence of hell

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Mark Peterson has given a brilliant and fascinating account of Galileo Galilei's early lectures (1588) on Dante's *Inferno* and their relation to Galileo's mature theory of the laws of scaling, presented in his *Two New Sciences* (1638).[1] Peterson uses texts long available in Galileo's collected works, but neglected by physicists and historians of science, probably because these texts fall in the category of "literary works."[2] His analysis shows the importance of going past the boundaries of modern disciplines in order to consider the larger coherence of such a wide-ranging mind as Galileo's.

I would like to add further comments that support and complement Peterson's work. He discusses Galileo's defense of the Florentine Antonio Manetti's account of the topography of Dante's hell against his non-Florentine opponent, Alessandro Vellutello.[3] In his lectures, Galileo assumes that the structural stability of Brunelleschi's dome in the cathedral of Florence [4 braccia (2.26 m) thick] can be scaled up to describe the "dome" of earth [100 miles (161 000 m) thick] that presumably covers the hollow inverted cone of Dante's subterranean Inferno.[4] Yet later Galileo should have realized that his account of the scale of hell was fundamentally unsound because it wrongly presumed that structures could be arbitrarily scaled up in size.

Peterson notes that Galileo's lectures were clearly meant to support the glory of the Medici and the Florentine Academy, so that such an egregious flaw in his argument could have been pounced on and caused much embarrassment. Hence, Peterson concludes that Galileo may have downplayed or ignored these flawed lectures, citing a letter of 1609 to show that by this time, Galileo had developed at least the basic idea that structures do not scale up arbitrarily. Yet one wonders why fifty years separated the 1588 lectures from Galileo's exposition of his corrected theory. Peterson speculates that Galileo was holding this theory in reserve to confute critics of his early lectures, in the way that he saved powerful counter-arguments for use at later stages in his polemics about the physics of floating bodies.

There may be another, simpler explanation of Galileo's reticence on questions of scale. By 1609, Galileo could have deduced that Dante's hell was structurally unsound, if he applied his new understanding of scaling to his earlier arguments. At the same time, he began the telescopic observations that led to the discovery of the satellites of Jupiter in 1610 and his adoption of Copernicanism.[5] He soon was confronted with disturbing signs of ecclesiastical opposition, which condemned Copernicus's book in 1616 and brought charges against Galileo himself in 1633.[6] In the midst of these dangers, Galileo would have had good reason to avoid showing that hell was physically impossible, at least the literal hell of Dante. The status of hell touches moral questions of punishment for sin and also the privilege of the popes, as successors of Peter, keepers of the keys of hell no less than of heaven. Furthermore, as we shall see, the Catholic description of hell was based on Ptolemaic cosmology, so that an attack on one would have involved the other.[7]

Indeed, in his 1588 lectures the young Galileo explicitly refers to the center of the Earth as being "at the center of the universe" (*nel centro del mondo*), showing his adherence to the Ptolemaic view.[8] He defends Manetti's view, in which a huge cone-shaped void would have to lie centered under Jerusalem, reaching the center of the Earth, 3200 miles down.[9] Would the physical improbability of such a chasm not have struck him immediately? It is difficult to imagine him believing naively that, as Manetti has it, there are entrances near Naples to that vast subterranean space. As Peterson discusses, Vellutello's alternative involved a far smaller inferno, only one-thousandth the size of Manetti's, according to Galileo, and hence less preposterous physically. Yet, as Peterson emphasizes, it seemed a given that Galileo must defend the Florentine Manetti at all costs, at least if he wished to gain the favor of the Medici. One wonders what Galileo thought of all this. Perhaps it was all a literary exercise in which it would be oafish to ask whether these poetic structures were physically possible.[10] Yet that was the very crux of Galileo's discussion. If indeed Galileo was aware of these issues (as seems likely), this might have been the first time that political realities affected his presentation of physics. The occasion seems significant because, as Peterson notes, it was an important first step in Galileo's professional career.

