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Representational measurement theory: Is its number up?

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Abstract

Representational measurement theory was proposed initially to solve problems caused by disciplinary aspirations of 19th-century mathematicians, who wanted to construe their subject as independent of its applications in empirical science. Half a century later, S. S. Stevens seized the opportunity provided by representational theory's reconstruction of measurement as numerical coding to rubber-stamp psychology's own aspirations to be counted as a quantitative science. Patrick Suppes' version of representational theory rectified defects in Stevens' theory, making it explicit that representational theory entails that mathematical structure is already *embedded in* empirical systems. However, Suppes' theory neglected the fact that attributes, not objects, are the focus of measurement and when that oversight is corrected, it follows that empirical systems sustaining measurement already instantiate positive real numbers. Thus, in measurement, real numbers are estimated, not assigned from without. Representational theory not only misrepresents measurement; it refutes itself.

Keywords

numbers, realism, representational theory, Stevens, Suppes

Patrick Suppes observed:

> The early history of mathematics shows how difficult it was to divorce arithmetic from particular empirical structures. The ancient Egyptians could not think of 2 + 3, but only of 2 bushels of wheat plus 3 bushels of wheat. Intellectually, it is a great step forward to realize that the assertion that 2 bushels of wheat plus 3 bushels of wheat equal 5 bushels of wheat involves the same mathematical considerations as the statement that 2 quarts of milk plus 3 quarts of milk equal 5 quarts of milk.
>
> From a logical standpoint there is just one arithmetic of numbers, not an arithmetic for bushels of wheat and a separate arithmetic for quarts of milk. (Suppes & Zinnes, 1963, p. 4)

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What is the relation between arithmetic and situations it successfully applies to? Two possibilities are mooted: instantiation versus representation. Is the fact that two bushels of wheat plus three bushels of wheat equal five bushels of wheat an instance of the arithmetic truth, $2 + 3 = 5$; or does that arithmetic formula represent that fact? If it is instantiation, then the fact about bushels of wheat is an instance of the arithmetic truth in a way that is similar to that in which the fact that Socrates is mortal is an instance of the general truth, all humans are mortal. That is, it is a two-term relation holding between a particular situation and a general truth. If it is representation, the fact about bushels of wheat is represented by the arithmetic formula in the same way sequences of dots and dashes represent messages in Morse code: a three-term relation of person *P* using formula *F* to represent situation *S*. Intriguing as these metaphysical alternatives are to philosophers, to psychologists, they may seem as inconsequential as the question, how many angels can dance on a needle's point?[1] However, it is far from inconsequential: it revolutionised the understanding of measurement in psychology.

Stanley Smith Stevens and Patrick Colonel Suppes were psychology's measurement theory gurus, dispensing the latest word on the philosophy of measurement to a discipline aching to be recognised as a quantitative science. Each spruiked the representational approach. Under their influence, most psychologists still endorse it. Consult any mainstream research methods text and an exposition of Stevens' version almost invariably appears (e.g., Kerlinger & Lee, 2000). Suppes' influence is not as ubiquitous, but was felt by those aspiring to greater mathematical rigour (e.g., van der Ven, 1980). Stevens was neither philosopher nor mathematician and used Carnap's (1939) philosophy of mathematics as his point of departure. Suppes was both, favouring Tarski's (1954a, 1954b) set-theoretical approach, and with a leaning towards Aristotle's idea that mathematics treats of the form of situations, not their matter. I use this same idea to expose and transcend the fallacy inherent to representational measurement theory. In doing this, I have presumed recent arguments by philosophers and others developing a naturalistic, realist philosophy of number (e.g., Armstrong, 1997; Bigelow, 1988; Franklin, 2014; Michell, 1994, 2005). That is, the general philosophical position presupposed here is not new,[2] but the argument presented against the representational theory of measurement is (although hinted at in Michell, 1997). Before that, however, features of Stevens' and Suppes' contributions are reviewed. But first, the philosophy of measurement before Stevens is sketched to provide context.

