Discrete Hartley transform

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The discrete Hartley transform (DHT) resembles the discrete Fourier transform (DFT) but is free from two characteristics of the DFT that are sometimes computationally undesirable. The inverse DHT is identical with the direct transform, and so it is not necessary to keep track of the $+i$ and $-i$ versions as with the DFT. Also, the DHT has real rather than complex values and thus does not require provision for complex arithmetic or separately managed storage for real and imaginary parts. Nevertheless, the DFT is directly obtainable from the DHT by a simple additive operation. In most image-processing applications the convolution of two data sequences f_1 and f_2 is given by DHT of [(DHT of f_1) × (DHT of f_2)], which is a rather simpler algorithm than the DFT permits, especially if images are to be manipulated in two dimensions. It permits faster computing. Since the speed of the fast Fourier transform depends on the number of multiplications, and since one complex multiplication equals four real multiplications, a fast Hartley transform also promises to speed up Fourier-transform calculations. The name discrete Hartley transform is proposed because the DHT bears the same relation to an integral transform described by Hartley [R. V. L. Hartley, Proc. IRE **30**, 144 (1942)] as the DFT bears to the Fourier transform.

A RECIPROCAL TRANSFORM

Given a real waveform $V(t)$, we can define the integral transform

$$\psi(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} V(t)(\cos \omega t + \sin \omega t)\mathrm{d}t, \quad (1)$$

where the integral exists. The waveform may be complex but will be taken as real in what follows and may contain generalized functions, such as delta functions and their derivatives. Clearly $\psi(\omega)$ is a sum of double-sided sine and cosine transforms from whose reciprocal properties one readily deduces the inverse relation

$$V(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi(\omega)(\cos \omega t + \sin \omega t)\mathrm{d}\omega. \quad (2)$$

These relations, which were presented by Hartley,[1] have never disappeared from electrical literature (see, for example, Ref. 2, p. 60, and Ref. 3, p. 178) and reappeared recently in a mathematical context.[4] The direct and inverse relations are identical in form, and, if the given waveform $V(t)$ is real, so is its transform $\psi(\omega)$.

To connect the transform $\psi(\omega)$ with the Fourier transform $S(\omega)$ of $V(t)$, it pays us to adopt the following definition:

$$S(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} V(t)\exp(-i\omega t)\mathrm{d}t$$

and its inverse

$$V(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} S(\omega)\exp(i\omega t)\mathrm{d}\omega.$$

Let $\psi(\omega) = e(\omega) + o(\omega)$, where $e(\omega)$ and $o(\omega)$ are the even and odd parts of $\psi(\omega)$, respectively. Then

$$e(\omega) = [\psi(\omega) + \psi(-\omega)]/2 = (2\pi)^{-1/2} \int_{-\infty}^{\infty} V(t)\cos \omega t \mathrm{d}t$$

and

$$o(\omega) = [\psi(\omega) - \psi(-\omega)]/2 = (2\pi)^{-1/2} \int_{-\infty}^{\infty} V(t)\sin \omega t \mathrm{d}t.$$

Given $\psi(\omega)$, we may form the sum $e(\omega) - io(\omega)$ to obtain the Fourier transform $S(\omega)$:

$$S(\omega) = e(\omega) - io(\omega) = (2\pi)^{-1/2} \times \int_{-\infty}^{\infty} V(t)(\cos \omega t - i \sin \omega t)\mathrm{d}t. \quad (3)$$

Thus we see that from $\psi(\omega)$ one readily extracts the Fourier transform of $V(t)$ by simple reflections and additions.

Conversely, given the Fourier transform $S(\omega)$, we may obtain $\psi(\omega)$ by noting that

$$\psi(\omega) = \mathrm{Re}[S(\omega)] - \mathrm{Im}[S(\omega)]. \quad (4)$$

Thus from $S(\omega)$ one finds $\psi(\omega)$ as the sum of the real part and the sign-reversed imaginary part of the Fourier transform.