Here many interesting questions open for further study. To what extent was the literal existence of hell crucial, in contrast to its status as a symbolic image of the state of souls after death? Even in 1913, *The Catholic Encyclopedia* stated that "theologians generally accept the opinion that hell is really within the earth," though noting that "the Church has decided nothing on this subject; hence we may say that hell is a definite place; but where it is, we do not know."[11] Thomas Aquinas argued in about 1270 that "after death souls have certain places for their reception," so that "those souls that have a perfect share of the Godhead are in heaven, and that those that are deprived of that share are assigned to a contrary place."[12] Here, Aquinas's argument requires that the Earth be the "middle of the whole world [cosmos]" and hence the only possible location of hell as the "contrary place" to heaven. In this view, the central Earth is a kind of "garbage heap," inferior to the heavens above it. Aquinas goes on to argue that the fires of hell are corporeal and of the same species as earthly fire, burning "within the bowels of the earth."[13] Though the exact location of hell was not a matter of faith, its existence was a tenet of Catholic belief and its negation thus heretical. Thus, in 1620 Giuseppe Rosaccio confidently described hell as being within the earth, noting that an enormous space was needed in view of the ever increasing number of the damned, who had no right to expect as much room as the blessed souls in heaven.[14] Explicit attacks on the orthodox doctrine of hell do not appear before about 1650; no pope before John Paul II held that hell is only a spiritual state, rather than a physical place.[15]

Dante himself, in a celebrated letter to Can Grande della Scala (1317), had noted that the literal level of meaning was one of four levels on which his poem worked, along with the allegorical, moral, and analogical levels.[16] If so, the literal existence of hell was on a par with its allegorical, moral, or analogical existence. To be sure, sophisticated readers, then as now, would weigh a literal interpretation against less literal readings.[17] Such problems haunted the reading of the Bible above all, but applied to Dante as well. Certainly the Church might not have wanted simple believers to depart too far from the literal. In any event, it would be disturbing to common beliefs if the traditional hell were shown to be physically impossible. Moreover, this would also deny the authoritative arguments of Thomas Aquinas, as well as the Ptolemaic cosmology on which they were based. As a Copernican, Galileo thought that the Earth was as noble as the other planets, not a "garbage heap" fit to contain hell. The impossibility of the subterranean inferno was dangerously supportive of Copernicus.

Pietro Redondi has argued that, behind the Galileo affair, lay the disturbing possibility that atomic theory contradicts the doctrine that wine is transubstanted into blood in the Eucharist, which is a far more explosive issue theologically than technical issues of astronomy.[18] As intriguing as Redondi's idea is, the existing documents do not give it explicit support; Galileo did not write on the matter at all. The question of the literal existence and structure of hell is another explosive issue that Galileo may have wanted to avoid. In this case, we have Galileo's extended physical description of Dante's inferno and also his detailed articulation of why such a structure could not exist. Galileo may well have dreaded writing down the conclusion: hell cannot exist. Even without his explicit statement, the evidence at hand shows that he would have drawn this inference and realized its danger.