Pre-Stevens

Even 20 years after Stevens' first measurement paper, Brian Ellis (1968), in *Basic Concepts of Measurement*, accused philosophers of indifference regarding measurement:

> The nature of measurement should . . . be a central concern of the philosophy of science. Yet, strangely, it has attracted little attention. If it is discussed at all in works on the philosophy of science, it is usually dismissed in a fairly short and standard chapter. There are notable exceptions; but for the most part the logic of measurement has been treated as though it were neither interesting nor important. (p. 1)

After Ellis, however, the philosophy of measurement blossomed.[3] While Stevens stimulated much of this, he knew little of the history of measurement theory beyond Russell (1903) and Campbell (1920). The immediate roots of their contributions lay in the 19th century. At that time, two movements stimulated interest: attempts to measure mental attributes, especially Fechner's (1860) psychophysics; and the modernist push to found mathematics as an independent discipline (Gray, 2008).[4]

Beneath these lay others, deeper still. Measurement is a primeval concept and probably there never was a time when humans did not measure (Morley & Renfrew, 2010). In early Greek thought, *μέτρον* (measure) was a pivotal concept (Prier, 1976) and profound insights into the logic of measurement appear in Aristotle's *Metaphysics* (McKeon, 1941; Sattler, 2017) and Book V of Euclid's *Elements* (Bostock, 1979; Heath, 1908; Stein, 1990). These spawned the traditional paradigm of measurement, which guided quantitative science from ancient times to the 20th century and culminated in the brilliant work of Otto Hölder (1901).[5] It was not that prior to Ellis the logic of measurement was treated as "neither interesting nor important" (1968, p. 1) as he suggested. Instead, it was widely considered all sewn-up and uncontroversial.

Three concepts framed the traditional paradigm: *quantity, magnitude*, and *ratio*. A *quantity* is an attribute of some kind, such as length, mass, or velocity, possessing internal structure sufficient for measurement, and the expression *quantitative attribute* is a synonym. A *magnitude* of a quantity is a specific degree of a quantitative attribute, such as the length of a football field. It is not a number. A magnitude is a specific, instantiated attribute of something. A *ratio* is a kind of relation holding between two magnitudes of the same kind. It is the relation of *relative magnitude*. For instance, the length of a football field relative to the length of the standard metre is a specific ratio (e.g., 100). The theory of ratios of magnitudes of quantities recorded in Book V of Euclid's *Elements* was the touchstone for the development of quantitative science. We think of ratios as numerical, but for Euclid and other ancient Greeks, for whom numbers were natural numbers only, ratios were not thought of as numerical, not even as ratios of natural numbers.[6] They were relations of their own kind between magnitudes.

Hölder's (1901)[7] achievement was to demonstrate that the system of all ratios of magnitudes of an unbounded continuous physical quantity[8] (such as length or mass) is isomorphic to the system of positive real numbers. Given that the system of real numbers is defined by its structure, these ratios instantiate real numbers. Consequently, within this paradigm, the concepts of number and quantity comprise a single package, and numbers are present in nature wherever quantities exist, which, in virtue of the ubiquity of space and time, is everywhere. This view of numbers was implicit in the thinking of Isaac Newton (1728/1967) when he wrote, "By *number* we understand not so much a multitude of Unities, as the abstracted[9] Ratio of any Quantity to another Quantity of the same kind, which we take for Unity" (p. 2); and, as Dedekind (1887/1909) showed, implicit in Euclid's concept of ratio (Bostock, 1979; Stein, 1990). According to this paradigm, real numbers are instantiated in nature in ratios of magnitudes and *measurement is the estimation of such ratios*. Within this paradigm, *number, quantity*, and *measurement* are interwoven concepts: *measurement* presupposes *number* and *quantity*; and *quantity* implies *number*. *Quantity* is the paradigm's pivotal concept. Importantly, there is nothing logically wrong with this paradigm. External circumstances caused its eclipse, namely, disciplinary aspirations in mathematics and psychology. It was never eclipsed in physics[10] because physicists were unmoved by these aspirations.[11]