NOTATION AND EXAMPLE

As an example, take

$$V(t) = \exp(-t)U(t),$$

where $U(t) = 1$ $(t > 0)$ and $U(t) = 0$ $(t < 0)$. Then

$$S(\omega) = (2\pi)^{-1/2}(1 - i\omega)/(1 + \omega^2)$$

and

$$\psi(\omega) = (2\pi)^{-1/2}(1 + \omega)/(1 + \omega^2).$$

Figure 1 shows $V(t)$ on the left-hand side and the Fourier transform $S(\omega)$ on the right; the real part of the transform is a dashed line, and the imaginary part is a dotted line. The imaginary part has been reversed in sign. Hartley's transform, shown by the solid line, is simply the sum of the real part and the sign-reversed imaginary part of $S(\omega)$. It is real and clearly unsymmetrical. From its even and odd parts, we could readily reverse the construction to recover the real and imaginary parts of the complex-valued Fourier transform $S(\omega)$. For historical continuity, we have retained the elegant factors $(2\pi)^{1/2}$ used by Hartley. But in what follows we drop these factors and move to the more familiar notation, in which 2π appears only in the combination 2π × frequency.

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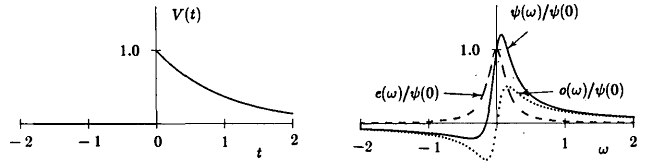

Fig. 1. Left, a waveform $V(t)$; right, the real part of the spectrum $S(\omega)$ (dashed line), the sign-reversed imaginary part of $S(\omega)$ (dotted line), and Hartley's transform of $V(t)$ (solid line).

THE DISCRETE HARTLEY TRANSFORM

Now consider a discrete variable τ that is like time but can assume only the N integral values from 0 to $N-1$. Given a function $f(\tau)$, which one could think of as the representation of a waveform, we define its discrete Hartley transform (DHT) to be

$$H(\nu) = N^{-1} \sum_{\tau=0}^{N-1} f(\tau)\text{cas}(2\pi\nu\tau/N), \tag{5}$$

where $\text{cas}\theta = \cos\theta + \sin\theta$, an abbreviation adopted from Hartley. For comparison, the discrete Fourier transform $F(\nu)$ is

$$F(\nu) = N^{-1} \sum_{\tau=0}^{N-1} f(\tau)\exp(-i2\pi\nu\tau/N).$$

The inverse DHT relation is

$$f(\tau) = \sum_{\nu=0}^{N-1} H(\nu)\text{cas}(2\pi\nu\tau/N). \tag{6}$$

To derive this result, we use the orthogonality relation

$$\sum_{\nu=0}^{N-1} \text{cas}(2\pi\nu\tau/N)\text{cas}(2\pi\nu\tau'/N) = \begin{cases} N, & \tau = \tau' \\ 0, & \tau \neq \tau' \end{cases}.$$

Substituting Eq. (5) into the right hand side of Eq. (6),

$$\begin{aligned}
&\sum_{\nu=0}^{N-1} H(\nu)\text{cas}(2\pi\nu\tau/N) \\
&= \sum_{\nu=0}^{N-1} N^{-1} \sum_{\tau'=0}^{N-1} f(\tau')\text{cas}(2\pi\nu\tau'/N)\text{cas}(2\pi\nu\tau/N) \\
&= N^{-1} \sum_{\tau'=0}^{N-1} f(\tau') \sum_{\nu=0}^{N-1} \text{cas}(2\pi\nu\tau'/N)\text{cas}(2\pi\nu\tau/N) \\
&= N^{-1} \sum_{\tau'=0}^{N-1} f(\tau') \times \begin{cases} N & \tau = \tau' \\ 0 & \tau \neq \tau' \end{cases} \\
&= f(\tau),
\end{aligned}$$

which verifies Eq. (6).

We see that the DHT is symmetrical, apart from the factor N^{-1}, which is familiar from the DFT, and it is real. To get the DFT from the DHT, split the latter into its even and odd parts,

$$H(\nu) = E(\nu) + O(\nu),$$

where

$$E(\nu) = [H(\nu) + H(N-\nu)]/2$$

and

$$O(\nu) = [H(\nu) - H(N-\nu)]/2.$$

Then the DFT is given by

$$F(\nu) = E(\nu) - iO(\nu).$$

Conversely, $H(\nu) = \mathcal{F}[f_{\text{even}}] - \mathcal{F}[f_{\text{odd}}]$.

EXAMPLES OF THE DISCRETE HARTLEY TRANSFORM

For comparison with Fig. 1, consider

$$f(\tau) = \begin{cases} 0.5, & \tau = 0 \\ \exp(-\tau/2), & \tau = 1, 2, \ldots 15 \end{cases},$$

which represents the earlier function of continuous t by $N = 16$ equispaced samples. The value at $\tau = 0$, since it falls on the discontinuity of $V(t)$, is assigned as $[V(0+) + V(0-)]/2 = 0.5$. The result for $H(\nu)$, which is shown in Fig. 2, closely resembles samples of the transform of Fig. 1 taken at intervals $\Delta\omega/2\pi = 1/16$.