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[1]Mark A. Peterson, "Galileo's discovery of scaling laws," Am. J. Phys. **70**(6), 575–580 (2002).
[2]Galileo Galilei, "Due lezioni all'Accademia Fiorentina circa la figura, sito e grandezza dell'Inferno di Dante," in *Le Opere di Galileo Galilei*, edited by G. Barbèra (Ristampa della Edizione Nazionale, Florence, 1933), Vol. 9, pp. 31–57, translated by Mark A. Peterson, http://www.mtholyoke.edu/~mpeterso/classes/galileo/inferno.html. *See also Mark Peterson, "Dante's Physics," in The Divine Comedy and the Encyclopedia of Arts and Sciences*, edited by Giuseppe C. DiScipio and Aldo Scaglione (Benjamins Publishing, Amsterdam, 1988), pp. 163–180.
[3]See Ref. 1 for a full description of these two views.
[4]For useful diagrams (which agree with the Manetti–Galileo description discussed below), see Dante Alighieri, *The Divine Comedy*, translation and commentary by Charles S. Singleton (Princeton U.P., Princeton, 1970), Vol. 1 (*Inferno*), pt. 2, pp. 43–44.
[5]See Galileo's "The Starry Messenger" (1610) in *Discoveries and Opinions of Galileo*, translated with notes by Stillman Drake (Doubleday, New York, 1957), pp. 21–58. Drake notes (p. 24, n. 2) that, though here Galileo intimates his acceptance of the Copernican system, it was not until 1613 that he unequivocally supported it in print.
[6]For a careful discussion, see Annibale Fantoli, *Galileo: For Copernicanism and for the Church*, translated by George V. Coyne (Vatican Observatory Publications, Vatican City, 1994), pp. 161–248.
[7]I thank Professor Fantoli for drawing my attention to the importance of this point. He also notes that, by Galileo's time, voyages to the southern hemisphere had disproved Isidore of Seville's hypothesis (critiqued by Thomas Aquinas) that hell might be located there.
[8]See Galileo, *Opere*, Vol. 9, p. 33.
[9]See Refs. 1 and 2.
[10]For a helpful overview, see Leonardo Olschki, "Galileo's literary formation," in *Galileo, Man of Science*, edited by Ernan McMullin (Basic Books, New York, 1967), pp. 140–159.
[11]"Hell," in *The Catholic Encyclopedia* (Encyclopedia Press, New York, 1913), Vol. 7, p. 207.
[12]St. Thomas Aquinas, *Summa Theologica*, Supplement to Part III, Question 69, Article 1, as translated in *The Summa Theologica of Saint Thomas Aquinas* (Encyclopaedia Brittanica, Chicago, 1952), pp. 885–886.
[13]Reference 12, Question 97, Article 7, pp. 1071–1072.
[14]Edward Moore, "The geography of hell," in *Studies in Dante* (Clarendon, Oxford, 1968), Vol. 3, p. 135.
[15]D. P. Walker, *The Decline of Hell* (University of Chicago Press, Chicago, 1964), p. 4.
[16]This letter is available in *Literary Criticism of Dante Alighieri*, edited by Robert S. Haller (University of Nebraska Press, Lincoln, NE, 1973), pp. 95–111; Dante had earlier set out his theory of the four levels of interpretation in his *Convivio* (1307), pp. 112–114.
[17]For instance, in his essay "The Divine Comedy" [*The Poets' Dante*, edited by Peter S. Hawkins and Rachel Jacoff (Farrar, Straus and Girous, New York, 2001), p. 119], Jorge Luis Borges notes that "Dante never presumed that what he was showing us corresponded to a real image of the world of death... I believe, nevertheless, in the usefulness of that ingenious concept: the idea that we are reading a true story." For a discussion of the "truth" of Dante's "fiction," see Charles S. Singleton, *Dante Studies 1* (Harvard U.P., Cambridge, MA, 1965), pp. 61–83.
[18]Pietro Redondi, *Galileo Heretic*, translated by Raymond Rosenthal (Princeton U.P., Princeton, 1987).

A remarkable mathematical property of the Landé factor in quantum mechanics

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We show that the Landé factor can, in principle, attain any positive or negative rational number.
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The splitting of the atomic spectra in a weak, homogeneous magnetic field **B** is determined by the Zeeman energy operator

$$H_B = -\mu_B(\mathbf{L}+2\mathbf{S})\mathbf{B}, \tag{1}$$

where the Bohr magneton $\mu_B = e\hbar/2mc$. For not too heavy atoms ($Z \lesssim 80$) and for not too highly excited states, the total orbital angular momentum L (in units of $\hbar$) and the total spin S of all electrons are good quantum numbers (LS coupling). Then the energy splitting is given to first-order in perturbation theory by the expectation value of H_B in the atomic eigenstates $|n,L,S;J,M_J\rangle$, with $\mathbf{J}=\mathbf{L}+\mathbf{S}$ and $M_J=-J,-J+1,\ldots,+J$. The evaluation of the matrix elements (usually with the Wigner–Eckart theorem) gives

$$\Delta E = -\mu_B g M_J B, \tag{2}$$

with the Landé factor

$$g = 1 + \frac{J(J+1)-L(L+1)+S(S+1)}{2J(J+1)}. \tag{3}$$

(For a derivation of this formula see any advanced textbook on quantum mechanics, for example, Ref. 1.)

