MONSTROUS MOONSHINE

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A quick summary of the recent amazing discoveries about the Fischer–Griess "MONSTER" simple group.

Section 1. *History*.

In 1973 Bernd Fischer and Bob Griess independently produced evidence for a new simple group M of order

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71$$

$$= 8080, 17424, 79451, 28758, 86459, 90496, 17107, 57005, 75436, 80000, 00000.$$

We proposed to call this group the MONSTER and conjectured that it had a representation of degree 196883. In a remarkable piece of work, Fischer, Livingstone and Thorne [**6**] have recently computed the entire character table on this assumption. The MONSTER has not yet been proved to exist, but Thompson [**18**] has proved its uniqueness on similar assumptions.

Here are some observations (roughly in chronological order) that are now known not to be mere coincidences:—

(A) M. J. T. Guy observed a certain symmetry in the character table of the monomial group $2^{12}M_{24}$ of [**5**].

(B) We pointed out long ago that the elements of M_{24} have "balanced" cycle-shapes, so that $a^{\alpha}b^{\beta}c^{\gamma}\ldots$ is the same as $(N/a)^{\alpha}(N/b)^{\beta}(N/c)^{\gamma}\ldots$ for some N. Example $1^2.2.4.8^2$, for which $N = 8$.

(C) For each prime p with $(p-1)|24$ there is a conjugacy class (called $p-$ below) of elements of M, with centraliser of form $p^{1+2d}.G_p$, where $p.G_p$ is the centraliser of a corresponding automorphism of the Leech Lattice.

[The symbol p^{1+2d} denotes an extraspecial p-group, and $2d = 24/(p-1)$.]

(D) For the same p, there is a second class $p+$, and the characters of $p+$ and $p-$ in the minimal faithful representation differ by p^d. (Similar properties were observed for elements of order $2p$.)

(E) Ogg [**15**] noticed that the primes p dividing $|M|$ are just those for which the function field determined by the normaliser of $\Gamma_0(p)$ in $PSL_2(\mathbb{R})$ has genus zero. (†)

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† Very recently A. Pizer [**16**] has shown that these primes are the only ones that satisfy a certain conjecture of Hecke (1936, op. cit.) relating modular forms of weight 2 to quaternion algebra theta-series.

(F) McKay noticed that one of the coefficients in the q-series

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots = \sum a_r q^r, \text{ say,}$$

is 196883+1, and Thompson [17] found that the later a_r are also simple linear combinations of the character degrees f_r of M:—

$$a_{-1} = f_1,\ a_1 = f_1 + f_2,\ a_2 = f_1 + f_2 + f_3,\ a_3 = 2f_1 + 2f_2 + f_3 + f_4.$$

Our Tables 1 and 1a, extracted from [6] and [19], give f_r and a_r.

(G) Finally, the Lie group E_8 has dimension 248 = 744/3.

Section 2. *The main conjectures*

As usual, we write Γ for the group $PSL_2(\mathbb{Z})$ of all linear fractional transformations

$$z \to \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{Z},\ ad-bc = 1)$$

and $\Gamma_0(N)$ for the congruence subgroup of all elements with $N|c$. The modular group Γ acts on the upper half-plane, and leaves invariant the field of rational functions of j. Various other discrete subgroups of $PSL_2(\mathbb{R})$ give rise to function fields that are of genus zero and so can be expressed in terms of a single function analogous to j. For instance this happens for $\Gamma_0(N)$ in the cases $N \leqslant 10$.

In all cases that concern us, the group contains the map $z \to z+1$, so that the functions can be written in terms of $q = e^{2\pi iz}$. We call such a function *normalised* if its q-series begins $q^{-1} + 0 + aq + bq^2 + \ldots$, so that the normalised function for Γ is not j, but $\mathrm{J} = j - 744$.

Thompson proposed that the coefficients in the q-series for J be replaced by the representations of M that they "suggest", so that we obtain a formal series

$$H_{-1}q^{-1} + 0 + H_1 q + H_2 q^2 + H_3 q^3 + \ldots$$

in which the H_r are certain representations of M that we call its *head representations*. H_r has degree a_r as in Table 1a, and, for example, H_{-1} is the trivial representation (degree 1), while H_1 is the sum of this and the degree 196883 representation.

Thompson also suggested that on replacing the H_r by their character values $H_r(m)$ for various elements m of M we obtain other functions that might be worth investigating. We have now evaluated these to the q^{10} term for every $m \in M$, and the results fully justify this idea. In fact we conjecture:—

The series

$$T_m = q^{-1} + 0 + H_1(m)q + H_2(m)q^2 + \ldots$$

is the normalised generator of a genus zero function field arising from a group between $\Gamma_0(N)$ *and its normaliser in* $PSL_2(\mathbb{R})$. *The modular groups that arise have a certain natural parameterisation, described later, and there are many formulae for the modular functions in terms of the eigenvalues of certain automorphisms of the Leech Lattice.*

The correspondence between MONSTER conjugacy classes and the genus zero function fields described above is quite remarkable, and at one stage we conjectured that it was essentially 1 to 1. Although this is now disproved, the following points deserve mention:—

(0) A MONSTER element and its inverse have the same Thompson series, as do the two distinct conjugacy classes of elements of order 27.

(1) Although there are no further equalities between these series, there are some linear dependences, for example

$$T_{6+}+2T_{6-} = T_{6+2}+T_{6+3}+T_{6+6}$$

(and similarly for other 4-groups found from Table 3). Oliver Atkin has verified our guess that there are exactly enough of these dependences to bring the dimension down to 163. (See Section 8.)

(2) From column 3 of "Antwerp IV's Table 5" **[1]** one can extract the genus of all function fields corresponding to involutory subgroups of the normaliser of $\Gamma_0(N)$ for $N \leqslant 300$. The last genus zero entry in that table is for the normaliser of $\Gamma_0(119)$, and indeed 119 is the largest order of any element of M. All genus zero cases but three, the "ghost elements" 25Z, 49Z, 50Z of our Table 2, correspond to elements of M. There is some hope of making the correspondence exact by adding functional conditions, because the modular functions in just these three cases have abnormal product formulae. Of course, this correspondence includes observation E.

(3) Our parametrisation for the modular groups suggests various relations between the classes, illustrated in our Table 3, and various identities between the corresponding modular functions, of which observations D and G are consequences.

(4) There are various correspondences between automorphisms of the Leech Lattice and MONSTER classes, which give rise to interesting formulae for the appropriate modular functions. Fixed-point-free automorphisms play a special role here, and there are connections with observation C. We can also deduce observation B from observation A and properties of the modular functions concerned.

(5) Of course the condition that H_r be a MONSTER representation of degree a_r does not determine H_r uniquely (for example, each H_r could be a multiple of the trivial representation). Even when we restrict attention to cases with small multiplicities ambiguities soon arise. However, the additional properties noted above have resolved these up to H_{10}, and, in principle, completely. One of the decompositions suggested in **[17]** has had to be altered.

(6) Had our conjectures been available some time ago they would have afforded an easy route to the computation of the MONSTER character table. It has not escaped our attention that the BABY MONSTER characters have not yet been found, and that the conjectures might help us to find them! Perhaps they could later be verified using the Brauer–Tate theorem.

(7) The resulting notations $T_m = T_{n|h+e,f,g,}$ and $t_m = T_m + \textit{constant}$ for certain modular functions are convenient in their own right, and happily generalise some that have already been used (e.g., Birch **[4]**).

Section 3. *The normaliser of* $\Gamma_0(N)$.

It is a curious fact that the divisors h of 24 are precisely those numbers h for which $xy \equiv 1 \pmod h$ implies $x \equiv y \pmod h$. We shall use this fact to give a simple description of the normaliser of $\Gamma_0(N)$ in $PSL_2(\mathbb{R})$ which does not seem to be generally known. Let h be the largest divisor of 24 for which $h^2|N$, and let $N = nh$.

Then from the rather complicated description of the normaliser in [3] it can be deduced that it consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cn & de \end{pmatrix} = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}_e, \text{ say, for } e \Big\| \frac{n}{h},$$

with the understandings that the determinant of the matrix is $e > 0$, and that $r \| s$ means that $r|s$ and $(r, s/r) = 1$. (We call r an *exact, unitary,* or *Hall* divisor of s.)

Since these matrices can be multiplied by the rule

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}_e \cdot \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix}_\varepsilon = u \text{ times } \begin{Bmatrix} au\alpha + bx\gamma & av\beta + bw\delta \\ cw\alpha + dv\gamma & cx\beta + du\delta \end{Bmatrix}_{vw}$$

(where $e = uv$, $\varepsilon = uw$, $n/h = uvwx$, and u, v, w, x are coprime) they do indeed form a group, up to scalar multiplication.

Moreover, the conditions for $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}_e$ to be in $\Gamma_0(N)$ are simply that

$$b \equiv c \equiv 0 \pmod h$$

and also

$$e = 1 = ad - bc(n/h), \quad \text{so that} \quad ad \equiv 1 \pmod h$$

whence

$$a \equiv d \pmod h$$

by our "defining property of 24".

So we see that $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}_e \in \Gamma_0(N)$ just when $e = 1$ and $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$ is congruent modulo h to an invertible scalar multiple of the identity. It follows that $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}_e$ and $\begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix}_\varepsilon$ lie in the same (left *or* right) coset of $\Gamma_0(N)$ just when

$$e = \varepsilon \quad \text{and} \quad a \equiv k\alpha, b \equiv k\beta, c \equiv k\gamma, d \equiv k\delta \pmod h$$

for some k invertible mod h, and so this set of matrices really does normalise $\Gamma_0(N)$.

From the indices given in the last theorem of Atkin–Lehner [3], we see that it is the full normaliser in $PSL_2(\mathbb{R})$, while the normaliser in $PSL_2(\mathbb{C})$ or $PGL_2(\mathbb{R})$ can be obtained simply by removing the condition $e > 0$.

Section 4. *Subgroups of the normaliser*

A number of subgroups deserve special mention.

(0) The map $z \to -1/Nz$, which we call *the Fricke involution*, is in the normaliser, and extends $\Gamma_0(N)$ to a group we call *the Fricke group*, in which $\Gamma_0(N)$ has index 2.

(1) Provided $N = nh$, where $h|24$ and $h^2|N$ the matrices

$$\begin{pmatrix} a & b/h \\ cn & d \end{pmatrix} = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}_1 \quad \text{of determinant 1}$$

form a group, even when h is not the largest divisor of 24 with this property. Since this group is a conjugate of $\Gamma_0(n/h)$ by $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ we shall call it $\Gamma_0(n|h)$.

(2) The set W_e of all matrices of the form

$$\begin{pmatrix} ae & b \\ cN & de \end{pmatrix} = \begin{Bmatrix} a & bh \\ ch & d \end{Bmatrix}_e, \text{ where } e\|N, \text{ and the determinant is } e$$

is a single coset of $\Gamma_0(N)$. We have the relations

$$W_e^2 \equiv 1,\ W_e W_f \equiv W_f W_e \equiv W_g \,(\text{mod } \Gamma_0(N)), \text{ where } g = \frac{e}{(e,f)}\cdot\frac{f}{(e,f)}$$

which show that these cosets form a subgroup of the normaliser that we call *the involutory normaliser*. They are called the *Atkin–Lehner involutions* for $\Gamma_0(N)$, and we can regard the Fricke involution as the special case W_N.

(3) Similarly the set w_e of all matrices $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}_e$ with a given value of e forms a single coset of $\Gamma_0(n|h)$, which is of course the conjugate by $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ of an Atkin–Lehner involution for $\Gamma_0(n/h)$. We shall therefore call w_e an Atkin–Lehner involution for $\Gamma_0(n|h)$, and this time the Fricke involution is $w_{n/h}$.

In this language we can summarise our results:—

The full normaliser of $\Gamma_0(N)$ *in* $PSL_2(\mathbb{R})$ *is obtained by adjoining to the group* $\Gamma_0(n|h)$ [*which is a conjugate of* $\Gamma_0(n/h)$] *its Atkin–Lehner involutions* [*which are conjugate to those of* $\Gamma_0(n/h)$].

(4) We shall use the notation

$$\Gamma_0(n|h)+e, f, g, \ldots$$

for the group obtained from $\Gamma_0(n|h)$ by adjoining its particular Atkin–Lehner involutions $w_e, w_f, w_g, \ldots$. We further abbreviate this notation (and similar notations later) by:—

(i) omitting "$|h$" when $h = 1$

(ii) writing $\Gamma_0(n|h)+$ when *all* $e\|n/h$ are present

(iii) writing $\Gamma_0(n|h)-$ when *no* e (except 1) is present.

Of course, the "$-$" in (iii) is optional, but is often included for greater clarity.

Section 5. The modular groups for elements of M

If $m \in M$ and $q = e^{2\pi iz}$, then the Thompson series

$$T_m = q^{-1} + 0 + H_1(m).q + H_2(m).q^2 + \dots$$

(in which $H_r(m)$ is the character of the rth head representation at m) defines a function of z which determines four subgroups of $PSL_2(\mathbb{R})$ with varying degrees of interest:

$F(m)$ consists just of the elements of $PSL_2(\mathbb{R})$ that fix T_m,

$E(m)$ of the elements that multiply it by hth roots of 1,

$D(m)$ of the elements that multiply it by *any* roots of 1,

$C(m)$ of elements that convert it to functions $(AT_m + B)/(CT_m + D)$.

We call $F(m)$ the *fixing group* and $C(m)$ the *converting group*, and use the term *eigengroup* for $E(m)$ rather than for the *distended eigengroup* $D(m)$ because the latter seems to be of less interest in this context.

Now an element $m \in M$ determines a number N in any of three ways:

(1) as the *level* of the group $F(m)$,

(2) as the *least* N with $z \to z/(Nz+1)$ in $F(m)$,

(3) as the *unique* N with $z \to -1/Nz$ in $C(m)$.