If this was the settled paradigm at the beginning of the 20th century, the modernist movement in mathematics unsettled it, at least in the eyes of philosophers.[12] As Russell (1903) announced, “The separation between number and quantity is now complete” (p. 158). The earlier view was that mathematics *is* the science of quantity. Quantities are natural attributes (e.g., distance, time, and mass) and 19th-century mathematicians hoped to distinguish themselves from physicists by defining their subject matter independently from the contexts of its application. They sought definitions of number logically detached from the concept of quantity. Three major schools of thought about the nature of numbers developed early in the 20th century within the philosophy of mathematics: logicism (which held that the concept of number could be defined in purely logical terms), formalism (which saw number as completely defined within formal axiomatic systems), and intuitionism (which was based on Kant’s idea that numbers are human intuitions).[13] During the 20th century, mathematics was increasingly understood as “the science of formal systems” (e.g., Curry, 1951, p. 56) and numbers were increasingly thought of as abstract entities (as existing independently of the spatiotemporal world).

This shift engendered a problem, namely, that of “the unreasonable effectiveness of mathematics in the natural sciences” (Wigner, 1960, p. 1). If numbers are not in the world, their effectiveness when applied to real-world situations appears unreasonable. No knowledge is applied more widely or effectively in the world than “other-worldly” (Hersh, 1997, p. 238) mathematics.

Russell’s solution captured the 20th-century imagination:

> Measurement of magnitudes is, in its most general sense, any method by which a unique and reciprocal correspondence is established between all or some of the magnitudes of a kind and all or some of the numbers, integral, rational, or real, as the case may be. (1903, p. 176)

This is the kernel of the *representational* paradigm of measurement, the idea that *measurement* is “the correlation, with numbers, of entities which are not numbers” (Russell, 1903, p. 158). Within this paradigm, measurement is seen as enabling *surrogative reasoning* (Swoyer, 1991). The theorems and valid argument forms of arithmetic are used to make inferences about nonnumerical entities by assigning numbers to them, performing valid arithmetic computations to arrive at numerical conclusions, and then, translating these numerical conclusions back into conclusions about the things to which numerical assignments were originally made. Surrogative reasoning depends upon structural similarity between the nonnumerical, real-world system represented and the other-worldly, numerical system representing it (Michell, 1986). The key to success is isomorphism between empirical and numerical systems.

Campbell’s (1920) contribution was to describe how he thought this isomorphism was obtained in physics. Because psychological quantitative practices did not conform to his description, he concluded, “nothing but confusion and error can result from using ‘measurement’ in any but its accepted sense. I call nothing measurement that does not possess the distinctive features of the processes physicists accept as measurement” (Campbell, 1933, p. 589).[14] This was the state of play when Stevens joined the fray in the 1930s keen to protect his field, psychophysics, from Campbell's ire. Campbell failed to grasp the full implications of Russell's letting of the representational cat out of the bag: Campbell wanted to endorse the representational approach while restricting the scope of measurement to the form it takes in physics. However, once measurement is defined in terms of numerical representation, the floodgates are opened to all forms of numerical representation, which potentially admits a wider class of empirical structures to that concept. Stevens grasped the full implications of Russell's position and, ever the opportunist, he seized the representationalist cat with both hands.

Stevens

Ask psychologists what measurement is and the answer will be "assignment of numerals to objects or events according to rule" (Stevens, 1946, p. 677) or some variant of this meme. Ask psychologists for examples and *nominal, ordinal, interval*, and *ratio* scales will be mentioned—these are also Stevens' terms. Fragmented as psychology is, it stands united behind Stevens' theory of measurement.

Theories of measurement purport to explain the position of measurement in science. In the 1940s, psychologists craved a theory meeting their aspirations and they felt besieged because some members of an expert committee investigating mental measurement (Ferguson et al., 1940), including Campbell, rejected its possibility. Stevens delivered his theory in the nick of time and the spirit of triumph pervading his announcement not only eased their minds; psychologists suddenly felt ahead of the game in understanding measurement.

And Stevens smuggled in a metaphysical dividend, blending operationism and logical positivism, both of which, despite disclaimers to the contrary,[15] are metaphysical doctrines.[16] He gave psychologists an apparently incontestable definition of measurement and a plausible rationale for it, which was actually metaphysics, masquerading as avant-garde, scientific philosophy. Psychologists welcomed their medicine, thinking it was an antiphilosophical prophylaxis; never dreaming it was metaphysical bromide.