Here L and $2S$ can in principle be any natural number $1,2,3,\ldots$, or zero. (In reality, L and S are limited from above because stable atoms have only a finite number of electrons, and because of the limits of validity of the LS coupling.) The quantity J can attain the values $J=|S-L|,|S-L|+1,\ldots,S+L$. For $J=0$, $g-1$ is of the indefinite form 0/0, but because $M_J=0$ in this case, ΔE is zero. Obviously g is a rational number in all cases. (It is remarkable that this property was conjectured as early as 1907 by Runge, after a careful analysis of the first systematic experimental data on the Zeeman splitting.[2]) For pure orbital angular momentum ($S=0$), we have $g=1$; for pure spin ($L=0$), $g=2$; and for $L=S$, $g=3/2$, a "mean value." For most atomic states g is in the interval $0.4 \leqslant g \leqslant 2$, but values outside this interval have been measured. The extremal values we have found in the literature are $g=-0.72$ and $g=3.35$.[3] It can also be seen that for nearly any particular value of g, at least in the range $0 \lesssim g \lesssim 2.5$, an atom exists in nature with g extremely close to this value. This experimental fact suggests the mathematical question if Eq. (3) for the Landé factor can produce *any* (positive or negative) rational number disregarding the upper limits on L and S.

In the following we give a simple illuminating proof of this conjecture. It is advantageous to consider instead of g, the quantity

$$f = g - \frac{3}{2} = \frac{(S-L)(S+L+1)}{2J(J+1)}, \tag{4}$$

the deviation of g from the mean value 3/2. The numerator of f is antisymmetric with respect to the exchange $S \leftrightarrow L$, and the denominator is symmetric, given the condition $|S-L| \leqslant J \leqslant S+L$. Therefore, it is sufficient to confine ourselves to positive values of $X=S-L$. (The value $f=0$ is trivially constructed by $X=0$, that is, $S=L$, and, for example, $J=L=1$.) To prove that f (and therefore g) can attain any (positive) rational number p/q, it turns out to be sufficient to consider only the limiting cases $J=X$, and $J=X+2L$. Furthermore, it is simpler to confine ourselves to integer spin values S, because then the quantities X, L, and J in $f=X(X+2L+1)/2J(J+1)$ appear on a more equal footing. For the limiting cases $J=X$, and $J=X+2L$, there are cancellations between the numerator and denominator of f, with the result that

$$f = \frac{X+2L+1}{2(X+1)} \quad (J=X), \tag{5}$$

$$f = \frac{X}{2(X+2L)} \quad (J=X+2L). \tag{6}$$

(For all other allowed values of J, the numerator and denominator of f remain quadratic in X and J, and it will be much more difficult to decide whether they can attain any natural numbers p and q.) Obviously, in Eq. (5), we have $f \geqslant 1/2$, and in Eq. (6) $f \leqslant 1/2$. To prove that f can be any rational number p/q, we will assume that $f=p/q$, and then show that suitable values for J, L, and S can be found as functions of p and q. For $f \geqslant 1/2$, the condition $f=(X+2L+1)/2(X+1)=p/q$ leads in general to $2(X+1)=2kq$, and $X+2L+1=2kp$, for some natural number k. Therefore we have $X+1=kq$, and $kq+2L=2kp$. Hence, k must be even if q is odd. It turns out that we may (always) take $k=2$. Then our solution for $f=p/q$ reads

$$J=X=2q-1\neq 0, \quad L=2p-q, \quad S=2p+q-1. \tag{7}$$

For $f \leqslant 1/2$, $f=X/2(X+2L)=p/q$ leads to $2(X+2L)=2kq$, and $X=2kp$, with the simplest solution ($k=2$):

$$L=q-2p, \quad X=4p, \quad J=2q\neq 0, \quad S=2p+q. \tag{8}$$

By this elementary reasoning, we have shown that f can attain any positive rational number. The negative rational numbers are constructed by the exchange $L \leftrightarrow S$. We find it remarkable that the values $f = \mp 1/2$, which divide the two cases of this derivation, belong to the physically preferred cases $g=1$ (pure orbital angular momentum), and $g=2$ (pure spin), and this although it was not necessary to consider half-integer S values in our proof.