Having determined N, we write $h = N/n$, where n is the order of m, and observe that in fact h is always an integer, $h|24$, and $h^2|N$.

Then we conjecture:—

$E(m)$ has the form $\Gamma_0(n|h) + e, f, g, \dots$

$F(m)$ is a certain subgroup of index h in this.

[It is easy to see that $C(m)$ is the normaliser of $F(m)$ in $PSL_2(\mathbb{R})$, and we are not very interested in $D(m)$, which is occasionally larger than $E(m)$.] To specify $F(m)$ exactly it will suffice of course to specify the eigenvalue λ by which a given element of $E(m)$ multiplies T_m. We believe:—

(0) $\lambda = 1$ for elements of $\Gamma_0(N)$, so is constant on cosets of $\Gamma_0(N)$,

(1) $\lambda = 1$ for all the Atkin–Lehner involutions of $\Gamma_0(N)$ inside $E(m)$,

(2) $\lambda = e^{-2\pi i/h}$ for the coset $\begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}_1$ (i.e., $e = 1$, $a \equiv b \equiv d \equiv 1$, $c \equiv 0 \pmod{h}$),

(3) $\lambda = e^{\pm 2\pi i/h}$ for the coset $\begin{Bmatrix} 1 & 0 \\ 1 & 1 \end{Bmatrix}_1$ (i.e., $e = 1$, $a \equiv c \equiv d \equiv 1$, $b \equiv 0 \pmod{h}$),

the sign in (3) being $+$ if $z \to -1/Nz$ is in $E(m)$, $-$ if not.

It can be checked that the cosets in (2) and (3) generate $\Gamma_0(n|h)$, so that (0–3) completely determine λ, and therefore the exact fixing group $F(m)$.

We shall use the symbol

$$n|h + e, f, g, \dots$$

as a name for the set of MONSTER elements m for which $E(m)$ has the form

$$\Gamma_0(n|h)+e,f,g,\ldots.$$

By the remarks in Section 2, this set is a union of one or two conjugacy classes, and we loosely call it a *class*. We abbreviate its name in similar ways to those in which we abbreviate the names for groups.

Section 6. Relations between the classes

Our parametrisation shows up a number of relations between the classes. In particular, the power maps are very simply expressed:—

The dth power of $n|h+e,f,g,\ldots$ *is of class* $n'|h'+e',f',g',\ldots$, *where* $n' = n/(n,d)$, $h' = h/(h,d)$, *and* $e',f',g',\ldots$ *are the divisors of* n'/h' *among the numbers* $e,f,g,\ldots$.

In the case that $d|h$ we call m the *dth harmonic* of m^d, and we call the elements with $h=1$ the *fundamental* elements. The general element, of class $n|h+e,f,g,\ldots$ is therefore the hth harmonic of a fundamental one of class $(n/h)+e,f,g,\ldots$. [In a slight divergence from musical terminology, fundamentals are their own first harmonics, rather than zeroth harmonics.]

If m' is the dth harmonic of m, then for an appropriate choice of the functions t_m and $t_{m'}$ we have

$t_{m'}(z) = [t_m(dz)]^{1/d}$

$E(m')$ is the conjugate of $E(m)$ by $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$

$F(m')$ contains the corresponding conjugate of $F(m)$ to index d.

If $M+e$ is the class obtained from class M under symmetrisation by a further Atkin–Lehner involution w_e, then for an appropriate choice of the functions t_M and t_{M+e} we have

$t_{M+e} = t_M(z)+k/t_M(z)$

$E(M+e) = E(M)$ extended by w_e

$F(M+e)$ contains $F(M)$ to index 2.

Each line of Table 3 illustrates a number of such relationships, and makes a simultaneous choice of the appropriate functions. The typical line starts with the symbol for a fundamental element of class m followed by a formula for the appropriate t_m, and then gives the Atlas names for all harmonics of m (including m itself), followed in parentheses by their symmetrisations (with the relevant constants k).

Some lines also give, after a semicolon, additional product formulae for the fundamental t_m (with different constants), and at the end of the table there are some cases of "pseudoharmonics", for which $t_{m'}(z) = [t_m(dz)]^{1/d}$, but our other conditions for harmonics are not all satisfied.

Thus the first line of the table tells us that $1A$ and $3C$ are the first and third harmonics of the identity, with functions $t_{1A} = j$, $t_{3C} = (j(3z))^{1/3}$. From the second line, $2B$, $4D$, $6F$, $8F$, $12J$, $24J$ are the harmonics of $2-$, with symmetrisations

$$t_{2A} = t_{2B}+4096/t_{2B}, \quad t_{4B} = t_{4D}+64/t_{4B}.$$

The symbol $1^{24}/2^{24}$ tells us that the appropriate function to use for class $2B$ is

$$\eta(z)^{24}/\eta(2z)^{24} = T_{2B}-24,$$

and so a product formula for its dth harmonic is $[\eta(dz)/\eta(2dz)]^{24/d}$.

The table is complete for all harmonics, pseudoharmonics, and symmetrisations within M, and displays all product formulae for the t_m in terms of factors $\eta(kz)$.

Section 7. Relations with the Leech Lattice

There are several correspondences between automorphisms of the Leech Lattice L (see [13], [5]) and elements of M. Because L is a rational lattice, we can use Frame's "generalised permutation" notation, [7], in which we say that an automorphism π has shape $a^{\alpha} b^{\beta} \ldots / c^{\gamma} d^{\delta} \ldots$ meaning that its eigenvalues can be obtained by removing those of a permutation of cycle-shape $c^{\gamma} d^{\delta} \ldots$ from those of one of shape $a^{\alpha} b^{\beta} \ldots$

We then write

$\eta_\pi(z)$ for $\eta(az)^{\alpha}\eta(bz)^{\beta}\ldots/\eta(cz)^{\gamma}\eta(dz)^{\delta}\ldots$

L_π for the sublattice of L fixed by π, and

$\theta_\pi(z)$ for the θ-function of L_π, namely $\sum u_n q^n$,

where u_n is the number of vectors of norm $2n$ in L_π.

Then it seems that there is always a class of elements π_1 in M whose Thompson series has a form $\theta_\pi(z)/\eta_\pi(z)$.

Some of the product formulae in Table 3 arise in this way from cases in which π acts fixed-point-freely on L, so that $\theta_\pi = 1$. But to understand some of the others, we shall need to study the groups $p.G_p$ of our observation (C) in more detail.

If π is a fixed-point-free automorphism of L of prime order p, its eigenvalues must be the $p-1$ primitive pth roots of unity, each repeated $24/(p-1) = 2d$ times. [The fact that this number is even follows either from a case-by-case analysis $\begin{pmatrix} p = & 2 & 3 & 5 & 7 & 13 \\ 2d = & 24 & 12 & 6 & 4 & 2 \end{pmatrix}$ or from later statements.] Now since π and the complex number $e_p = e^{2\pi i/p}$ have the same minimal polynomial $t^{p-1}+t^{p-2}+\ldots+t+1$ over $\mathbb{Z}$, we can define $v.f(e_p) = v.f(\pi)$ for each $v \in L$ and each polynomial $f(t) \in \mathbb{Z}[t]$, and so turn L into a $2d$-dimensional lattice over $\mathbb{Z}[e_p]$. The automorphisms of L that preserve this structure are just those that commute with π, and they form the group $p.G_p$ of observation (C).

[We use an extended form of the notation first introduced in [5], under which the symbols

n		cyclic	
p^n	denote	elementary abelian	groups of those orders.
p^{1+2d}		extraspecial	
$[n]$		unspecified	

while $A.B$ just means a group with a normal subgroup of type A whose quotient has type B.]

When we factor by the ideal generated by $e_p - 1$, $\mathbb{Z}[e_p]$ becomes the field of order p, and so L becomes a vector space $L(p)$ of dimension $2d$ over this, or equivalently an elementary abelian group of order p^{2d}. It follows that G_p has an action on this group, so that there exists a group $p^{2d}.G_p$. Moreover, a suitable complex multiple of the inner product on L yields a symplectic inner product on $L(p)$, so that there exists a group $p^{1+2d}.G_p$ with the property that for $x, y \in p^{1+2d}$ we have

$$x^{-1}y^{-1}xy = m^{\bar{x}\cdot\bar{y}}$$

where m is the central element of p^{1+2d} and $\bar{x}.\bar{y}$ the symplectic inner product of the images of x and y in $L(p)$.

Now it happens that the centraliser in M of an element m of class $p-$ is a group $p^{1+2d}.G_p$ of just this form. The particular cases are

$2^{1+24}.G_2$ (G_2 the Conway simple group $\cdot 1$)

$3^{1+12}.G_3$ ($G_3 = 2.\mathrm{Sz}$, Sz = Suzuki's sporadic simple group)

$5^{1+6}.G_5$ ($G_5 = 2.HJ$, HJ = the Hall–Janko simple group)

$7^{1+4}.G_7$ ($G_7 = 2A_7$, the Schur double cover of A_7)

$13^{1+2}.G_{13}$ ($G_{13} = SL_2(3)$, the double cover of A_4).

It appears that if π is an automorphism of L whose p-part is the central element of $p.G_p$, then the element π_1 of M considered above can be taken as an element of $p^{1+2d}.G_p$ whose p-part is the central element of this group, and which has the same image in G_p as π does.

However, it seems that the correspondence between π and π_1 is not the only one of interest. If π is an element of $p.G_p$ of Frame-shape $a^\alpha b^\beta \ldots$, there is usually an element π_p of M with product formula

$$a^\alpha b^\beta \ldots / (pa)^\alpha (pb)^\beta \ldots$$

Thus, if π is the automorphism $x \to -x$ of L, regarded as an element of $2.G_2$, it has Frame-shape $2^{24}/1^{24}$, so that π_2 has product formula

$$(2^{24}/1^{24})/(4^{24}/2^{24}) = 2^{48}/1^{24}\,4^{24}$$

which we see from Table 3 corresponds to MONSTER class 4+. In these calculations, the Frame-shape to use is that corresponding to the representation of $p.G_p$ on a $2d$-dimensional lattice over $\mathbb{Z}[e_p]$.

There appear to be similar correspondences $\pi \to \pi_n$ for non-prime n—we shall not go into more detail here. Since $p.G_p$ (or more generally $n.G_n$) is not always a rational group, the automorphisms involved are not always expressible in Frame's notation, and the obvious generalisation of our remarks (which works!) involves η-functions evaluated at points of the form $nz + (a/b)$. A number of such formulae, not all obtainable in this way, are given in Table 3a.

We show in Section 11 that the number of product formulae of this type for a given function T_m(+constants) is at most equal to the number of finite values taken by T_m at cusps. The formulae of Tables 3 and 3a show that this bound is attained for all functions T_m except those corresponding to the three ghost elements 25Z, 49Z, 50Z. The "missing" formulae in these cases are also given in Table 3a; they involve a slightly generalised kind of η-function.

Michael Guy's symmetry of $2^{12} M_{24}$ shows that that group has an element of shape $(2a)^{\alpha}(2b)^{\beta}\ldots/a^{\alpha} b^{\beta}\ldots$ whenever M_{24} has one of shape $a^{\alpha} b^{\beta}\ldots$. The "balance" property of M_{24} now follows from the fact that the corresponding η-function product formula is inverted by the Fricke involution $z \to -1/2Nz$. (See Section 11.)

A more complicated formula apparently enables us to compute t_m for any m in the centraliser $G_1 = 2^{1+24}.C_1$ of an element of class $2-$. We regard m as the image of two elements π and $-\pi$ of the central extension $G_0 = 2^{1+24}.C_0$ of G_1, and can then define $\eta_\pi, L_\pi, \theta_\pi$ as above, since the quotient group C_0 acts on the Leech Lattice. Also, since any vector v of L_π has a natural image in the group 2^{24}, it may be called symmetric or skew according as the two corresponding elements of 2^{1+24} are fixed or interchanged on conjugating by π. If we define

$$\theta_\pi^-(z) = \textstyle\sum_{v \in L_\pi} \pm q^{\mathrm{norm}(v)} \quad (+ \text{ for symmetric } v, \ - \text{ for skew } v)$$

then our formula is

$$t_m(z) = \tfrac{1}{2}\left(\frac{\theta_\pi^-(z)+\delta_\pi.\theta_\pi(2z)}{\eta_\pi(z)} + \frac{\theta_{-\pi}^-(z)+\delta_{-\pi}.\theta_{-\pi}(2z)}{\eta_{-\pi}(z)}\right)$$

where δ_π is the value at π of the unique character of degree 2^{12} for G_0 that restricts irreducibly to the extraspecial group 2^{1+24}.

We make the following remarks:

(0) t_m is well-defined by the formula, since it is symmetric in π and $-\pi$.

(1) Classes of G_1 that fuse in M should yield the same function T_m, but the formula may well give different constant terms for t_m.

(2) It follows from its expression in terms of eigenvalues that the coefficient of q^n in $1/\eta_\pi(z)$ is a character of G_0, and it can be seen that the coefficients in $\theta_\pi^-(z)+\delta_\pi.\theta_\pi(2z)$ are characters of G_0 obtained by induction from linear characters of various subgroups. These remarks entail that the coefficients in our formula are characters of G_1.

(3) We have not been able to find similar formulae for the centralisers of elements of classes $3-$, $5-$, $7-$, and $13-$. If this could be done, it would, in virtue of the Brauer–Tate theorem, go a long way towards establishing our main conjecture that the coefficients of the T_m are characters of M.