The discrepancies, which are small in this example, are due partly to the truncation of the exponential waveform and partly to aliasing, exactly as with the DFT. As a final example, take the binomial sequence 1, 6, 15, 20, 15, 6, 1 representing samples of a smooth pulse. To obtain the simplest result, assign the peak value at $\tau = 0$. Thus

τ, ν	=	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15,
$f(\tau)$	=	20	15	6	1	0	0	0	0	0	0	0	0	0	1	6	15,
$H(\nu)$	=	4	3.56	2.49	1.32	0	0.12	0.01	0	0	0	0	0.12	0.5	1.32	2.49	3.56.

The result in Fig. 3 is the expected smooth pulse peaking at $\nu = 0$.

For numerical checking, it is useful to know that, as for the discrete Fourier transform, the sum of the DHT values $\Sigma H(\nu)$ is equal to $f(0)$. Conversely, the sum of the data values $\Sigma f(\tau)$ is equal to $NF(0)$.

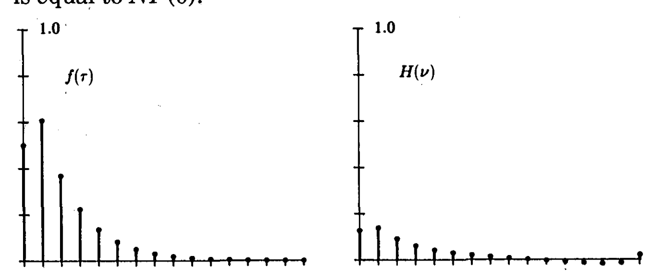

Fig. 2. A 16-point representation of the truncated exponential waveform used in Fig. 1 (left) and its DHT (right).

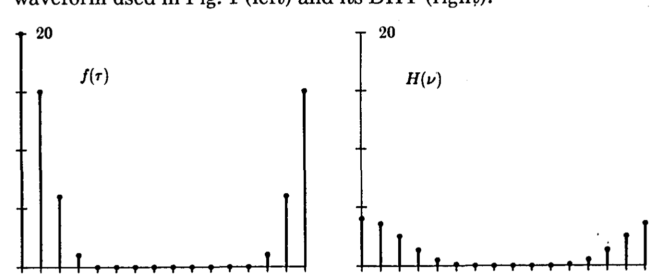

Fig. 3. Samples representing a smooth binomial hump (left) and its DHT (right).

DISCUSSION

At first sight it may seem strange that N real values of the DHT can substitute for the N complex values of the DFT, a total of $2N$ real numbers. We can understand this, however, by remembering that the Hermitian property of the DFT means redundancy by a factor of 2. The $N/2$ real numbers that suffice to specify the cosine transform combine with the $N/2$ needed for the sine transform to form a total of N DHT coefficients containing no degeneracy that is due to symmetry.

The function cas θ, which may be thought of as a sine wave shifted 45°, automatically responds to cosine and sine components equally. If, as a kernel, we use $2^{1/2}\sin(\theta+\alpha)$, where α is an arbitrary shift, the responses will be unequal, but no information will be lost unless $\alpha = 0, \pi/2, \ldots$. Consequently, one would expect to be able to invert; the inversion kernel is $\cot^{1/2}\alpha \sin\theta + \tan^{1/2}\alpha \cos\theta$.

A CONVOLUTION ALGORITHM IN ONE AND TWO DIMENSIONS

The convolution theorem obeyed by the DHT is as follows. If $f(\tau)$ is the convolution of $f_1(\tau)$ with $f_2(\tau)$, i.e.,

$$f(\tau) \equiv f_1(\tau) * f_2(\tau) = \sum_{\tau'=0}^{N-1} f_1(\tau')f_2(\tau-\tau'),$$

then

$$H(\nu) = H_1(\nu)H_{2e}(\nu) + H_1(-\nu)H_{2o}(\nu),$$

where $H(\nu)$, $H_1(\nu)$, and $H_2(\nu)$ are the DHT's of $f(\tau)$, $f_1(\tau)$, and $f_2(\tau)$, respectively, and $H_2(\nu) = H_{2e}(\nu) + H_{2o}(\nu)$, the sum of its even and odd parts.