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[1]A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1962), Vol. II, Chap. XVI, Secs. 10–12.
[2]C. Runge, ‘‘Über die Zerlegung von Spektrallinien im magnetischen Felde,’’ Phys. Z. **8**, 232–237 (1907).
[3]*Atomic Energy Levels*, edited by C. E. Moore (Circular of the National Bureau of Standards No. 467), Vol. I (1949), Vol. II (1952), Vol. III (1958).

Comment on ‘‘The Thomas rotation,’’ by John P. Costella *et al.* [Am. J. Phys. 69 (8), 837–847 (2001)]

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An interesting method for studying a facet of special relativity called the Thomas rotation was provided recently by Costella *et al.*[1] One of the most important points discovered by the authors of Ref. 1 is the surprising effect that arises from the composition of two perpendicular boosts. The outcome of two such transformations is that, from the point of view of a boosted observer, the second boost changes the velocity of the *first* one.[2] This unexpected result is considered and commented on by the authors to show that it makes sense from the energetic point of view. However, its direct origin is not revealed and for many readers it may still seem to be counterintuitive and inexplicable by a straightforward calculation. The strange reduction in velocity of the first boost deserves to be examined more thoroughly, because it appears to be just one of the causes of the Thomas rotation.

In this comment we first give a simple explanation for why the second boost reduces the velocity of the first one. Additionally, we show that the strangeness of this effect is even greater than expected. Namely, using the same method as Ref. 1, we find that, from the point of view of a body at rest during the passive boosts (for example, a star in Ref. 1), it is just the *second* boost that is changed while the first remains unaltered! Thus, *the same two boosts* are evaluated differently by the boosted observer and by someone who is at rest. This asymmetry is not mentioned in Ref. 1. However, it is very instructive to reflect on the difference between the two viewpoints because the indicated reduction of boost velocity *together with* the asymmetry is equivalent to the Thomas rotation. Because the mathematical method introduced in Ref. 1 is very convenient for tracing the asymmetry, we apply their method to visualize the Thomas rotation for two perpendicular boosts (in Ref. 1 four boosts are required).

Finally, we show how we can make use of the velocity reduction effect noticed in Ref. 1 to obtain the standard expression for the Thomas *precession* without any complicated calculations. We believe that it is a useful and important example of the elegant formal method contained in their paper.

Let us first elucidate the reduction of velocity suffered by the first boost after the second perpendicular boost is performed. Following the hypothetical situation introduced in Ref. 1, let the starship *Enterprise*, initially at rest relative to some particular star, be boosted in the x direction by a velocity v_x. After the boost an astronaut in the Enterprise observes the star moving with the velocity $-v_x$ which may be calculated in a frame S_x moving (at the moment) together with the Enterprise as $-v_x = \Delta x/\Delta t$ (the x axis of S_x is parallel to the direction x of the space ship motion). It is important that the time Δt be measured by clocks lying along the x axis of the frame S_x. Next, the second boost is accomplished on the Enterprise, this time in the y direction and by a velocity v_y. Now we notice that while in the ‘‘abandoned’’ frame S_x, the displacement of the star during the time Δt is equal to Δx, so in the *new* rest frame of the Enterprise, the respective shift of the star in the x direction also is equal to Δx (because the second boost is *perpendicular* to the x axis of S_x), but the time $\Delta t'$ of the star displacement observed in the new rest frame differs from Δt. To show the difference and find the relation between $\Delta t'$ and Δt, we notice that the synchronized clocks that measure the interval Δt in the frame S_x are *still synchronized* for the astronaut, now additionally boosted in the y direction. If so, the clocks act as a ‘‘single’’ clock extended along the x axis in S_x and moving with velocity $-v_y$ with respect to the Enterprise. Due to the time dilation effect, for the observer in the space ship they seem to run slowly and

$$\Delta t' = \gamma_{v_y} \Delta t. \tag{1}$$

We see then that the velocity of the star in its motion along Δx after the two boosts appears to be reduced and equals

$$\frac{\Delta x}{\Delta t'} = -\frac{v_x}{\gamma_{v_y}}. \tag{2}$$

Thus the twice boosted astronaut registers the velocity of the star to be

$$\mathbf{v}_S = \left(-\frac{v_x}{\gamma_{v_y}}, -v_y \right). \tag{3}$$

Equation (3) it is just the intriguing result that sprang out from the formal procedure introduced by Costella *et al.*[1]