Section 8. *The replication and other formulae*

Let us write $J(z) = T_1(z) = j(z) - 744 = q^{-1} + a_1 q + a_2 q^2 + \ldots$. Then for any prime number p the expression

$$K(z) = J(pz) + J\left(\frac{z}{p}\right) + J\left(\frac{z+1}{p}\right) + \ldots + J\left(\frac{z+p-1}{p}\right)$$

$$= q^{-p} + a_1 q^p + a_2 q^{2p} + \ldots + p\{a_p q + a_{2p} q^2 + \ldots\}$$

is invariant under Γ, which permutes its $p+1$ arguments. Since it has no poles inside the upper half-plane, it is actually a polynomial in J, whose coefficients can be found from the leading terms. We deduce that there is an identity of the form

$$\frac{1}{p}\{J^p(z)-J(pz)\} = f(J)+a_p q+a_{2p}q^2+\dots .$$

Such identities exist also for composite multipliers, and apparently always have character valued versions, which we call *replication formulae*, obtained by replacing each a_r by a suitable value of H_r. Here are the first few cases:—

$$\tfrac{1}{2}\{T^2-T_{(2)}(2z)\} = \{H_2 q+H_4 q^2+\dots\}+H_1 \quad \text{(duplication)}$$

$$\tfrac{1}{3}\{T^3-T_{(3)}(3z)\} = \{H_3 q+H_6 q^2+\dots\}+H_1 T+H_2 \quad \text{(triplication)}$$

$$\tfrac{1}{4}\{T^4-T_{(4)}(4z)-T_{(2)}(z)-T_{(2)}(z+\tfrac{1}{2})\}$$
$$= \{H_4 q+H_8 q^2+\dots\}+H_1 T^2+H_2 T+(H_3-\tfrac{1}{2}H_1{}^2)$$

$$\tfrac{1}{5}\{T^5-T_{(5)}(5z)\}$$
$$= \{H_5 q+H_{10}q^2+\dots\}+H_1 T^3+H_2 T^2+(H_3-H_1{}^2)T+(H_4-H_2 H_1).$$

where $T = T_m(z)$, $T_{(n)} = T_{m^n}$, and $H_r = H_r(m)$. To obtain the n-plication formula, set

$$K(z) = \sum J\left(\frac{nz+dr}{d^2}\right), \text{ summed over } d|n \text{ and } 0 \leqslant r \leqslant d$$

and replace a_r by $H_r(m^{n/d})$ inside $K(z)$ and by $H_r(m)$ outside (including appearances in the coefficients of f).

Comparing powers of q we obtain many identities, for example

$$H_4 = H_3+H_1{}^{2-}$$

$H_6 = H_4+H_2 H_1$	$H_6 = H_4+H_2 H_1$
$H_8 = H_5+H_3 H_1+H_2{}^{2-}$	$H_9 = H_5+H_3 H_1+H_2{}^2+H_1{}^{[21]}$
$H_{10} = H_6+H_4 H_1+H_3 H_2$	$H_{12} = H_6+H_4 H_1+2H_3 H_2+H_2 H_1{}^2$
$H_{12} = H_7+H_5 H_1+H_4 H_2+H_3{}^{2-}$	

from the duplication and triplication formulae, where

$$H_r{}^{2-} = \tfrac{1}{2}\{H_r{}^2(m)-H_r(m^2)\}\, H_r^{[21]} = \tfrac{1}{3}\{H_r{}^3(m)-H_r(m^3)\}.$$

Although it seems that the a-plication and b-plication formulae agree at H_{ab}, we get new relations in other cases when there are two formulae for the same H_n. For example, our two formulae for H_{12} show that H_7 can be expressed in terms of H_1, H_2, H_3, H_5, and similar methods show that the same is in fact true of all H_n. We can also find a number of relations, of increasing complexity, between H_1, H_2, H_3, and H_5.

When the fixing group $F(m)$ contains the involution w_p, the two sides of

$$T_m(z)+T_m\left(\frac{z}{p}\right)+T_m\left(\frac{z+1}{p}\right)+\dots+T_m\left(\frac{z+p-1}{p}\right) = T_{m^p}(z)$$

will have the same invariance group and leading terms, and must therefore be equal. Between this *compression formula* and the corresponding replication formula we can eliminate some terms to get an *expansion formula*, of which the prototype is

$$J(z)+J(2z) = T_{2+}{}^2(z)+T_{2+}(z)-2H_1(2+).$$

Some other instances, providing expressions for T_m as roots of polynomials involving T_{m^n}, have been used in Table 4a.

There are other types of compression formula in which the symmetrisation involves some Atkin–Lehner involution or is achieved in other ways, for example

$$T_{2+}(z) = T_{2-}(z)-T_{2-}\left(\frac{z}{2}\right)-T_{2-}\left(\frac{z+1}{2}\right)$$

$$T_{4+}(z) = T_{4-}(z)+T_{8+}\left(\frac{z}{2}\right)+T_{8+}\left(\frac{z+1}{2}\right).$$

The same type of argument can be used to establish the linear relations between the T_m that were mentioned in Section 2. If N is one of 6, 10, 12, 18, and W_a, W_b, W_c are the three Atkin–Lehner involutions for $\Gamma_0(N)$, then the functions

$$T_{N-} \qquad T_{N+} \qquad T_{N+a} \qquad T_{N+b} \qquad T_{N+c}$$

have the respective forms

$$f, \quad f+f(W_a z)+f(W_b z)+f(W_c z), \quad f+f(W_a z), \quad f+f(W_b z), \quad f+f(W_c z)$$

and so we have

$$2T_{N-}+T_{N+} = T_{N+a}+T_{N+b}+T_{N+c}.$$

The relation

$$2T_{30+15}+T_{30+} = T_{30+6,10,15}+T_{30+3,5,15}+T_{30+2,15,30}$$

is exactly similar, and there are two more equations

$$2T_{8-} = T_{4-}+T_{4|2} \qquad 2T_{16-} = T_{8-}+T_{8|2}$$

which can be regarded as four-group relations of this type in which the missing terms correspond to functions that symmetrise to zero. The last relation

$$T_{12+3}+T_{12|2+}-T_{12|2+2}-T_{12|2+6} = 2(T_{12-}+T_{24+}-T_{24+8}-T_{24+24})$$

is more difficult, but since $\Gamma_0(24)$ has genus 1 we can use the theory of elliptic functions to show that the difference between the two sides has no poles, and so is zero. The last three relations were discovered by Atkin, who has also shown that there are no more.

Section 9. Moonshine for other groups.

Various groups G, often derived in some way from centralisers of elements of M, have moonshine properties of their own. In other words, to each element $g \in G$ there corresponds a series

$$t_g = q^{-1}+h_0(g)+h_1(g).q+h_2(g).q^2+\dots$$

defining the modular function for which the fixing group $F(g)$ contains some $\Gamma_0(N)$

and determines a function field of genus zero. Most of the properties we found for M extend, though there are some differences:

(1) The fixing group does not always contain $\Gamma_0(N)$ *normally*.

(2) The Fricke involution need not lie in the converting group.

(3) The replication formulae need certain modifications.

(4) There are additional multiplicative properties for certain groups, and for these the most natural $h_r(g)$ are *generalised* characters.

Multiplicative moonshine. We discuss (4) first. The group G_p of Section 7 has a central element -1, and two algebraically conjugate representations ϕ_+ and ϕ_- of degree $2d = 24/(p-1)$, except that $\phi_+ = \phi_-$ for $p = 2$. For this group, *every* t_g has a multiplicative formula:

$$t_g = \frac{1}{q} \prod_{p \nmid n} \operatorname{char}_{\pm}(q^n)$$

in which

$$\operatorname{char}_{\pm}(q) = (1-q\varepsilon_1)(1-q\varepsilon_2) \dots (1-q\varepsilon_{2d})$$

where the ε's are the eigenvalues of g in the representation $\phi_{\pm}$, and the sign is the Legendre symbol (n/p). With this definition it can readily be shown that $h_r(-g)$ is a character, while $h_r(g)$ is only a *generalised* character, but certain properties of the replication formulae show us that it would be wrong to exchange the two functions. However, there is a bonus: for these groups $h_0(-g) = -h_0(g)$ is also a character of G, namely that afforded by the basic representation ϕ_+.

Immaterial moonshine. For the groups $2B$, $3F_{24}'$, E, F, H, M_{12} of Table 2a, we seem always to get proper characters, and the constant term $h_0(g)$ is immaterial, just as in M.

The fixing groups of the new modular functions are less restricted than those that arise from M. For example, in $2B$ there is an involution corresponding to the function $(j-1728)^{\frac{1}{2}}$, and which we therefore call $2|2$, but although W_4 is in the eigengroup, it is not in the fixing group and therefore has eigenvalue -1 rather than $+1$. A seventh root, $14|2$, of this element arises in both $2B$ and H, and has similar properties. In F there is an element we call $5|5$, since its fixing group has index 5 in $\Gamma_0(5|5)$, but the latter is not the eigengroup, and does not even contain the fixing group normally. Our naming system rapidly breaks down, and in fact it seems that the possible fixing groups are *all* the discrete extensions of $\Gamma_0(N)$ for which the corresponding function field has genus zero. We shall say more about such groups in a moment.

When algebraic irrationalities arise in the coefficients, there are new problems, like the need to distinguish between ϕ_+ and ϕ_- above. We have noticed that when T_g involves quadratic irrationalities, and G is derived from the centraliser of an element of order p in M, then the part $\{H_n q + H_{2n} q^2 + \dots\}$ of the n-plication formula must be replaced by its algebraic conjugate whenever $(n/p) = -1$. Presumably this rule has a natural extension to other groups and higher degree irrationalities, if indeed these arise.

Abstract replication. If $T = q^{-1}+H_1q+H_2q^2+\ldots$ generates a genus zero function field corresponding to some group containing $\Gamma_0(N)$, there will usually be several groups G with elements g for which $T = T_g$. We say that these elements have *type* T, and call T a *type* (even if there is no g with $T = T_g$).

The replication formulae, as just amended, can now be used to *define* certain functions $T_{(2)}$, $T_{(3)}$, ..., $T_{(n)}$, which we call the *duplicate, triplicate, ,..., n-plicate* of T. Of course, if T is the type of some $m \in M$, these will just be the types of the square, cube, ..., nth power of m, so that our abstract definition has captured at least something of the multiplication in M.

If G is derived from the centraliser of some element of M of order s, and $T = T_g$, then it seems that indeed $T_{(n)} = T_{g^n}$ whenever $(n, s) = 1$. But if $(n, s) > 1$, then n-plication often yields an element in another group, usually the Monster itself. For example, the quintuplicate of the type $T_{5|5}$ is $J(z)$, and so corresponds to the identity element, not of F, but of M. Since $J(z)$ is its own n-plicate for every n, we call it the identity type, and say that $T_{5|5}$ has replication order 5, while elements of any order n prime to 5 in F have replication order $5n$.

Many questions arise about this abstract replication of modular functions. Is the a-plicate of the b-plicate equal to the ab-plicate? Does every type have a well-defined and finite replication order? What is the proper treatment of algebraic irrationalities? And so on.

It is a famous assertion of Galois that $PSL_2(p)$ has a subgroup of index p only for $p = 2, 3, 5, 7, 11$. We have already mentioned the types $t_{2|2}$, $t_{3|3}$, $t_{5|5}$, and remark that $t_{7|7}$ and $t_{11|11}$ arise respectively in Held's group and M_{12}. The exact correspondence of these with the Galois exceptions appears to be significant.

Finally, we ask whether the sporadic simple groups that may not be involved in M (those discovered by Lyons, O'Nan, Rudvalis, and the three Janko groups J_1, J_3 J_4) have moonshine properties. There is an exceptional involutory automorphism of the algebraic curve for $\Gamma_0(37)$ that might be relevant for the Lyons group. Is there a similar period three automorphism for the case $\Gamma_0(67)+$?

Section 10. *The genus zero problem*

Helling [9] has shown that the groups $\Gamma_0(n)+$ for square-free n are maximal discrete groups, and that every discrete group Δ commensurable with Γ can be conjugated into one of these groups. Moreover, if the function field for Δ is of genus zero, so is that for $\Gamma_0(n)+$, and it is easy to see that the conjugating element can be taken in the form $z \to (pz+q)/r$ where p, q, r are integers with no common factor.

The question as to which groups between $\Gamma_0(N)$ and its involutory normaliser $\Gamma_0(N)+$ give genus zero is an old one. Fricke ([**8**], p. 367, but accidentally omitting 59) lists cases when the Fricke normaliser $\Gamma_0(N)+N$ gives genus zero, and Ogg [**15**] has used techniques from algebraic geometry to show that Fricke's list (with 59 inserted) is complete for primes (and offers a bottle of Jack Daniels' for an explanation of why the primes that arise are just those dividing $|M|$!) More recently, Kluit [**12**] has shown that there are no cases other than those appearing in Table 5 of [**1**], which are of course just the cases with $h = 1$ in our Table 2. See also Kluit [**11**].

The corresponding discussion for the non-involutory part of the normaliser does not seem to be available in the literature. However, our remark that the full normaliser of $\Gamma_0(N)$ is conjugate to the involutory normaliser of $\Gamma_0(n/h)$ makes a fairly elegant discussion possible. In particular, the largest N for which the full normaliser of $\Gamma_0(N)$ has genus zero is $N = 24^2.119 = 68544$.

However, we are concerned also with groups not containing any $\Gamma_0(N)$ normally, and the correct requirements seem to be:—

(1) Δ contains some $\Gamma_0(N)$.

(2) the function field for Δ has genus zero.

(3) the translation $z \to z+k$ is in Δ exactly when k is an integer.

(4) the coefficients in the canonical Hauptmodul T for Δ are algebraic integers.

