UNSOLVED PROBLEMS

Edited by: **Richard Guy**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics & Statistics, The University of Calgary, Alberta, Canada T2N 1N4.

A mod-*n* Ackermann Function, or What's So Special About 1969?

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One of computer scientists' favorite functions is the *Ackermann function*, first studied by David Hilbert and Wilhelm Ackermann about 75 years ago [**2**]. It is recursive (i.e., computable), but it grows too fast to be primitive recursive (i.e., computable without using dirty tricks like double recursion or the operator "the least n such that"). A kind of inverse for this function (which grows excruciatingly slowly—it makes something like $\ln \ln n$ look like the U.S. national debt by comparison) enters into the efficiency analysis of some important algorithms, such as keeping track of the components of a graph as new edges are added.

If we restrict the range of the Ackermann function to a finite set (with a suitable "mod"-ification of its definition), then we might expect the exuberance of the original function to be reflected in rather chaotic behavior within this set. In fact we seem to find just the opposite, with the finitized Ackermann function petering out very quickly. We have many partial results about this mod-n Ackermann function, obtained using fairly straightforward ad hoc arguments as well as a little elementary number theory. We also have some intriguing experimental data. Perhaps some readers of this article can provide a more definitive description of what's going on.

To be specific, let $\mathbf{N}$ denote the set $\{0, 1, 2, 3, \dots\}$ of natural numbers, and for each integer $n > 2$ let $\mathbf{N}_n$ denote the set $\{0, 1, 2, \dots, n-1\}$ of natural numbers less than n. Define the **standard mod-n Ackermann function** from $\mathbf{N} \times \mathbf{N}_n$ to $\mathbf{N}_n$ by

$$A_n(i,j) = \begin{cases} (j+1) \bmod n & \text{if } i = 0 \\ A_n(i-1, 1) & \text{if } i > 0 \quad \text{and} \quad j = 0 \\ A_n(i-1, A_n(i, j-1)) & \text{if } i > 0 \quad \text{and} \quad j > 0. \end{cases}$$

A (possibly) **nonstandard** mod-n Ackermann function is defined in the same way, except that the values $A_n(0, j)$ for $j = 0, 1, 2, \ldots, n-1$ are arbitrary. We will write A_n^s to refer specifically to the standard function. The value $A_n(i, j)$ is said to be in the ith column and jth row; we picture these values arranged as in Figures 1 and 2.

12	0	1	1	5	3	5	9	9	...
11	12	0	12	1	2	9	9	9	...
10	11	12	10	12	6	5	9	9	...
9	10	11	8	11	5	9	9	9	...
8	9	10	6	4	0	5	9	9	...
7	8	9	4	7	1	9	9	9	...
6	7	8	2	2	11	5	9	9	...
5	6	7	0	6	9	9	9	9	...
4	5	6	11	8	3	5	9	9	...
3	4	5	9	9	2	9	9	9	...
2	3	4	7	3	6	5	9	9	...
1	2	3	5	0	5	9	9	9	...
$j = 0$	1	2	3	5	0	5	9	9	...
	$i = 0$	1	2	3	4	5	6	7	...

Figure 1. The standard mod-13 Ackermann function A_{13}^s.

6	6	3	0	0	0	0	...
5	0	4	0	4	0	4	...
4	3	0	0	0	0	0	...
3	5	5	0	4	0	4	...
2	6	3	0	0	0	0	...
1	0	4	0	4	0	4	...
$j = 0$	4	0	4	0	4	0	...
	$i = 0$	1	2	3	4	5	...

Figure 2. A nonstandard mod-7 Ackermann function A_7.

If we set $n = \infty$, then we obtain in the standard case one of the usual versions of the nonfinitized Ackermann function. It grows monotonically (and wildly) as i and j increase; for example, $A_\infty^s(2, 3) = 9$, $A_\infty^s(3, 3) = 61$, and $A_\infty^s(4, 3)$ has about 10^{20000} digits.

Let us adopt the following terminology. Denote the set of values that appear in the ith column by $P_n(i)$. Clearly $P_n(0) \supseteq P_n(1) \supseteq P_n(2) \supseteq \cdots$; denote the intersection of this sequence, $\bigcap_{i=0}^{\infty} P_n(i)$, by P_n. If A_n becomes constant in some column i, i.e., $A_n(i, 0) = A_n(i, 1) = \cdots = A_n(i, n-1)$, then the function is said to have **stabilized** in column i (and clearly remains constant in all subsequent columns). The smallest i, if any, such that A_n has stabilized in column i is called the **stability number** of A_n, denoted by $s(n)$ in the case of the standard mod-n Ackermann function.

For $n < \infty$ only two kinds of asymptotic behavior are possible (since there are only finitely many different columns, and each column is uniquely determined by the one before it): either A_n stabilizes, or the columns are (nontrivially) **periodic**, i.e., for some $t > 1$, $A_n(i, j) = A_n(i + t, j)$ for all j and large enough i. In the nonstable case the smallest t for which this occurs is called the **period**.