The metaphysics began with Stevens' view of numbers. As he summarised this view later, it was,

> For millennia the number system and the empirical operations of measurement were so closely linked that even the greatest thinkers did not discern the crucial difference between them. It now is generally agreed that the number system is a formal, syntactical system, defined by a set of arbitrary assumptions. (Stevens, 1967, p. 733)

By a "formal, syntactical system," Stevens meant one composed of symbols, possessing rules for combinations into "well-formed formulae," for transforming strings of symbols into others, and that exhibits a definite structure or form (but with unspecified material content). By "defined by a set of arbitrary assumptions," he meant that the axioms of the system could be altered and providing they remain syntactically well formed, still give rise to a consistent system. For Stevens (1951),

> Measurement is possible in the first place only because there is a kind of isomorphism between (1) the empirical relations among objects and events and (2) the properties of the formal game in which numerals[17] are the pawns, and operators the moves. (pp. 1–2)

In this schema, numbers are not present in the naturally occurring phenomena comprising the empirical context of measurement. They are conceived as elements in the "formal game" of mathematics. In Stevens' (1951) view, "mathematics is a human invention, like language, or like chess, and men not only play the game, they also make the rules" (p. 2). In other words, numbers are not part of the furniture of the universe and did not exist before humans "invented" them. He quoted with approval the mathematician E. T. Bell's assessment, "In the same way that a novelist invents characters, dialogues and situations of which he is both author and master, the mathematician devises at will the postulates upon which he bases his mathematical systems" (as cited in Stevens, 1951, p. 10). Just as Tom Sawyer and Sherlock Holmes are not real people, so numbers are not real, causally active things located in the natural world. This is a metaphysical thesis: it purports to answer the question of what numbers really are; an issue philosophers had long investigated (e.g., Bostock, 2009).

Stevens (1974) did not invent this view. His "own central problem throughout the 1930s was measurement" (p. 409) and being at Harvard when not only Bridgman[18] (who had first proposed operationism in 1927), but also Carnap (the leading logical positivist), were present afforded the opportunity to forge a solution from their philosophies. He participated in a "Science of Science" discussion group (Hardcastle, 2003), which included a host of philosophers, mathematicians, and scientists engaged in working out the implications of these new philosophies. Their spirit captured Stevens' imagination and his papers on measurement theory have the tone of one who has just come down from the mountain having communed with God.

He resolved his own central problem by concluding that measurement involves numerically representing relations between objects and these relations are defined by the operations performed on the relevant objects. As he put it:

> Scales are possible in the first place only because there is a certain isomorphism between what we can do with the aspects of objects and the properties of the numerical series. In dealing with the aspects of objects we invoke empirical operations for determining equality (classifying), for rank-ordering, and for determining when differences and when ratios between the aspects of objects are equal. (Stevens, 1946, p. 677)

For operationists, concepts are defined by operations used to identify them. According to Stevens, if it is stipulated that relations, such as equality of ratios, equality of differences, order, or class membership are established by certain, prespecified operations, these relations are thereby operationally defined into existence.[19] As he put it in an earlier paper, considering classification, "the concept of that class is defined by the operations which determine inclusion within the class" (Stevens, 1939, p. 234). In fashioning his interpretation of operationism, Stevens (1935) rejected the idea of objective truth, asserting that science is a "social convention" (p. 327) and "from the social criterion of truth there is no appeal" (1936, p. 97). He was a relativist regarding truth long before relativism became popular in the 1970s.

Stevens' theory broadened the concept of measurement to cover classifications and orderings, and, so, appeared to qualify psychology for the ranks of quantitative science.[20] His nominal, ordinal, interval, and ratio scales covered most cases of numerical representation and the traditional concept of measurement was repackaged as merely one form, namely, *ratio scaling*. Psychologists exploited Stevens' (1951) blind eye regarding the boundary between ordinal and interval scales, whereby he classified standardised testing as interval scale measurement simply because "the assumption of normality has the advocacy of a certain pragmatic usefulness in the measurement of many human traits" (1951, p. 28), which licensed psychologists to market their meretricious "metrics" as measurement and to ride "natural scientists' coattails onto the endless frontier" (Solovey, 2004) of advancing knowledge.