We conjecture that there are only finitely many groups with these properties. Larissa Queen has computed the first few terms of T_g for all g in various finite groups (E, F, H, M_{12}, ...) and for the elements of smallest order in the infinite Lie group $E_8(\mathbb{R})$. In most cases the corresponding modular groups are easily identified. On the basis of these results we conjecture that there will be three or four hundred cases in all (171 of which appear in M). It would be very interesting to have a complete list, and to study the replication maps between them.

Section 11. *Description of the tables*

Table 1, copied from [6], gives the degrees f_i of the irreducible characters of M.

Table 1a gives first, copied from [19], the coefficients $a_0 - a_{24}$ in the q-series for j. Beside this are given the decomposition numbers for the Head characters H_{-1} to H_9 in terms of the MONSTER irreducibles ordered as in Table 1.

Table 2 is our class list for the MONSTER. Its columns give:—

column heading	*contents*
name	ATLAS name of the conjugacy class of m
primepowers	ATLAS names for the prime powers of m
F	letter assigned to the class of m in [6]
symbol	our parametrisation $n\|h+e, f, g, \ldots$
centraliser	order of the centraliser of m
D	$D/12$ is the "Euler characteristic" of $F(m)$, i.e., $2\pi/3D$ = area of fundamental region of $F(m)$.
C	cusp number.

The term ATLAS refers to the Atlas of Finite Groups that we are preparing with R. T. Curtis and R. A. Parker, in which classes of elements of order n in any group are named nA, nB, nC, ... in descending order of their centraliser sizes. The number D can be used to find the index of one of the groups $F(m)$ in another that contains it.

Table 2a supplements Table 2 by giving structural details of the centralisers of elements of small order. It also gives decimal forms for the centraliser orders that were too long to fit in Table 2 itself.

Table 3 gives all products for t_m expressible only in terms of $\eta(kz)$ for various k. It also illustrates various relations between the classes in a way described in more detail in Section 6. This table can be used to derive a formula for any class in any line from one for the fundamental class.

Table 3a gives additional product formulae involving $\eta(kz+c)$, $c \neq 0$, and some transformation rules for such functions. The three formulae for 25Z, 49Z, 50Z involve a further generalisation explained in the table.

If $\pi(z)$ is one of the product formulae for m in Table 3 or 3a, then $\pi(z) = T_m - k$, and since $\pi(z)$ does not vanish in the interior of the upper half-plane, T_m must take the value k at a *cusp*. Since T_m takes the value ∞ at the cusp $i\infty$, the number of such product formulae for a given m is therefore at most $C-1$, where C, given in Table 2, is the number of equivalence classes of cusps under $F(m)$. Study of Tables 3 and 3a shows that the bound is always attained, so that no more such formulae are to be expected.

It is also possible to see from these tables how the Atkin–Lehner involutions transform the t_m. The well-known formulae

$$\eta(z+1) = \varepsilon.\eta(z), \quad \eta(-1/z) = \eta(z)\Big/\sqrt{\left(\frac{z}{i}\right)} = \eta(z).z^{\frac{1}{2}}.\varepsilon^{-3},$$

where $\varepsilon = e^{2\pi i/24}$, imply that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ we have

$$\eta\left(\frac{az+b}{cz+d}\right) = \eta(z).(cz+d)^{\frac{1}{2}}.\varepsilon^{f(a,\,b,\,c,\,d)}.$$

and using this one can show that to within algebraic factors that largely cancel in our calculations, the elements of $\Gamma_0(N)$ leave all $\eta(ez)$ $(e\|N)$ fixed, while the Atkin–Lehner involutions of $\Gamma_0(N)$ permute them in the obvious way. For instance, when $N = 6$, we find that W_2 interchanges $\eta(z)$ with $\eta(2z)$ and $\eta(3z)$ with $\eta(6z)$ to within such algebraic factors, and therefore fixes the product formula $1^4\,2^4/3^4\,6^4 = t_{6+2}$ but inverts $1^6\,3^6/2^6\,6^6 = t_{6+3}$ of Table 3. [A more detailed calculation shows that W_2 takes t_{6+3} to $81/t_{6+3}$.]

Table 4 gives numbers $H_{-1}(m), \ldots, H_{10}(m)$ for each $m \in M$. For $r \neq 0$, $H_r(m)$ is the coefficient of q^r in T_m (i.e. the value of the rth head character at m), while $H_0(m)$ is the Rademacher constant for T_m. The Rademacher constant of a modular function f is the complex number c for which $f+c$ lies in a certain complex vector space. This is the unique space that is invariant under the positive elements of $PGL_2(\mathbb{Q})$ and has codimension 1 in the space of *all* modular functions belonging to groups commensurable with Γ.

Table 4a provides sufficient additional formulae to identify T_m for every $m \in M$. Several of these are consequences of our expansion and compression formulae (Section 8), while others involve the θ-functions of certain 2-dimensional lattices. The Table is self-explanatory. Some of the formulae are due to Fricke, and some to Atkin.

Table 1

1	5578077210155766091776
196883	6566555764392010419123
21296876	7226910362631220625000
842609326	10145274012943412428800
18538750076	12810005542623250817856
19360062527	19795913912408993711352
293553734298	21803647757861753437500
3879214937598	24670833602960142274950
36173193327999	31714653744947491918600
125510727015275	41209556844092914062500
190292345709543	42940402913709544921875
222879856734249	42940402913709544921875
1044868466775133	60683762052057587326065
1109944460516150	70660346341309333984375
2374124840062976	70660346341309333984375
8980616927734375	86551489469233273849000
8980616927734375	91068387388302451493925
15178147608537368	114212876389603002704448
39660520552077425	115192831837135016250000
60359800576579350	146575737439884098045700
251098487132187500	149614794149226010902528
290568421805921077	149614794149226010902528
336041615485626050	161649111002260792968750
2500435234254428856	161649111002260792968750
2986480825407204125	191259085113459945312500
3503434660075044981	191259085113459945312500
3503434660075044981	218028402153522030021875
3605718753596953125	220326476909636307378168
8456836343580310400	260799524107083767968750
8754193822112578125	260799524107083767968750
28585990950721640625	261575621299360905468750
30815545786259524745	277540481294528814140625
31569817307122699605	303379039247015811718750
47377503606648784400	331150814995116217581480
49609712911192813665	351532203382732066094400
77316619273928125000	391009081837477378329600
130415350420342968750	392611651975065600000000
155943076739182582850	433528694560596978525184
172399434201593354756	597787522207315571077947
172399434201593354756	597787522207315571077947
286243267692724486144	600020772685064502392907
286243267692724486144	626877403613887304040448
379913824694312370176	626877403613887304040448
640558364167263622626	655159231073705404921875
640558364167263622626	689763222744895005949242
643356925889917747200	689763222744895005949242
691170144025469730622	689766726179555080994223
691170144025469730622	689766726179555080994223
776097192277137500000	1037605886984697481755304
918438233727730974720	1361549126105752982272875
1201241700908448332364	1599110387863558882812500
1353006807137391674268	1662686180483865572016128
1480279477146615234375	2181694185821505680397072
1480279477146615234375	2216343020913351966796875
1768130802583126953125	2477548750555298068681032
1768130802583126953125	3282510540283631442175104
2351753641814605348320	3537292796538741415074900
2382987417506242421875	3619209050774375426792424
4567199176912486400000	4004308274823270400000000
4567199176912486400000	4239315652979009728125000

4926670174323484069683200
5334046162569208352215625
5514132424881463208443904
5514132424881463208443904
5514132424881463208443904
7118465328761788475375616
7375892500409609408203125
7567151576542452425781250
7567151576542452425781250
7850934959207940600000000
8394037047155083487634450
8874260875527017936065100
9416031858681585751556096
9479495745805305653125000
9592298143650890255171584
9592298143650890255171584
9592584386918582979657728
10023854998171489083984375
12650882100466187033706250
14930164283563048960000000
16109407269221032565630370
22626621365160537099927552
24546384719289825598186695
27501917609709102247187500
29734941419909382162874368
33684388830359981044531200
33722191327002668157047380
37310715211546624000000000
38471795739256565080575180
41738941151243953804687500
41762322738385820195625000
42001454087954515167503490
42601474860639579669896397
43527130990147981755651072
50572542024949598403750000
51324350389558097414062500
56356433273146675005489152
58437394633227526183321600
62038057486792249132974080
63750812845035828079008441
64326163427522624205703125
66550339514356152000000000
69084959008005036431224066
74612213529720383654779356
77423398454853064646282250
83974774459050335630859375
86206621680977834911875000
88943820620288343261672393
103354104243912727763091456
115165062362004433625000000
121170799240938738783416925
124058385593021471188320256
124982156072747647257292800
125517264890136048242396811
129572518017902934396764160
130287135266837289237316743
135226984222789977095703125
136107644194473772613203125
136574874874360806036041889
136574874874360806036041889

138988549876584520148320256
161561864971171113287540625
163216709667196367710937500
172248852397651745653437500
173865305251972140447265625
175867626988794162227008203
177966317773633111141787C812
198203900044423845494482560
200390867219082687273984375
203314261261157852274218750
207467089840006711558593750
212490247553365721772656250
241866941438795926688759808
258823477531055064045234375

Table 1a. Coefficients of the j-function, and head character decompositions.

degree = a_n		decomposition															
1	a_{-1},	H_{-1} =	1														
(744)	a_0,	H_0 =	?														
1 96884	a_1,	H_1 =	1														
214 93760	a_2,	H_2 =	1	1	1												
8642 99970	a_3,	H_3 =	2	2	1	1											
2 02458 56256	a_4,	H_4 =	2	3	2	1	0	1									
33 32026 40600	a_5,	H_5 =	4	5	3	2	1	1	1								
425 20233 00096	a_6,	H_6 =	4	7	5	3	1	3	1	1							
4465 69940 71935	a_7,	H_7 =	7	11	7	6	3	4	2	2	1						
40149 08866 56000	a_8,	H_8 =	8	15	12	8	4	8	4	4	1	1	0	1			
3 17644 02297 84420	a_9,	H_9 =	12	23	16	14	8	12	7	7	3	2	1	1	0	0	1
22 56739 33095 93600	a_{10}		f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
146 21191 14995 19294	a_{11}		(f_n = n'th MONSTER character.)														
874 31371 96857 75360	a_{12}																
4872 01011 17981 42520	a_{13}																
25497 82738 94105 25184	a_{14}																
1 26142 91646 57818 43075	a_{15}																
5 93121 77242 14450 58560	a_{16}																
26 62842 41315 07752 45160	a_{17}																
114 59912 78844 47865 13920	a_{18}																
474 38786 80123 41688 13250	a_{19}																
1894 49976 24889 33900 28800	a_{20}																
7318 11377 31813 75192 45696	a_{21}																
27406 30712 51362 46549 29920	a_{22}																
99710 41659 93718 26935 33820	a_{23}																
3 53074 53186 56142 70998 77376	a_{24}																

Table 2. Class list of M.