A customary way of performing convolution numerically is to take the discrete Fourier transform of each of the two given sequences and then to do complex multiplication, element by element, in the Fourier domain, which will require four real multiplications per element. Then one inverts the Fourier transform, remembering to change the sign of i. Analogous use of the Hartley transform for convolution will require only two real multiplications per element.

However, it often happens, especially in image processing but also in digital filtering in general, that one of the convolving functions, say, $f_2(\tau)$, is even. In that case $H_{2o}(\nu)$ is zero, and the convolution theorem therefore simplifies to $H(\nu) = H_1(\nu)H_2(\nu)$. In view of this simplified form of the theorem, one need only take the DHT's of the two data sequences, multiply together term by term the resulting two real sequences, and take one more DHT. Thus the proposed procedure is

$$f_1(\tau) * f_2(\tau) = \text{DHT of } [(\text{DHT of } f_1) \times (\text{DHT of } f_2)].$$

Before taking the DHT, one extends the given sequences to twice the original length with zeros in order to have space for the convolution. The zeros may lead, trail, or bracket the data.

For image processing in two dimensions, exactly the same procedure is available, and the advantages of avoiding complex arithmetic and nonreciprocal subprograms are increased.

TWO DIMENSIONS

Manipulation of two-dimensional images may also benefit from the existence of a real transform. An image $f(\tau_1, \tau_2)$ represented by an $M \times N$ matrix does indeed possess a two-dimensional DHT, which is itself an $M \times N$ matrix $H(\nu_1, \nu_2)$ of real numbers. The transformation and its inverse are

$$H(\nu_1, \nu_2) = M^{-1}N^{-1} \sum_{\tau_1=0}^{M-1} \sum_{\tau_2=0}^{N-1} f(\tau_1, \tau_2) \times \text{cas}(2\pi\nu_1\tau_1)\text{cas}(2\pi\nu_2\tau_2),$$

$$f(\tau_1, \tau_2) = \sum_{\nu_1=0}^{M-1} \sum_{\nu_2=0}^{N-1} H(\nu_1, \nu_2)\text{cas}(2\pi\nu_1\tau_1)\text{cas}(2\pi\nu_2\tau_2).$$

The transform formulas generalize regularly to three (and more) dimensions.

THEOREMS

There is a Hartley-transform theorem for every theorem that applies to the Fourier transform, some of the theorems corresponding exactly, as with $\Sigma H(\nu) = f(0)$ and $\Sigma f(\tau) = NF(0)$. Likewise, the Hartley transform of most convolutions is the product of the separate Hartley transforms, as mentioned above. In other cases there are differences. For example, the shift theorem, needed in implementation of the fast Hartley algorithm, is

$$\text{DHT of } f(\tau + a) = H(\nu)\cos(2\pi a\nu/N) - H(-\nu)\sin(2\pi a\nu/N).$$

There is a quadratic content theorem

$$\sum_{\tau=0}^{N-1} [f(\tau)]^2 = N \sum_{\nu=0}^{N-1} [H(\nu)]^2$$

that resembles the analogous theorem for the discrete Fourier transform except that no complex conjugates enter, only real numbers. The cross-correlation theorem is

$$\text{DHT of} \left[\sum_{\tau'=0}^{N-1} f_1(\tau')f_2(\tau+\tau')\right] = NH_1(\nu)H_2(-\nu).$$

To interpret $H_2(-\nu)$, or any other $f(\ )$ or $H(\ )$ where the argument falls outside the range 0 to $N-1$, add or subtract multiples of N as needed. Thus $H_2(-1)$ is interpreted as $H_2(N-1)$. According to the reversal theorem, $f(-\tau)$ transforms into $H(-\nu)$.

CONCLUSION

The properties of the DHT commend themselves for application to numerical analysis. It goes without saying that on some computers it will be faster to replace cas θ by $2^{1/2}\sin(\theta + \pi/4)$, but where speed is of the essence one may use a fast Hartley transform (FHT). In the author's opinion, many users' programs would run significantly faster with a FHT than with the fast Fourier transform (FFT) in applications such as convolution; on personal computers, the simplicity of the Hartley transform would be an advantage. Also, in spite of the perfection toward which FFT algorithms for a variety of purposes have evolved, one complex multiplication equals four real multiplications. So it is quite possible that a Hartley transform on a large central computer would prove faster than the FFT for taking the Fourier transform of a large data set, such as an image. Experience is reported elsewhere. Of course, it may well be that there exist among the diverse FFT schemes some that really are already implementations of the DHT presented here or could profitably be so viewed.

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