According to Edwin B. Newman (1974), his colleague at Harvard, Stevens' theory "has stood like the Decalogue . . . No single more recent statement has replaced it" (p. 137) and that assessment remains true for psychology. He wrote that in adopting Stevens' theory,

> We are asserting thereby that there exists a significant isomorphism between the two, the mathematics on the one hand and empirical science on the other. Yet if they are isomorphic they are both instances of some common structure. This should tell us something. (Newman, 1974, p. 143)

Indeed, it does! However, that "something" is more sharply seen in the light of Suppes' version of representational theory.

Suppes

Ernest Nagel introduced Suppes to measurement theory[21] and to Hölder's paper.[22] Suppes' version of representational theory,[23] definitively expressed in the three volumes of *Foundations of Measurement*, in collaboration with the psychologists, Duncan Luce, Amos Tversky, and David Krantz (Krantz et al., 1971; Luce et al., 1990; Suppes et al., 1989), is the high water mark of the representational paradigm. Suppes advocated a set-theoretical approach to philosophy and employed it to great effect in measurement theory. Central to this approach is Tarski's (1954a, 1954b) concept of a relational system. A relational system is constituted by (a) its *domain*, which is a nonempty set of *elements* of some kind, (b) a finite assortment of *relations* holding between these elements, and (c) a series of *laws* (sometimes called *axioms*) taken to characterise these relations on those elements. Measurement, as construed in this paradigm, requires an empirical relational system, where the domain is "a set of identifiable entities" (Suppes & Zinnes, 1963, p. 7), the relations are said to be "qualitative" ones[24] holding between these, and the laws are empirical conditions that these relations are believed to conform to; and a numerical relational system, in which the domain is a set of numbers, the relations are numerical ones, and the laws are the familiar laws of arithmetic (or a subset of them). A measurement scale is an isomorphic mapping between these systems. The advantage of making their structure explicit set-theoretically is that it facilitates proof of the existence of mappings (what Suppes called *representation theorems*; e.g., Suppes & Zinnes, 1963, p. 4) and specification of the class of mapping functions, any member of which would equally well map the empirical system onto (or into) the same numerical system (what Suppes called the *uniqueness theorem*; e.g., Suppes & Zinnes, 1963, p. 8).

Such a precise specification of the empirical relational system, however, inevitably raised questions about much of what is routinely considered measurement in psychology. Rarely is any attempt made to characterise the relevant empirical relational system and prove that it may be numerically represented and, thus, Suppes' style of presentation exposed Stevens' vagueness regarding the systems allegedly numerically represented in psychological measurement. One psychometrician, reviewing *Foundations of Measurement*, concluded,

> The lack of empirical relations other than order in the behavioural sciences severely limits the scope of interpretation of numerical data. It would be a good thing in my opinion if we could restrain our use of terms such as "measurement" and "instrument" in the context of fields such as mental testing in deference to this fact. (Ramsay, 1991, p. 357)

Even this apparently draconian recommendation does not go far enough because, generally, psychologists fail to establish the existence of order relations within the relevant empirical context. Suppes' theory exposed gaping holes in psychologists' thinking about their measurement practices. Representational theory only appeared to work for psychologists because of defects in Stevens' presentation. For his part, Stevens (1968), defending against Suppes' rigor, attempted to marginalise it by suggesting, "measurement models sometimes drift off into the vacuum of abstraction" (p. 854), forgetting that abstraction is a necessary step in any intellectual work, science included.