name (primepowers)	F	symbol	D	C	centraliser		
1A( )	*A*	1	2	1	$2^{46}3^{20}5^{9}7^{6}11^{2}13^{3}17.19$ $23.29.31.41.47.59.71$		
2*A*(1*A*)	*A*	2+	3	1	$2^{42}3^{13}5^{6}7^{2}11.13.17.19.23.31.47$		
2*B*(1*A*)	*B*	2−	6	2	$2^{46}3^{9}5^{4}7^{2}11.13.23$		
3*A*(1*A*)	*B*	3+	4	1	$2^{21}3^{17}5^{2}7^{3}11.13.17.23.29$		
3*B*(1*A*)	*A*	3−	8	2	$2^{14}3^{20}5^{2}7.11.13$		
3*C*(1*A*)	*C*	$3\mid 3$	6	1	$2^{15}3^{11}5^{3}7^{2}13.19.31$		
4*A*(2*B*)	*B*	4+	6	2	$2^{34}3^{7}5^{3}7.11.23$		
4*B*(2*A*)	*A*	$4\mid 2+$	6	1	$2^{27}3^{6}5^{2}7^{2}13.17$		
4*C*(2*B*)	*C*	4−	12	3	$2^{34}3^{4}5.7$		
4*D*(2*B*)	*D*	$4\mid 2-$	12	2	$2^{27}3^{3}5^{2}7.13$		
5*A*(1*A*)	*B*	5+	6	1	$2^{14}3^{6}5^{7}7.11.19$		
5*B*(1*A*)	*A*	5−	12	2	$2^{8}3^{3}5^{9}7$		
6*A*(3*A*, 2*A*)	*D*	6+	6	1	$2^{19}3^{10}5^{2}7.11.13$		
6*B*(3*B*, 2*B*)	*B*	6+6	12	2	$2^{14}3^{8}5^{2}7.11.13$		
6*C*(3*A*, 2*B*)	*E*	6+3	12	2	$2^{21}3^{8}5.7$		
6*D*(3*B*, 2*A*)	*A*	6+2	12	2	$2^{14}3^{13}5$		
6*E*(3*B*, 2*B*)	*C*	6−	24	4	$2^{14}3^{9}5$		
6*F*(3*C*, 2*B*)	*F*	$6\mid 3$	18	2	$2^{15}3^{5}5.7$		
7*A*(1*A*)	*B*	7+	8	1	$2^{10}3^{3}5^{2}7^{4}17$		
7*B*(1*A*)	*A*	7−	16	2	$2^{4}3^{2}5.7^{6}$		
8*A*(4*C*)	*D*	8+	12	2	$2^{22}3^{3}7$		
8*B*(4*A*)	*B*	$8\mid 2+$	12	2	$2^{19}3^{3}5.11$		
8*C*(4*B*)	*A*	$8\mid 4+$	12	1	$2^{14}3^{3}5^{2}13$		
8*D*(4*C*)	*E*	$8\mid 2-$	24	4	$2^{19}3^{2}5$		
8*E*(4*C*)	*C*	8−	24	4	$2^{22}3$		
8*F*(4*D*)	*F*	$8\mid 4-$	24	2	$2^{14}3^{3}7$		
9*A*(3*B*)	*A*	9+	12	2	$2^{6}3^{11}5$		
9*B*(3*B*)	*B*	9−	24	4	$2^{4}3^{11}$		
10*A*(5*A*, 2*A*)	*D*	10+	9	1	$2^{11}3^{2}5^{4}7.11$		
10*B*(5*A*, 2*B*)	*E*	10+5	18	2	$2^{14}3^{2}5^{3}$		
10*C*(5*B*, 2*A*)	*A*	10+2	18	2	$2^{8}3.5^{6}$		
10*D*(5*B*, 2*B*)	*B*	10+10	18	2	$2^{8}3^{3}5^{3}7$		
10*E*(5*B*, 2*B*)	*C*	10−	36	4	$2^{8}3.5^{4}$		
11*A*(1*A*)	*A*	11+	24	1	$2^{6}3^{3}5.11^{2}$		
12*A*(6*C*, 4*A*)	*G*	12+	12	2	$2^{15}3^{6}5$	=	1194 39360
12*B*(6*E*, 4*A*)	*B*	12+4	24	4	$2^{11}3^{7}5$	=	223 94880
12*C*(6*A*, 4*B*)	*F*	$12\mid 2+$	12	1	$2^{11}3^{5}5.7$	=	174 18240
12*D*(6*F*, 4*A*)	*I*	$12\mid 3+$	18	2	$2^{11}3^{4}7$	=	11 61216
12*E*(6*C*, 4*C*)	*H*	12+3	24	3	$2^{15}3^{3}$	=	8 84736
12*F*(6*B*, 4*D*)	*E*	$12\mid 2+6$	24	2	$2^{9}3^{3}5.7$	=	4 83840
12*G*(6*D*, 4*B*)	*A*	$12\mid 2+2$	24	2	$2^{9}3^{6}$	=	3 73248
12*H*(6*E*, 4*C*)	*C*	12+12	24	3	$2^{11}3^{3}5$	=	2 76480
12*I*(6*E*, 4*C*)	*D*	12−	48	6	$2^{10}3^{4}$	=	82944
12*J*(6*F*, 4*D*)	*J*	$12\mid 6$	36	2	$2^{9}3^{2}5$	=	23040
13*A*(1*A*)	*B*	13+	14	1	$2^{4}3^{3}13^{2}$	=	73008
13*B*(1*A*)	*A*	13−	28	2	$2^{3}3.13^{3}$	=	52728
14*A*(7*A*, 2*A*)	*B*	14+	12	1	$2^{9}3^{2}5.7^{2}$	=	11 28960
14*B*(7*A*, 2*B*)	*C*	14+7	24	2	$2^{10}3.7^{2}$	=	1 50528
14*C*(7*B*, 2*B*)	*A*	14+14	24	2	$2^{4}3^{2}5.7^{2}$	=	35280
15*A*(5*A*, 3*A*)	*D*	15+	12	1	$2^{6}3^{5}5^{2}7$	=	27 21600
15*B*(5*A*, 3*B*)	*C*	15+5	24	2	$2^{3}3^{6}5^{2}$	=	1 45800
15*C*(5*B*, 3*B*)	*A*	15+15	24	2	$2^{4}3^{3}5^{2}$	=	10800
15*D*(5*B*, 3*C*)	*B*	$15\mid 3$	36	2	$2^{3}3^{2}5^{3}$	=	9000
16*A*(8*A*)	*C*	$16\mid 2+$	24	2	$2^{12}3$	=	12288
16*B*(8*E*)	*A*	16−	48	6	2^{13}	=	8192
16*C*(8*E*)	*B*	16+	24	3	2^{13}	=	8192
17*A*(1*A*)	*A*	17+	18	1	$2^{3}3.7.17$	=	2856
18*A*(9*B*, 6*D*)	*C*	18+2	36	4	$2^{4}3^{7}$	=	34992
18*B*(9*A*,6*D*)	*A*	18+	18	2	$2^{5}3^{6}$	=	23328
18*C*(9*A*, 6*E*)	*B*	18+9	36	4	$2^{6}3^{5}$	=	15552
18*D*(9*B*, 6*E*)	*E*	18−	72	8	$2^{4}3^{5}$	=	3888
18*E*(9*B*, 6*E*)	*D*	18+18	36	4	$2^{4}3^{5}$	=	3888

name (primepowers)	F	symbol	D	C	centraliser		
19A(1A)	A	19+	20	1	$2^{2}3.5.19$	=	1140
20A(10B, 4A)	E	20+	18	2	$2^{10}3.5^{2}$	=	76800
20B(10A, 4B)	D	20|2+	18	1	$2^{7}3^{2}5^{2}$	=	28800
20C(10E, 4A)	A	20+4	36	4	$2^{6}3.5^{3}$	=	24000
20D(10B, 4D)	F	20|2+5	36	2	$2^{8}3.5^{2}$	=	19200
20E(10D, 4D)	C	20|2+10	36	2	$2^{4}3.5^{2}$	=	1200
20F(10E, 4C)	B	20+20	36	3	$2^{6}3.5$	=	960
21A(7A, 3A)	C	21+	16	1	$2^{3}3^{3}5.7^{2}$	=	52920
21B(7B, 3A)	B	21+3	32	2	$2.3^{2}7^{3}$	=	6174
21C(7A, 3C)	D	21|3+	24	1	$2^{3}3^{2}7^{2}$	=	3528
21D(7B, 3B)	A	21+21	32	2	$2^{3}3^{2}7$	=	504
22A(11A, 2A)	A	22+	18	1	$2^{4}3.5.11$	=	2640
22B(11A, 2B)	B	22+11	36	2	$2^{6}3.11$	=	2112
*23AB(1A)	AB	23+	24	1	$2^{3}3.23$	=	552
24A(12A, 8B)	F	24|2+	24	2	$2^{8}3^{3}$	=	6912
24B(12E, 8A)	G	24+	24	2	$2^{9}3^{2}$	=	4608
24C(12I, 8A)	D	24+8	48	4	$2^{7}3^{3}$	=	3456
24D(12E, 8D)	H	24|2+3	48	4	$2^{8}3^{2}$	=	2304
24E(12D, 8B)	I	24|6+	36	2	$2^{7}3^{2}$	=	1152
24F(12F, 8F)	E	24|4+6	48	2	$2^{5}3^{3}$	=	864
24G(12G, 8C)	A	24|4+2	48	2	$2^{5}3^{3}$	=	864
24H(12H, 8D)	B	24|2+12	48	4	$2^{6}3^{2}$	=	576
24I(12I, 8E)	C	24+24	48	4	$2^{7}3$	=	384
24J(12J, 8F)	J	24|12	72	2	$2^{5}3^{2}$	=	288
25A(5B)	A	25+	30	3	2.5^{3}	=	250
26A(13A, 2A)	B	26+	21	1	$2^{4}3.13$	=	624
26B(13B, 2B)	A	26+26	42	2	$2^{3}3.13$	=	312
27A(9B)	A	27+	36	3	2.3^{5}	=	486
27B(9B)	B	27+	36	3	3^{5}	=	243
28A(14A, 4B)	B	28|2+	24	1	$2^{5}3.7^{2}$	=	4704
28B(14B, 4A)	C	28+	24	2	$2^{7}3.7$	=	2688
28C(14B, 4C)	D	28+7	48	3	$2^{7}7$	=	896
28D(14C, 4D)	A	28|2+14	48	2	$2^{3}3.7$	=	168
29A(1A)	A	29+	30	1	3.29	=	87
30A(15C, 10D, 6B)	B	30+6,10,15	36	2	$2^{4}3^{3}5^{2}$	=	10800
30B(15A, 10A, 6A)	F	30+	18	1	$2^{5}3^{2}5^{2}$	=	7200
30C(15A, 10B, 6C)	G	30+3, 5, 15	36	2	$2^{6}3^{2}5$	=	2880
30D(15B, 10B, 6B)	E	30+5, 6, 30	36	2	$2^{3}3^{2}5^{2}$	=	1800
30E(15D, 10D, 6F)	D	30|3+10	54	2	$2^{3}3^{2}5$	=	360
30F(15C, 10C, 6D)	A	30+2, 15, 30	36	2	$2^{4}3.5$	=	240
30G(15C, 10E, 6E)	C	30+15	72	4	$2^{4}3.5$	=	240
*31AB(1A)	AB	31+	32	1	$2.3.31$	=	186
32A(16B)	A	32+	48	4	2^{7}	=	128
32B(16C)	B	32|2+	48	4	2^{7}	=	128
33A(11A, 3B)	B	33+11	48	2	$2.3^{3}11$	=	594
33B(11A, 3A)	A	33+	24	1	$2^{2}3^{2}11$	=	396
34A(17A, 2A)	A	34+	27	1	$2^{3}17$	=	136
35A(7A, 5A)	B	35+	24	1	$2^{2}3.5^{2}7$	=	2100
35B(7B, 5B)	A	35+35	48	2	$2.5.7$	=	70
36A(18C, 12B)	B	36+	36	4	$2^{4}3^{4}$	=	1296
36B(18D, 12B)	C	36+4	32	8	$2^{3}3^{4}$	=	648
36C(18B, 12G)	A	36|2+	36	2	$2^{3}3^{3}$	=	216
36D(18D, 12I)	D	36+36	72	6	$2^{3}3^{2}$	=	72
38A(19A, 2A)	A	38+	60	1	$2^{2}19$	=	76
39A(13A. 3A)	C	39+	28	1	$2.3^{3}13$	=	702
39B(13A, 3C)	D	39|3+	42	1	$3^{2}13$	=	117
*39CD(13B, 3B)	AB	39+39	56	2	$2.3.13$	=	78
40A(20B, 8C)	C	40|4+	36	1	$2^{4}5^{2}$	=	400
40B(20A, 8B)	D	40|2+	36	2	$2^{6}5$	=	320
*40CD(20F, 8D)	AB	40|2+20	72	4	$2^{4}5$	=	80
41A(1A)	A	41+	42	1	41	=	41

name (primepowers)	F	symbol	D	C	centraliser		
42*A*(21*A*, 14*A*, 6*A*)	*C*	42+	24	1	$2^3 3^2 7$	=	504
42*B*(21*D*, 14*C*, 6*B*)	*A*	42+6, 14, 21	48	2	$2^3 3^2 7$	=	504
42*C*(21*C*, 14*B*, 6*F*)	*D*	42|3+7	72	2	$2^3 3.7$	=	168
42*D*(21*B*, 14*C*, 6*C*)	*B*	42+3, 14, 42	48	2	$2.3^2 7$	=	126
*44*AB*(22*B*, 4*A*)	*AB*	44+	36	2	$2^5 11$	=	352
45*A*(15*B*, 9*A*)	*A*	45+	36	2	$3^3 5$	=	135
*46*AB*(23*AB*, 2*B*)	*BD*	46+23	72	2	$2^3 23$	=	184
*46*CD*(23*AB*, 2*A*)	*AC*	46+	36	1	$2^2 23$	=	92
*47*AB*(1*A*)	*AB*	47+	48	1	2.47	=	94
48*A*(24*B*, 16*A*)	*A*	48|2+	48	2	$2^5 3$	=	96
50*A*(25*A*, 10*C*)	*A*	50+	45	3	2.5^2	=	50
51*A*(17*A*, 3*A*)	*A*	51+	36	1	3.17	=	51
52*A*(26*A*, 4*B*)	*B*	52|2+	42	1	$2^3 13$	=	104
52*B*(26*B*, 4*D*)	*A*	52|2+26	84	2	$2^2 13$	=	52
54*A*(27*A*, 18*A*)	*A*	54+	54	3	2.3^3	=	54
55*A*(11*A*, 5*A*)	*A*	55+	36	1	$2.5.11$	=	110
56*A*(28*C*, 8*A*)	*C*	56+	48	2	$2^4 7$	=	112
*56*BC*(28*D*, 8*F*)	*AB*	56|4+14	96	2	$2^3 7$	=	56
57*A*(19*A*, 3*C*)	*A*	57|3+	60	1	3.19	=	57
*59*AB*(1*A*)	*AB*	59+	60	1	59	=	59
60*A*(30*B*, 20*B*, 12*C*)	*D*	60|2+	36	1	$2^3 3^2 5$	=	360
60*B*(30*C*, 20*A*, 12*A*)	*E*	60+	36	2	$2^4 3.5$	=	240
60*C*(30*G*, 20*C*, 12*B*)	*A*	60+4, 15, 60	72	4	$2^3 3.5$	=	120
60*D*(30*G*, 20*F*, 12*H*)	*B*	60+12, 15, 20	72	3	$2^3 3.5$	=	120
60*E*(30*D*, 20*D*, 12*F*)	*F*	60|2+5, 6, 30	72	2	$2^2 3.5$	=	60
60*F*(30*E*, 20*E*, 12*J*)	*C*	60|6+10	108	2	$2^2 3.5$	=	60
*62*AB*(31*AB*, 2*A*)	*AB*	62+	48	1	2.31	=	62
66*A*(33*B*, 22*A*, 6*A*)	*A*	66+	36	1	$2^2 3.11$	=	132
66*B*(33*A*, 22*B*, 6*B*)	*B*	66+6, 11, 66	72	2	$2.3.11$	=	66
68*A*(34*A*, 4*B*)	*A*	68|2+	54	1	$2^2 17$	=	68
*69*AB*(23*AB*, 3*A*)	*AB*	69+	48	1	3.23	=	69
70*A*(35*A*, 14*A*, 10*A*)	*B*	70+	36	1	$2^2 5.7$	=	140
70*B*(35*B*, 14*C*, 10*D*)	*A*	70+10, 14, 35	72	2	$2.5.7$	=	70
*71*AB*(1*A*)	*AB*	71+	72	1	71	=	71
78*A*(39*A*, 26*A*, 6*A*)	*C*	78+	42	1	$2.3.13$	=	78
*78*BC*(39*CD*, 26*B*. 6*B*)	*AB*	78+6, 26, 39	84	2	$2.3.13$	=	78
84*A*(42*A*, 28*A*, 12*C*)	*B*	84|2+	48	1	$2^2 3.7$	=	84
84*B*(42*B*, 28*D*, 12*F*)	*A*	84|2+6, 14, 21	96	2	$2^2 3.7$	=	84
84*C*(42*C*, 28*B*, 12*D*)	*C*	84|3+	72	2	$2^2 3.7$	=	84
*87*AB*(29*A*, 3*A*)	*AB*	87+	60	1	3.29	=	87
*88*AB*(44*BA*, 8*B*)	*AB*	88|2+	72	2	$2^3 11$	=	88
*92*AB*(46*AB*, 4*A*)	*AB*	92+	72	2	$2^2 23$	=	92
*93*AB*(31*BA*, 3*C*)	*AB*	93|3+	96	1	3.31	=	93
*94*AB*(47*AB*, 2*A*)	*AB*	94+	72	1	2.47	=	94
*95*AB*(19*A*, 5*A*)	*AB*	95+	60	1	5.19	=	95
*104*AB*(52*A*, 8*C*)	*AB*	104|4+	84	1	$2^3 13$	=	104
105*A*(35*A*, 21*A*, 15*A*)	*A*	105+	48	1	$3.5.7$	=	105
110*A*(55*A*, 22*A*, 10*A*)	*A*	110+	54	1	$2.5.11$	=	110
*119*AB*(17*A*, 7*A*)	*AB*	119+	72	1	7.17	=	119
25*Z*(5*B*)	?	25−	60	6	???	=	???
49*Z*(7*B*)	?	49+	56	4	???	=	???
50*Z*(25*Z*, 10*E*)	?	50+50	90	6	???	=	???