Figures 1 and 2 illustrate the only two known ways in which any mod-n Ackermann function behaves asymptotically. In Figure 1 we see that $s(13) = 6$. This behavior, in which the function stabilizes fairly quickly, seems to happen in almost all cases, standard or not. On the other hand, in Figure 2, we see a nonstable situation for a nonstandard mod-n Ackermann function, in which the period is 2. It is easy to construct an example of this type for any even positive integer m, i.e., a nonstandard mod-n Ackermann function with period 2, whose columns eventually alternate between $(m, 0, 0, 0, \dots)$ and $(0, m, 0, m, \dots)$, in fact starting with a permutation of $\mathbf{N}_n$ in column 0 as long as $m > 2$.

Here is what we have found computationally. The only value of $n < 1{,}000{,}000$ for which the standard mod-n Ackermann function does not stabilize is $n = 1969$. (The first author's older child has been searching for some mystical significance to this property of his birth year.) For $n = 1969$ the period 2 behavior starts in column 8, with the columns alternating between $(1698, 0, 0, 0, \dots)$ and $(0, 1698, 0, 1698, \dots)$. For all other $n < 500{,}000$, the stability number for the standard function is at most 15, and is usually much less (for example, it often happens that $s(n) = 5$ and $P_n(5) = \{65533\}$). On the other hand, since $\lim_{n \to \infty} A_n^s(i, j) = A_\infty^s(i, j)$ for any fixed i and j, the function $s(n)$ is unbounded. We have also tried all possible starting columns for all $n \leqslant 10$, and there are no other patterns.

Here is some of what we know theoretically. First, P_n cannot be all of $\mathbf{N}_n$; in other words, at least some numbers have to disappear as we move from column to column. To prove this, suppose that $P_n = \mathbf{N}_n$. Since $A_n(i+1, 0) = A_n(i, 1)$, the number 1 cannot appear in column $i+1$ except in row $n-1$, or else $A_n(i+1, 0)$ would be repeated. Hence 1 must appear in row $n-1$ in every column from 1 on. But the only way that $A_n(i+2, n-1)$ gets to be 1 is for $A_n(i+2, n-2)$ to be $n-1$ (because 1 appears only in row $n-1$ of column $i+1$). Hence $n-1$ must appear in row $n-2$ in every column from 2 on. Similarly, $n-2$ must appear in row $n-3$ in every column from 3 on. Eventually this says that 2 must appear in row 1 in every column from $n-1$ on, which is absurd, since if 2 appears in row 1 in column i, then it appears in row 0 in column $i+1$. The "line-'em-up" argument used in this proof seems useful in deriving other results as well.

Once we know at least that $P_n \neq \mathbf{N}_n$, under what conditions can we go the whole distance and prove that $|P_n| = 1$ (i.e., A_n stabilizes)? On the one hand, we can prove that $|P_n| = 1$ if $0 \notin P_n$ or $1 \in P_n$. Our strongest result is that the standard mod-n Ackermann function stabilizes if n has a prime factor p such that $2^{j+3} \equiv 3 \pmod{p}$ has no solutions; this is the case for $p = 2, 3, 7, 17, 31, 41$ and 43, to name the first few. From still another perspective, we can show that the two situations discussed above (and illustrated in Figures 1 and 2) are the only possible asymptotic behaviors when $|P_n| \leq 4$ or the period is 2. Open questions abound, such as whether 1969 is the only counterexample to stability in the standard case, or how to compute $s(n)$ efficiently.

As a final variation, we can run the Ackermann function "in reverse" to generate for each n a canonical but random-looking permutation of $\mathbf{N}_n - \{1\}$, somewhat in the spirit of the shuffles reported on by David Gale [1]. Again we start with $A(0, j) = j + 1$ for all $j > 0$, but we set $A(0,0) = 0$. The procedure for producing column $i+1$ from column i is as follows: $A(i+1, 1) = A(i, 0)$, and for $j \neq 1$, $A(i+1, j) = A(i, k+1)$, where $A(i, k) = j$. The first few columns are shown in Figure 3.

Note that each column can be obtained from the column *following* it by applying our original construction. It is easy to show that this function is well-

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
8	9	9	9	9	9	9	9	9	...
7	8	8	8	8	8	8	8	3	...
6	7	7	7	7	7	7	5	0	...
5	6	6	6	6	6	0	0	8	...
4	5	5	5	5	3	5	6	2	...
3	4	4	4	2	0	6	2	4	...
2	3	3	0	0	5	4	4	6	...
1	2	0	2	3	4	2	3	7	...
$j = 0$	0	2	3	4	2	3	7	5	...
	$i = 0$	1	2	3	4	5	6	7	...

Figure 3. The Ackermann function in reverse.

defined in each column; it gives a permutation of $\mathbf{N} - \{1\}$ that leaves $A(i, j) = j + 1$ for all $j > i$. Here one might ask, for example, whether every positive integer $j \neq 1$ appears infinitely often in each row other than row j. As of yet, we have no answers.

REFERENCES

1. D. Gale, Mathematical entertainments, *The Mathematical Intelligencer*, 14, no. 1 (1992) 54–57.
2. J. W. Grossman and R. S. Zeitman, An inherently iterative computation of Ackermann's function, *Theoretical Computer Science*, 57 (1988) 327–330.

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Erratum: Contrary to the information we received some months ago and which was published in the October issue of this MONTHLY, we have just been advised that Professor Emeritus G. H. Hunt is alive and well: our deepest apologies and our very best wishes to him for a long life ahead.

—*American Mathematical Monthly* 75, (1968) p. 1145.