Now let us consider the same two boosts from the point of view of the star. Imagine that the Enterprise is already twice boosted and as such is at rest in some reference frame. Now we pass from that frame to the one connected with the star. To do so we have to cancel the two performed boosts, that is, proceed in the opposite direction with respect to the original order of transformations. First we apply to the rest frame of the Enterprise the reversed y boost and then the reversed x one. By using the method of Ref. 1, we find that after the action of the two reversed boosts, the energy-momentum four-vector of the Enterprise is

$$B_x(-v_x)B_y(-v_y)\begin{pmatrix} m_E \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_E\gamma_{v_x}\gamma_{v_y} \\ m_E\gamma_{v_x}\gamma_{v_y}v_x \\ m_E\gamma_{v_y}v_y \\ 0 \end{pmatrix}, \tag{4}$$

where m_E is the mass of the Enterprise. We calculate the ratio p^i/E and obtain the velocity of the Enterprise as seen by the observer on the star (we omit the z component because it is equal to zero):

$$\mathbf{v}_E = \left(v_x, \frac{v_y}{\gamma_{v_x}} \right). \tag{5}$$

Contrary to Eq. (3), from the point of view of the star, it is the x boost that is unchanged while the y one is reduced in its velocity. Certainly, the decrease of the y component of the velocity can also be explained in the same manner as was done above for the boosted Enterprise where the reduction in velocity is shown to emerge from the time dilation effect.

At the moment, however, we pay attention to the contrast between the results (3) and (5): the velocities $\mathbf{v}_S$ and $\mathbf{v}_E$ differ in the absolute values of their components. Certainly, the relative velocity between the star and the Enterprise must have the same magnitude both for the observer on the star and for the astronaut. Using Eqs. (3) and (5) we may verify that $v_S = v_E$. However, if the system of coordinates connected with the star (in which $\mathbf{v}_E$ is measured) and that associated with the Enterprise (where $\mathbf{v}_S$ is determined) have their respective axes parallel with respect to each other (as they were at the beginning), the respective components of $\mathbf{v}_S$ and $\mathbf{v}_E$ should differ only in their signs. That the components are completely different follows from the fact that the two frames after the two boosts appear to be rotated with respect to each other (see Fig. 1). In other words, although the two boosts of the *Enterprise* were performed without any angular rotation, the resultant rest frame of the spaceship *is* rotated with respect to the initial one (that is, with respect to the frame in which the star rests). So, from the point of view of the star, the two pure boosts make the Enterprise turn around. As promised, by explaining the asymmetry of the results (3) and (5), we have arrived at the Thomas rotation.

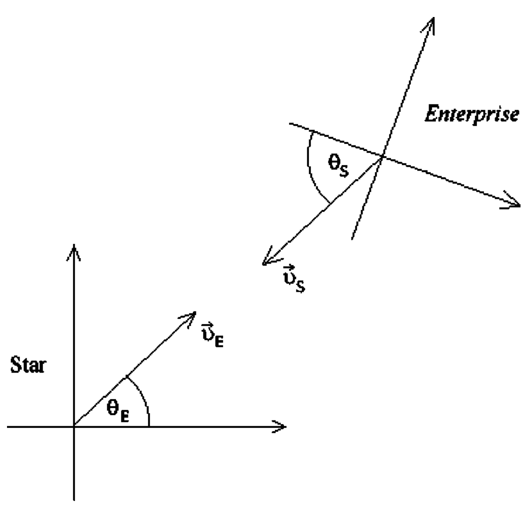

Fig. 1. After the two perpendicular boosts along the x and the y axes, the rest frame of the Enterprise is rotated with respect to the frame of the star by the angle $\Delta\theta = \theta_E - \theta_S$.