Table 2a. Additional information for small order elements.

	centraliser structure	and order	class
(MONSTER)		8080 17424 79451 28758 86459 90496 17107 57005 75436 80000 00000	1*A*
(BABY)	2.B	8305 96296 24528 52382 35516 10880 00000	2*A*
(Conway)	$2^{1+24}C_1$	13 95118 39126 33632 81715 20000	2*B*
(Fischer)	$3.F'_{24}$	37656 17127 57198 51638 78400	3*A*
(Suzuki)	$3^{1+12}.2.Sz$	1429 61507 75402 49600	3*B*
(Thompson, Smith)	$3\times E$	272 23783 16636 16000	3*C*
(Conway)	$4.2^{22}.C_3$	8317 58427 33096 96000	4*A*
	$\{4\times F_4(2)\}.2$	26 48901 28269 31200	4*B*
	$4.2^{15}.2^8.S_6(2)$	4870 49291 36640	4*C*
	$4.2^{12}.G_2(4).2$	824 43239 42400	4*D*
(Harada, Norton)	$5\times F$	1 36515 45600 00000	5*A*
(Hall, Janko)	$5^{1+6}.2.HJ$	9 45000 00000	5*B*
(Fischer)	$3\times 2.F_{22}.2$	77474 10198 52800	6*A*
(Suzuki)	$6.Sz$	269 00729 85600	6*B*
	$2^{1+12}.3^2.U_4(3).2$	48 15794 99520	6*C*
	$2.3^{1+8}.2^{1+6}.U_4(2)$	13 06069 40160	6*D*
	$2.3^{1+4}.2^{1+6}.U_4(2)$	16124 31360	6*E*
	$3\times 2^{1+8}.A_9$	2786 91840	6*F*
(Held)	$7\times H$	2 82127 10400	7*A*
	$7^{1+4}.2.A_7$	847 07280	7*B*
	$8.2^7.2^6.U_3(3).2$	7927 23456	8*A*
(Mathieu)	$8.2^{10}.M_{12}$	7785 67680	8*B*
(Tits)	$8\times {}^2F'_4(2)$	1437 69600	8*C*
	$8.2^9.2^4.A_6$	235 92960	8*D*
	$[2^{22}3]$	125 82912	8*E*
	$8.2^6.U_3(3)$	30 96576	8*F*
	$9.3^{1+4}.S_4(3)$	566 87040	9*A*
	$[2^4 3^{11}]$	28 34352	9*B*
(Higman, Sims)	$5\times 2.HS.2$	8870 40000	10*A*
	$5\times 2^{1+8}.(A_5\times A_5).2$	184 32000	10*B*
	$2.5^{1+4}.2^{1+4}.A_5$	120 00000	10*C*
(Hall, Janko)	$5\times 2.HJ$	60 48000	10*D*
	$2.5^{1+2}.2^{1+4}.A_5$	4 80000	10*E*
(Mathieu)	$11\times M_{12}$	10 45440	11*A*

Table 3. Relations between the classes

Symbol	Formula	Harmonies and Symmetrisations
1	$j = T+744$	$1A$, $3C$
2−	$1^{24}/2^{24}$	$2B(4096:2A)$, $4D(64:4B)$, $6F$, $8F$, $12J$, $24J$
2+	$T+104$	$2A$, $4B$, $8C$
3−	$1^{12}/3^{12}$	$3B(729:3A)$
4−	$1^{8}/4^{8}$	$4C(256:4A)$, $8D(16:8B)$
4+	$2^{48}/1^{24}4^{24}$	$4A$, $8B$, $12D$, $24E$
5−	$1^{6}/5^{6}$	$5B(125:5A)$, $15D$
6−	$2^{3}3^{9}/1^{3}6^{9}$	$6E(-8:6D)$
	$2^{8}3^{4}/1^{4}6^{8}$	$6E(9:6C)$
	$1^{5}3/2.6^{5}$	$6E(72:6B)$
6+2	$1^{4}2^{4}/3^{4}6^{4}$	$6D(81:6A)$, $12G(9:12C)$, $24G$
6+3	$1^{6}3^{6}/2^{6}6^{6}$	$6C(64:6A)$
6+6	$2^{12}3^{12}/1^{12}6^{12}$	$6B(1:6A)$, $12F(1:12C)$, $24F$
6+	$T+14$	$6A$, $12C$
7−	$1^{4}/7^{4}$	$7B(49:7A)$
7+	$T+9$	$7A$, $21C$
8−	$1^{4}4^{2}/2^{2}8^{4}$	$8E(32:8A)$; $4^{12}/2^{4}8^{8}$, $2^{10}/1^{4}4^{2}8^{4}$
8+	$2^{8}4^{8}/1^{8}8^{8}$	$8A$, $16A$
9−	$1^{3}/9^{3}$	$9B(27:9A)$
9+	$3^{12}/1^{6}9^{6}$	$9A$
10−	$2.5^{5}/1.10^{5}$	$10E(-4:10C)$
	$2^{4}5^{2}/1^{2}10^{4}$	$10E(5:10B)$
	$1^{3}5/2.10^{3}$	$10E(20:10D)$
10+2	$1^{2}2^{2}/5^{2}10^{2}$	$10C(25:10A)$
10+5	$1^{4}5^{4}/2^{4}10^{4}$	$10B(16:10A)$, $20D(4:20B)$
10+10	$2^{6}5^{6}/1^{6}10^{6}$	$10D(1:10A)$, $20E(-1:20B)$, $30E$, $60F$
10+	$T+4$	$10A$, $20B$, $40A$
12−	$4^{4}6^{2}/2^{2}12^{4}$	$12I(-3:12E)$; $1.4^{2}6^{9}/2^{3}3^{3}12^{6}$
	$3^{3}4/1.12^{3}$	$12I(4:12B)$; $2^{7}3/1^{3}4^{2}6.12^{2}$
	$1^{3}4.6^{2}/2^{2}3.12^{3}$	$12I(12:12H)$
12+3	$1^{2}3^{2}/4^{2}12^{2}$	$12E(16:12A)$, $24D(4:24A)$; $2^{6}6^{6}/1^{2}3^{2}4^{4}12^{4}$
12+4	$1^{4}4^{4}6^{4}/2^{4}3^{4}12^{4}$	$12B(9:12A)$; $2^{14}/1^{5}3.4^{5}6^{2}12$, $1^{3}4^{3}6^{18}/2^{6}3^{9}12^{9}$
12+12	$3^{4}4^{4}/1^{4}12^{4}$	$12H(1:12A)$, $24H(1:24A)$; $2^{6}6^{6}/1^{5}3.4.12^{5}$
12+	$2^{12}6^{12}/1^{6}3^{6}4^{6}12^{6}$	$12A$, $24A$
13−	$1^{2}/13^{2}$	$13B(13:13A)$
13+	$T+3$	$13A$, $39B$
14+7	$1^{3}7^{3}/2^{3}14^{3}$	$14B(8:14A)$, $42C$
14+14	$2^{4}7^{4}/1^{4}14^{4}$	$14C(1:14A)$, $28D(1:28A)$, $56BC$
14+	$T+6$	$14A$, $28A$
15+5	$1^{2}5^{2}/3^{2}15^{2}$	$15B(9:15A)$
15+15	$3^{3}5^{3}/1^{3}15^{3}$	$15C(1:15A)$
16−	$1^{2}8/2.16^{2}$	$16B(8:16C)$; $8^{6}/4^{2}16^{4}$, $2^{5}8/1^{2}4^{2}16^{2}$
16+	$2^{6}8^{6}/1^{4}4^{4}16^{4}$	$16C$, $32B$; $4^{10}/1^{2}2^{3}8^{3}16^{2}$
18−	$6.9^{3}/3.18^{3}$	$18D(-2:18A)$
	$2^{2}9/1.18^{2}$	$18D(3:18C)$
	$1^{2}6.9/2.3.18^{2}$	$18D(6:18E)$
18+2	$1.2/9.18$	$18A(9:18B)$
18+9	$1^{3}6^{2}9^{3}/2^{3}3^{2}18^{3}$	$18C(4:18B)$; $1^{2}6^{8}9^{2}/2^{4}3^{4}18^{4}$, $3^{6}/1.2.6^{2}9.18$
18+18	$2^{3}9^{3}/1^{3}18^{3}$	$18E(1:18B)$
18+	$3^{4}6^{4}/1^{2}2^{2}9^{2}18^{2}$	$18B$, $36C$
19+	$T+3$	$19A$, $57A$
20+4	$1^{2}4^{2}10^{2}/2^{2}5^{2}20^{2}$	$20C(5:20A)$; $1.4.10^{10}/2^{2}5^{5}20^{5}$, $2^{8}/1^{3}4^{3}5.20$
20+20	$4^{2}5^{2}/1^{2}20^{2}$	$20F(1:20A)$, $40CD(1:40B)$; $2^{4}10^{4}/1^{3}4.5.20^{3}$
20+	$2^{8}10^{8}/1^{4}4^{4}5^{4}20^{4}$	$20A$, $40B$
21+3	$1.3/7.21$	$21B(7:21A)$
21+21	$3^{2}7^{2}/1^{2}21^{2}$	$21D(1:21A)$
22+11	$1^{2}11^{2}/2^{2}22^{2}$	$22B(4:22A)$
24+8	$1^{2}6.8^{2}12/2.3^{2}4.24^{2}$	$24C(3:24B)$; $2^{2}4^{2}/1.3.8.24$, $1.6^{3}8.12^{3}/2.3^{3}4.24^{3}$
24+24	$2.3^{2}8^{2}12/1^{2}4.6.24^{2}$	$24I(-1:24B)$; $4^{4}6^{4}/1.2^{2}3.8.12^{2}24$, $2^{5}3.8.12^{5}/1^{3}4^{3}6^{3}24^{3}$
24+	$2^{2}4^{2}6^{2}12^{2}/1^{2}3^{2}8^{2}24^{2}$	$24B$, $48A$

Symbol	Formula	Harmonies and Symmetrisations
25 –	1/25	25*Z*(5 : 25*A*)
26 + 26	$2^2 13^2/1^2 26^2$	26*B*(1 : 26*A*), 52*B*(– 1 : 52*A*)
26 +	*T*	26*A*, 52*A*, 104*AB*
28 + 7	1 . 7/4 . 28	28*C*(4 : 28*B*); $2^3 14^3/1 . 4^2 7 . 28^2$
28 +	$2^6 14^6/1^3 4^3 7^3 28^3$	28*B*, 84*C*
30 + 15	3 . 5/2 . 30	30*G*(2 : 30*A*)
	$1 . 6^2 10^2 15/2^2 3 . 5 . 30^2$	30*G*(– 1 : 30*C*)
	$1^2 6 . 10 . 15^2/2^2 3 . 5 . 30^2$	30*G*(2 : 30*F*)
30 + 6, 10, 15	$1^3 6^3 10^3 15^3/2^3 3^3 5^3 30^3$	30*A*(1 : 30*B*)
30 + 3, 5, 15	1 . 3 . 5 . 15/2 . 6 . 10 . 30	30*C*(4 : 30*B*)
30 + 2, 15, 30	3 . 5 . 6 . 10/1 . 2 . 15 . 30	30*F*(1 : 30*B*)
30 +	*T* + 4	30*B*, 60*A*
30 + 5, 6, 30	$2^2 3^2 10^2 15^2/1^2 5^2 6^2 30^2$	30*D*(1 : 30*B*), 60*E*(1 : 60*A*)
31 +	*T*	31*AB*, 93*AB*
32 +	$2^3 16^3/1^2 4 . 8 . 32^2$	32*A*
33 + 11	1 . 11/3 . 33	33*A*(3 : 33*B*)
34 +	*T* + 2	34*A*, 68*A*
35 + 35	5 . 7/1 . 35	35*B*(– 1 : 35*A*)
36 + 4	1 . 4 . 18/2 . 9 . 36	36*B*(3 : 36*A*); $3 . 12 . 18^6/6^2 9^3 36^3$,
		$2^5 3 . 12 . 18/1^2 4^2 6^2 9 . 36$
36 + 36	4 . 9/1 . 36	36*D*(1 : 36*A*); $4 . 6^8 9/2^2 3^3 12^3 18^2$,
		$2^3 3 . 12 . 18^3/1^2 4 . 6^2 9 . 36^2$
36 +	$1 . 4 . 6^{16} 9 . 36/2^4 3^6 12^6 18^4$	36*A*; $2^6 3^2 12^2 18^6/1^3 4^3 6^4 9^3 36^3$,
		$2^2 3^4 12^4 18^2/1^2 4^2 6^4 9^2 36^2$
39 + 39	3 . 13/1 . 39	39*CD*(1 : 39*A*)
42 + 6, 14, 21	$1^2 6^2 14^2 21^2/2^2 3^2 7^2 42^2$	42*B*(1 : 42*A*), 84*B*(1 : 84*A*)
42 + 3, 14, 42	2 . 6 . 7 . 21/1 . 3 . 14 . 42	42*D*(1 : 42*A*)
42 +	*T* – 2	42*A*, 84*A*
44 +	$2^4 22^4/1^2 4^2 11^2 44^2$	44*AB*, 88*AB*
45 +	$3^2 15^2/1 . 5 . 9 . 45$	45*A*
46 + 23	1 . 23/2 . 46	46*AB*(2 : 46*CD*)
50 + 50	2 . 25/1 . 50	50*Z*(1 : 50*A*)
56 +	2 . 4 . 14 . 28/1 . 7 . 8 . 56	56*A*
60 + 12, 15, 20	1 . 12 . 15 . 20/3 . 4 . 5 . 60	60*D*(1 : 60*B*); 2 . 6 . 10 . 30/3 . 4 . 5 . 60
60 + 4, 15, 60	2 . 3 . 5 . 12 . 20 . 30/1 . 4 . 6 . 10 . 15 . 60	60*C*(– 1 : 60*B*); $6^3 10^3/2 . 3 . 5 . 12 . 20 . 30$,
		$2^4 3 . 5 . 12 . 20 . 30^4/1^2 4^2 6^2 10^2 15^2 60^2$
60 +	$2^2 6^2 10^2 30^2/1 . 3 . 4 . 5 . 12 . 15 . 20 . 60$	60*B*
66 + 6, 11, 66	2 . 3 . 22 . 33/1 . 6 . 11 . 66	66*B*(1 : 66*A*)
70 + 10, 14, 35	1 . 10 . 14 . 35/2 . 5 . 7 . 70	70*B*(1 : 70*A*)
78 + 6, 26, 39	1 . 6 . 26 . 39/2 . 3 . 13 . 78	78*BC*(1 : 78A)
92 +	$2^2 46^2/1 . 4 . 23 . 92$	92*AB*
2 –	*T* + 40	2*B*, 4*C*
3 –	*T* + 15	3*B*, 9*B*
4 –	$2^{24}/1^8 4^{16}$	4*C*, 8*E*, 16*B*
6 –	$2^8 3^4/1^4 6^8$	6*E*(9 : 6*C*), 12*I*(– 3 : 12*E*)
	$2^3 3^9/1^3 6^9$	6*E*(– 8 : 6*D*), 18*D*(– 2 : 18.4)
6 + 3	*T* – 2	6*C*, 12*E*
12 + 4	$1^3 4^3 6^{18}/2^6 3^9 12^9$	12*B*, 36*B*
14 + 7	*T* – 2	14*B*, 28*C*

Table 3a. Additional product formulae.

class	formulae	
$8D$	$(1\tfrac{1}{4})^4 2^2/4^4 8^2$	
$9B$	$(1\tfrac{1}{3})^3/9^3$	
$16B$	$(1\tfrac{1}{4})^2 2.8^2/4^3 16^2$	
$18A$	$(1\tfrac{1}{3})^3(2\tfrac{1}{3})^3 1.2/3^4 6^4$	
$18D$	$(1\tfrac{1}{3})^2 6.9/(2\tfrac{2}{3})3.18$	
$18D$	$(2\tfrac{2}{3})^2 9/(1\tfrac{1}{3})18^2$	
$18E$	$(1\tfrac{1}{3})(2\tfrac{2}{3})2.9/1^2 18^2$	
$24D$	$(1\tfrac{1}{4})^2(3\tfrac{1}{4})^2 2^2 6^2 8.24/4^5 12^5$	
$24H$	$(1\tfrac{1}{4})(3\tfrac{3}{4})6.8/2.4.12.24$	
$25A$	$(1\tfrac{1}{5})(1\tfrac{4}{5})/1.25$	
$27AB$	$(1\tfrac{1}{3})(3\tfrac{2}{3})/1.27$	
$32A$	$(1\tfrac{1}{4})(2\tfrac{3}{4})2.16/1.4.8.32$	
$32B$	$(1\tfrac{1}{8})(1\tfrac{7}{8})4^2 16^2/2.8^4 32$	
$36B$	$(1\tfrac{1}{3})(4\tfrac{1}{3})18/(2\tfrac{2}{3})9.36$	
$36B$	$(2\tfrac{2}{3})^5 3.12.18/(1\tfrac{1}{3})^2(4\tfrac{1}{3})^2 6^2 9.36$	
$36D$	$(1\tfrac{1}{3})(4\tfrac{2}{3})2.18/1.6^2 36$	
$40CD$	$(1\tfrac{1}{4})(5\tfrac{3}{4})8.10/4^2 20^2$	
$50A$	$(1\tfrac{1}{5})(1\tfrac{4}{5})(2\tfrac{1}{5})(2\tfrac{4}{5})/5^2 10^2$	
$54A$	$(1\tfrac{1}{3})(2\tfrac{1}{3})(3\tfrac{2}{3})(6\tfrac{2}{3})/3.6.9.18$	
$25Z$	$(1\tfrac{1}{5})(1\tfrac{2}{5})/(1\tfrac{3}{5})25$	(4 forms)
$49Z$	$(1\tfrac{2}{7})(1\tfrac{3}{7})(1\tfrac{4}{7})(1\tfrac{5}{7})/7^4$	(3 forms)
$50Z$	$5^2 10^2/(1\tfrac{1}{5})(2\tfrac{4}{5})1.50$	(4 forms)

All formulae except those for $25Z$, $49Z$, $50Z$ are valid up to a constant factor when $\left(N\,\frac{a}{b}\right)$ is interpreted as $\eta\left(Nz+\frac{a}{b}\right)$. *All* formulae are valid under the interpretation

$$\left(N\,\frac{a}{b}\right) = q^{N/24}\prod_{n=1}^{\infty}(1-u^{\bar{n}}q^{nN})$$

where

$$u = e^{2\pi i a/b}$$

and

$$n\bar{n} \equiv 1 \pmod b \text{ if } (n,b)=1\,,$$
$$\bar{n} \equiv n \pmod b \text{ if } (n,b)\neq 1\,.$$

All formulae except those for $25Z$, $49Z$, $50Z$ have two algebraically conjugate forms, while these cases yield the numbers indicated. The conjugate forms can be found by applying the permutations

$$\left(\tfrac{1}{3}\;\tfrac{2}{3}\right),\ \left(\tfrac{1}{4}\;\tfrac{3}{4}\right),\ \left(\tfrac{1}{5}\;\tfrac{2}{5}\;\tfrac{4}{5}\;\tfrac{3}{5}\right),$$
$$\left(\tfrac{1}{7}\;\tfrac{3}{7}\;\tfrac{2}{7}\;\tfrac{6}{7}\;\tfrac{4}{7}\;\tfrac{5}{7}\right),\ \left(\tfrac{1}{8}\;\tfrac{3}{8}\right)\left(\tfrac{5}{8}\;\tfrac{7}{8}\right).$$

There are some useful transformations:—

$$(N\tfrac{1}{2}) = (2N)^3/(N)(4N)$$
$$(N\tfrac{1}{3})(N\tfrac{2}{3}) = (3N)^4/(N)(9N)$$
$$(N\tfrac{1}{4})(N\tfrac{3}{4}) = (4N)^8/(2N)^3(8N)^3$$
$$(N\tfrac{1}{5})(N\tfrac{2}{5})(N\tfrac{3}{5})(N\tfrac{4}{5}) = (5N)^6/(N)(25N)$$
$$(N\tfrac{1}{7})(N\tfrac{2}{7})(N\tfrac{3}{7})(N\tfrac{4}{7})(N\tfrac{5}{7})(N\tfrac{6}{7}) = (7N)^8/(N)(49N)$$
$$(N\tfrac{1}{8})(N\tfrac{3}{8})(N\tfrac{5}{8})(N\tfrac{7}{8}) = (8N)^{18}/(4N)^7(16N)^7$$

Table 4. Values of head characters.

1A	2A	2B	3A	3B	3C
1	1	1	1	1	1
24	8	−8	6	−3	0
196884	4372	276	783	54	0
21493760	96256	−2048	8672	−76	248
864299970	1240002	11202	65367	−243	0
20245856256	10698752	−49152	371520	1188	0
333202640600	74428120	184024	1741655	−1384	4124
4252023300096	431529984	−614400	7161696	−2916	0
44656994071935	2206741887	1881471	26567946	11934	0
401490886656000	10117578752	−5373952	90521472	−11580	34752
3176440229784420	42616961892	14478180	288078201	−21870	0
22567393309593600	1665[illegible]4106240	−37122048	864924480	79704	0

4A	4B	4C	4D	5A	5B	6A	6B	6C	6D	6E	6F
1	1	1	1	1	1	1	1	1	1	1	1
8	0	0	0	4	−1	2	7	−2	−1	1	0
276	52	20	−12	134	9	79	78	15	−2	6	0
2048	0	0	0	760	10	352	364	−32	28	4	−8
11202	834	−62	66	3345	−30	1431	1365	87	−27	−3	0
49152	0	0	0	12256	6	4160	4380	−192	−52	−12	0
184024	4760	216	−232	39350	−25	13015	12520	343	136	−8	28
614400	0	0	0	114096	96	31968	32772	−672	−108	12	0
1881471	24703	−641	639	307060	60	81162	80094	1290	−162	30	0
5373952	0	0	0	776000	−250	183680	185276	−2176	620	20	−64
14478180	94980	1636	−1596	1867170	45	412857	409578	3705	−486	−30	0
37122048	0	0	0	4298600	−150	864320	871272	−6336	−760	−72	0

7A	7B	8A	8B	8C	8D	8E	8F	9A	9B
1	1	1	1	1	1	1	1	1	1
3	$-\frac{1}{2}$	4	0	0	0	0	0	3	0
51	2	36	12	0	−4	4	0	27	0
204	8	128	0	0	0	0	0	86	5
681	−5	386	66	26	2	2	−6	243	0
1956	−4	1024	0	0	0	0	0	594	0
5135	−10	2488	232	0	8	−8	0	1370	−7
12360	12	5632	0	0	0	0	0	2916	0
28119	−7	12031	639	79	−1	−1	15	5967	0
60572	8	24576	0	0	0	0	0	11586	3
125682	46	48308	1596	0	−20	20	0	21870	0
251040	−36	91904	0	0	0	0	0	39852	0

10A	10B	10C	10D	10E	11A	12A	12B	12C	12D	12E	12F	12G	12H	12I	12J
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	$\frac{11}{3}$	$\frac{1}{3}$	2	2	−1	0	0	0	0	0	3	0	0
22	6	−3	21	1	17	15	6	7	0	−1	6	−2	14	2	0
56	−8	6	62	2	46	32	−4	0	8	0	0	0	36	0	0
177	17	2	162	2	116	87	−3	15	0	7	21	−3	85	1	0
352	−32	2	378	−2	252	192	12	0	0	0	0	0	180	0	0
870	54	−5	819	−1	533	343	−8	71	28	−9	56	8	360	0	−4
1584	−80	−16	1680	0	1034	672	−12	0	0	0	0	0	684	0	0
3412	116	12	3276	−4	1961	1290	30	106	0	10	126	−2	1246	−2	0
5952	−192	2	6138	−2	3540	2176	−20	0	64	0	0	0	2196	0	0
11442	290	17	11145	5	6253	3705	−30	273	0	−23	258	−6	3754	−2	0
19240	−408	−10	19662	2	10654	6336	72	0	0	0	0	0	6264	0	0

13A	13B	14A	14B	14C	15A	15B	15C	15D	16A	16B	16C	17A
1	1	1	1	1	1	1	1	1	1	1	1	1
12	$-\frac{1}{7}$	1	-1	$\frac{5}{2}$	1	$-\frac{1}{2}$	2	0	0	0	2	$\frac{4}{3}$
12	-1	11	3	10	8	-1	9	0	4	0	8	7
28	2	20	-4	24	22	4	19	-2	0	0	16	14
66	1	57	9	51	42	-3	42	0	10	2	34	29
132	2	92	-12	100	70	-2	78	0	0	0	64	50
258	-2	207	15	190	155	11	146	-1	24	0	112	92
468	0	312	-24	340	246	-6	249	0	0	0	192	148
843	-2	623	39	585	421	-11	429	0	47	-1	319	246
1428	-2	932	-52	984	722	20	695	2	0	0	512	386
2406	1	1674	66	1606	1101	-15	1125	0	84	0	808	603
3900	0	2464	-96	2564	1730	-16	1749	0	0	0	1248	904