On the basis of Fig. 1, the angle of rotation suffered by the rest frame of the Enterprise after the two boosts is

$$\Delta\theta = \theta_E - \theta_S, \tag{6}$$

where

$$\theta_S = \arctan\left(\frac{v_y\gamma_{v_y}}{v_x} \right), \tag{7}$$

and

$$\theta_E = \arctan\left(\frac{v_y}{v_x\gamma_{v_x}} \right), \tag{8}$$

as follows from Eqs. (3) and (5). To check that our reasoning is consistent with that in Ref. 1, we choose $v_x = v_0$ and $v_y = v_1$, where v_0 and v_1 are the velocities of the boosts considered in Ref. 1 [see Eq. (17)]. For that case we find:

$$\Delta\theta = -\arctan\left(\frac{\sqrt{2\gamma_0^2 - 1} - 1}{\sqrt{2\gamma_0^2 - 1} + 1} \right). \tag{9}$$

It can be verified that the double value of $\Delta\theta$ is given by

$$2\Delta\theta = -\arctan\left(\frac{\gamma_0^2 - 1}{\sqrt{2\gamma_0^2 - 1}} \right), \tag{10}$$

which agrees with the result (22) in Ref. 1 obtained for *two pairs* of perpendicular boosts by velocities v_0 and v_1.

Although our presentation of the Thomas rotation may seem to be less concise than that in Ref. 1, we believe that it allows one to understand the relativistic effect more immediately. However, our approach has its own intrinsic benefit as discussed in the Appendix, where we calculate the angle of rotation in the special and important case for which the Enterprise moves in a circle around a star. In this way we obtain the desired standard expression for the Thomas precession without excessive effort.

APPENDIX

Let the Enterprise move in a circle around a star with a speed v. At any moment the movement may be regarded as the effect of a boost of the star by the velocity $-v$ along the x direction of the instantaneous velocity followed by the second boost of velocity $-dv$ perpendicular to the first one. If we make use of the result (5), the velocity of the Enterprise as observed from the point of view of the star is

$$\mathbf{v}_E = \left(v, \frac{dv}{\gamma_v} \right). \tag{A1}$$

We see that $\mathbf{v}_E$ is inclined to the x axis in the frame of the star by the angle

$$\theta_E = \frac{1}{\gamma}\frac{dv}{v}, \tag{A2}$$

where $\gamma \equiv \gamma_v$.

In turn, by applying the reasoning leading to Eq. (3), we find the velocity of the star observed by the astronaut sitting in the Enterprise:

$$\mathbf{v}_S = \left(-\frac{v}{\gamma_{dv}}, -dv\right) \approx (-v, -dv). \tag{A3}$$

The vector $\mathbf{v}_S$ is inclined to the x axis in the frame of the spacecraft by the angle:

$$\theta_S = \frac{dv}{v}. \tag{A4}$$

The difference between θ_E and θ_S is the measure of the Thomas precession suffered by the system of coordinates moving with the Enterprise:

$$d\theta = \theta_E - \theta_S = -\frac{\gamma - 1}{\gamma}\frac{dv}{v}. \tag{A5}$$

If during the time interval dt, the change of the velocity of the Enterprise in the y direction observed from the point of view of the star is dv/γ [see Eq. (A1)], then the angular velocity ω_T of the Thomas precession is

$$\omega_T \equiv \frac{d\theta}{dt} = -(\gamma - 1)\omega, \tag{A6}$$

where ω is the angular velocity of the Enterprise in its circular motion around the star. Despite the simplicity of our approach, the result (A6) is exact and we may compare it with the same outcome delivered by Muller.[3] The negative value of ω_T reflects the fact that the Thomas precession proceeds in the opposite direction with respect to the rotation (assumed to be positive) performed by the Enterprise around the star.

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[1]J. P. Costella, B. H. McKellar, A. A. Rawlinson, and G. J. Stephenson, Jr., ''The Thomas rotation,'' Am. J. Phys. **69**, 837–847 (2001).
[2]Reference 1, pp. 840, 842.
[3]R. A. Muller, ''Thomas precession: Where is the torque?,'' Am. J. Phys. **60**, 313–317 (1992).