18A	18B	18C	18D	18E	19A	20A	20B	20C	20D	20E	20F
1	1	1	1	1	1	1	1	1	1	1	1
0	1	-1	0	2	$\frac{6}{5}$	$\frac{4}{3}$	0	$-\frac{1}{3}$	0	0	$\frac{5}{3}$
-2	7	3	0	6	6	6	2	1	-2	3	5
1	10	-2	1	13	10	8	0	-2	0	0	10
0	27	3	0	24	21	17	9	2	1	6	18
2	38	-6	0	42	36	32	0	2	0	0	30
1	82	10	1	73	61	54	10	-1	-2	13	51
0	108	-12	0	120	96	80	0	0	0	0	80
0	207	15	0	192	156	116	28	-4	4	24	124
-1	278	-22	-1	299	232	192	0	2	0	0	190
0	486	30	0	456	357	290	30	5	-6	39	281
-4	644	-36	0	684	522	408	0	-2	0	0	410

21A	21B	21C	21D	22A	22B	23A	24A	24B	24C	24D	24E	24F	24G	24H	24I	24J
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\frac{3}{4}$	$-\frac{1}{8}$	0	$\frac{11}{8}$	$\frac{2}{3}$	$-\frac{2}{3}$	1	0	1	$-\frac{1}{2}$	0	0	0	0	0	$\frac{3}{2}$	0
6	-1	0	5	5	1	4	3	3	0	-1	0	0	0	2	4	0
6	-1	3	8	6	-2	7	0	8	2	0	0	0	0	0	6	0
15	1	0	16	16	4	13	3	11	-1	-1	0	3	-1	5	11	0
30	2	0	26	20	-4	19	0	16	-2	0	0	0	0	0	18	0
41	-1	8	44	41	5	33	7	31	4	-1	4	0	0	8	28	0
66	3	0	66	50	-6	47	0	40	-2	0	0	0	0	0	42	0
111	-1	0	104	97	9	74	18	58	-2	2	0	6	-2	14	62	0
146	-1	11	152	116	-12	106	0	96	6	0	0	0	0	0	90	0
222	-2	0	229	197	13	154	21	125	-4	1	0	0	0	22	128	0
336	0	0	324	246	-18	214	0	176	-4	0	0	0	0	0	180	0

25A	26A	26B	27A	27B	28A	28B	28C	28D	29A	30A	30B	30C	30D	30E	30F	30G
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	$\frac{4}{7}$	$\frac{9}{7}$	1	1	0	1	0	0	$\frac{4}{5}$	$-\frac{4}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{7}{6}$	0	$\frac{2}{3}$	$-\frac{2}{3}$
4	4	3	3	3	3	3	-1	2	3	3	4	0	3	0	3	1
5	4	6	5	5	0	4	0	0	4	-1	2	-2	4	2	3	-1
10	10	9	9	9	1	9	1	3	7	0	6	2	5	0	8	2
16	12	14	12	12	0	12	0	0	10	0	10	-2	10	0	8	-2
25	26	22	20	20	7	15	-1	6	17	0	15	3	15	3	16	2
36	28	32	27	27	0	24	0	0	22	-3	18	-2	22	0	17	-3
55	51	46	42	42	7	39	3	9	32	9	37	5	29	0	33	5
75	60	66	57	57	0	52	0	0	44	-9	30	-6	36	6	35	-5
110	102	93	81	81	18	66	-2	14	62	3	57	5	53	0	59	5
150	116	128	108	108	0	96	0	0	80	-3	70	-6	72	0	65	-7

31A	32A	32B	33A	33B	34A	35A	35B	36A	36B	36C	36D	38A	39A	39B	39C
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\frac{3}{4}$	1	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{4}{9}$	$\frac{1}{2}$	$\frac{3}{4}$	1	0	0	1	$\frac{2}{5}$	$\frac{3}{7}$	0	$\frac{5}{7}$
3	2	2	−1	2	3	1	2	3	0	1	2	2	3	0	2
3	4	0	1	4	2	4	3	2	−1	0	3	2	1	1	2
6	6	2	−1	5	5	6	5	3	0	3	4	5	3	0	4
9	8	0	0	6	6	6	6	6	0	0	6	4	6	0	5
13	12	4	2	14	12	10	10	10	1	2	9	9	6	3	7
18	16	0	−1	14	12	10	12	12	0	0	12	8	9	0	9
27	23	7	−1	20	22	19	18	15	0	7	16	16	15	0	13
34	32	0	3	30	22	22	23	22	1	0	21	16	15	3	16
48	42	10	−2	37	39	32	31	30	0	6	28	25	21	0	22
63	56	0	−2	46	40	40	39	36	0	0	36	26	30	0	27

40A	40B	40C	41A	42A	42B	42C	42D	44A	45A	46A	46C	47A	48A
1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	$\frac{4}{7}$	$\frac{1}{4}$	$-\frac{7}{8}$	0	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	0
0	2	1	2	2	1	0	1	1	2	0	2	1	1
0	0	0	2	2	0	−1	3	2	1	−1	1	2	0
1	1	2	3	3	0	0	3	4	3	1	3	3	1
0	0	0	4	2	−2	0	4	4	4	−1	3	3	0
0	2	3	7	9	4	0	7	5	5	1	5	5	3
0	0	0	8	6	−2	0	7	6	6	−1	5	5	0
4	4	4	11	11	0	0	9	9	7	2	10	8	2
0	0	0	14	14	0	−1	15	12	11	−2	8	9	0
0	6	5	19	18	1	0	16	13	15	2	14	12	3
0	0	0	22	16	−4	0	20	18	17	−2	14	14	0

50A	51A	52A	52B	54A	55A	56A	56B	57A	59A
1	1	1	1	1	1	1	1	1	1
$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	$\frac{2}{5}$
2	1	0	1	1	2	1	0	0	1
1	2	0	0	1	1	2	0	1	1
2	2	2	1	3	1	1	1	0	2
2	2	0	0	2	2	2	0	0	2
5	5	2	2	4	3	3	0	1	3
4	4	0	0	3	4	4	0	0	3
7	6	3	2	6	6	5	1	0	4
7	8	0	0	5	5	6	0	1	5
12	9	2	3	9	8	8	0	0	6
10	10	0	0	8	9	8	0	0	7

60A	60B	60C	60D	60E	60F	62A	66A	66B	68A	69A
1	1	1	1	1	1	1	1	1	1	1
0	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{7}{12}$	0	$\frac{1}{4}$
2	0	1	−1	1	0	1	2	1	1	1
0	2	1	1	0	0	1	0	1	0	1
0	2	2	0	1	0	2	1	1	1	1
0	2	2	0	0	0	1	2	2	0	1
1	3	2	0	1	1	3	2	2	0	3
0	2	3	−1	0	0	2	2	3	0	2
1	5	5	1	1	0	5	4	3	2	2
0	6	5	1	0	0	4	2	3	0	4
3	5	5	−1	3	0	6	5	4	1	4
0	6	7	−1	0	0	5	6	6	0	4

70A	70B	71A	78A	78B	84A	84B	84C	87A	88A	92A	93A	94A	95A
1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\frac{1}{6}$	$-\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{7}$	$-\frac{3}{7}$	0	0	0	$\frac{1}{5}$	0	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{1}{5}$
1	0	1	1	0	0	−1	0	0	1	0	0	1	1
0	−1	1	1	0	0	0	1	1	0	1	0	0	0
2	1	1	1	0	1	0	0	1	0	1	0	1	1
2	0	1	0	−1	0	0	0	1	0	1	0	1	1
2	0	2	2	1	1	0	0	2	1	1	1	1	1
2	0	2	1	−1	0	0	0	1	0	1	0	1	1
3	0	3	3	1	1	0	0	2	1	2	0	2	1
2	−1	3	3	0	0	0	1	2	0	2	1	1	2
4	1	4	3	0	0	−1	0	2	1	2	0	2	2
4	−1	4	2	−1	0	0	0	2	0	2	0	2	2

104A	105A	110A	119A	25Z	49Z	50Z
1	1	1	1	1	1	1
0	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{2}{3}$
0	1	0	0	−1	2	1
0	1	1	0	0	1	2
0	0	1	1	0	2	2
0	0	0	1	1	3	3
0	1	1	1	0	4	4
0	1	0	1	1	5	5
1	1	2	1	0	7	6
0	1	1	1	0	8	8
0	2	2	1	0	11	10
0	1	1	1	0	13	12

Table 4a. Further formulae for the t_m.

class	formula for $f = t_m$	
1A	$f = j = (t+256)^3 t^{-2}$	, $t = T_2 - 24$
3C	$f = t + 256t^{-2}$	, $t = T_{6\|3}$
3C	$f = t - 27t^{-1}$	, $t = T_{9+} - 6$
11A	$f = t + 16t^{-1} + 16t^{-2}$	, $t = T_{22+11} - 2$
21C	$f = t + 4t^{-2}\|$	, $t = T_{42\|3+7}$
23AB	$f = t + 4t^{-1} + 4t^{-2}$	, $t = T_{46+23} - 1$
17A, 27AB	$f = t(z) + t\left(\frac{z}{2}\right) + t\left(\frac{z+1}{2}\right)$	, $t = \begin{cases} T_{34+} \\ T_{54+} \end{cases}$
27AB	$f^2 + 3f + 3 = \eta_3^3\eta_9^3/\eta_1^3\eta_{27}^3$	
54A	$f^2 + f + 1 = \eta_3\eta_6\eta_9\eta_{18}/\eta_1\eta_2\eta_{27}\eta_{54}$	
49Z	$f^3 + 2f^2 - f - 1 = \eta_7^4/\eta_1^2\eta_{49}^2$	
49Z	$f = (\eta_7^4 + 7\eta_1^2\eta_{49}^2)/\eta_1\eta_{49}(\eta_1^2 + 7\eta_1\eta_{49} + \eta_{49}^2)$	

11A	$\theta(2,2,6)$	$= \eta_1\eta_{11}f^{\frac{1}{2}}$	34A	$S(2,17) = f^2 + f - 6$
17A	$\theta_x(\frac{1}{2},0,\frac{17}{2}) - \theta_y(\frac{1}{2},0,\frac{17}{2})$	$= 2\eta_1\eta_{17}f^{\frac{1}{2}}$	38A	$S(2,19) = f^2 + f - 4$
19A	$\theta(1,2,20) - \theta(4,2,5)$	$= 2\theta(2,2,10)f^{-\frac{1}{2}}$	51A	$S(3,17) = f^3 - 2f - 6$
23AB	$\theta(2,2,12)$	$= \eta_1\eta_{23}f$	54A	$S(2,27) = f^2 + f - 2$
29A	$\theta_x(\frac{1}{2},0,\frac{29}{2}) - \theta_y(\frac{1}{2},0,\frac{29}{2})$	$= 2\eta_1\eta_{29}f$	55A	$S(5,11) = f^5 - 10f^3 - 5f^2 + 16f$
31AB	$\theta(2,2,16) - \theta(4,2,8)$	$= 2\eta_1\eta_{31}f^{1/3}$		
34A	$\theta_x(\frac{1}{2},1,9) - \theta_x(\frac{9}{2},1,1)$	$= 2\eta_1\eta_{17}f^{\frac{1}{2}}$	62AB	$S(2,31) = f^2 + f - 2$
41A	$\theta_x(\frac{3}{2},2,\frac{15}{2}) - \theta_y(\frac{3}{2},2,\frac{15}{2})$	$= 2\eta_1\eta_{41}f$	69AB	$S(3,23) = f^3 - 2f - 3$
47AB	$\theta(2,2,24) - \theta(4,2,12)$	$= 2\eta_1\eta_{47}f$	87AB	$S(3,29) = f^3 + f - 3$
59AB	$\theta(2,2,30) - \theta(6,2,10)$	$= 2\theta(6,2,10)f^{-1}$	94AB	$S(2,47) = f^2 + f - 2$
71AB	$\theta(4,2,18) - \theta(6,2,12)$	$= 2\eta_1\eta_{71}f$	95AB	$S(5,19) = f^5 - 3f^3 + f - 2$
			105A	$S(3,35) = f^3 - 2f - 3$
			110A	$S(2,55) = f^2 + f$
			119AB	$S(7,17) = f^7 - 7f^3 - 7f^2 - 6f - 7$

Here η_n denotes $\eta(nz)$, and $\theta(a,b,c)$ denotes $\Sigma q^{\frac{1}{2}(ax^2+bxy+cy^2)}$, while $\theta_x(a,b,c)$ or $\theta_y(a,b,c)$ would be the same sum restricted to odd values of x or y respectively. We use $S(d,N)$ for $T_{N+}(z) + T_{N+}(dz)$.

Section 12. Postscript.

It seems to follow from Kac [**10**] that the properties of E_8 noted in Section 9 are suitably generalised forms of certain identities of MacDonald [**14**], for which the appropriate framework is the theory of Lie superalgebras, which are a kind of graded Lie algebra.

Is there a Lie superalgebra that "explains" the MONSTER? Our own tentative investigations of this possibility have not yet proved fruitful, but it is at least consistent with the discovery made by one of us some time ago that the 196883-dimensional representation admits a natural commutative algebra structure. There are difficulties concerned with the portion of the Lie superalgebra corresponding to the q^0 term, which should either be 0-dimensional or infinite-dimensional. Perhaps a more "twisted" kind of algebra is needed?

Most explanations of M along these lines suggest that it is embedded in an infinite group M^1 that should be more "natural". Unfortunately there are difficulties with this possibility as well. M^1 can hardly be an infinite Lie group, and we can find no real evidence for the existence of an infinite discrete group with the required properties.

Another possibility is that M is a Galois group. However, although there are many pairs of mutually algebraic fields in sight, for example $\mathbb{C}(j)$ and $\mathbb{C}(j, t_2, \ldots, t_{119+})$, all the most obvious pairs, including this one, have either been rendered extremely unlikely or actually disproved. However, such an explanation could carry with it an understanding of the "genus zero" property, which would follow if all the Riemann surfaces corresponding to the T_m were quotients of a universal surface of genus zero.

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