Another attempt to marginalise Suppes' theory is the widespread misunderstanding that the laws characterising empirical relational systems apply only to "error-free data" (e.g., Borsboom, 2005, p. 89; Boumans, 2016, p. 305). This overlooks that data, being fallible, are never exactly as scientific laws predict in any area. Suppes' theory is really one of measurability (Baccelli, in press; Michell, 2014, p. 115), intended to describe empirical structures necessary and sufficient for measurement. An empirical relational system is not an unvarnished description of data but, rather, is posited, theoretically, as an objective, independently existing structure able to be numerically represented. To think that theories must predict data without residue is a positivist conceit still prevalent in psychology. Data are used to assess the truth of laws hypothesised to characterise empirical systems and because data are fallible, there is always give-and-take and hit-and-miss in judgements made about this. This is the position of *Foundations of Measurement* (see Krantz et al., 1971, p. 13; Luce et al., 1990, section 21.8).

However, once it is admitted that laws hypothesised to characterise empirical relational systems are theories about the relevant phenomena, not unvarnished descriptions of data, it makes sense to reinterpret the domains of such systems as classes of attributes (e.g., all possible lengths for the case of length measurement). In the first instance, measurement is of attributes and only concerns objects because attributes only occur as attributes of objects. Brent Mundy (1994) argued that what is numerically represented is "not necessarily a directly implementable *empirical* process" (p. 64) involving objects. If anything is represented, it is attributes. To take length as an example: it is specific lengths and relationships between them that are represented in measurement according to representational theory; not objects, such as rigid, straight rods and physical operations upon such objects. The attribute, length, is a relational system, the domain of which is the set of all possible magnitudes of length (understood as linear extensions in space). It is characterised by a greater-than relation (which is transitive, asymmetric, and connected) and is additive (where the relation of additivity is associative, commutative, etc.). The distinction between objects and attributes is important because attributes are more general than objects and a relational system composed of a set of objects may not capture every possible magnitude of the relevant attribute and, so, any proposed numerical representation of the objects in the set will leave magnitudes not instantiated in those objects unmeasured. Suppes always thought of empirical structures numerically represented as being composed of finite sets of objects, but no finite set of objects ever instantiates all possible magnitudes (all possible lengths, for example). Hence, a general account of the measurement of length, for example, must go beyond objects and consider attributes.[25] So, more satisfactory versions of representational theory, such as those of Mundy (1987, 1994) and Chris Swoyer (1987) shift focus to the structure of attributes.[26] This, however, immediately redirects attention back to Hölder's axioms for quantitative structure, for these are conditions hypothesised to hold for the attributes measured in physics.

Hölder gave a more adequate account than Mundy and Swoyer, both of whom balked at the prospect of including the axiom of continuity.[27] Continuity[28] is not an idealisation introduced into the specification of quantitative attributes to enable application of the differential calculus, as Pap (1959) suggested. The axiom of continuity means that no possible magnitudes of the relevant attribute are excluded a priori from the domain of the attribute and therefore from the range of measurement. To introduce a restriction on the set of magnitudes would add to the complexity of the structure of quantitative attributes. The axiom of continuity universally generalises (i.e., ensures that all possible magnitudes are included) and, so, completes membership of the domain of the attribute.

Suppes' theory improved on Stevens' by exposing the structure of the empirical relational system supposedly represented numerically in measurement. Mundy and Swoyer improved on Suppes' theory by reconceptualising the domain of the empirical relational system to be of attributes rather than objects. But, Hölder had already completed that picture. If Hölder is thought of as a representational theorist, which Suppes was inclined to do, measurement involves proposing a theory regarding the relevant attribute, namely, that it possesses the structure of an unbounded continuous quantity. If it does possess this structure, it is isomorphic to the system of positive real numbers. Up to this point, this understanding of representational theory is faithful to the character of quantitative science. But just when all bases seem covered, representational theory self-destructs.

The fallacy of representational measurement

According to representational theory, measurement is possible only because the empirical system represented and the numerical system representing it, possess the same *mathematical* structure. This is the basis of the theory, necessary to sustain surrogative reasoning. However, if the empirical and numerical systems have the same mathematical structure, it follows that mathematical structure is present in empirical systems. This implication is not controversial and representational theorists have noted it: for example, Nagel (1931) wrote, "if mathematics is applicable to the natural world, the formal properties of the symbolic operations of mathematics must also be predicable of many segments of the world" (p. 314); Swoyer (1991) put it more clearly: "I believe that the best explanation why a mathematical theory applies to the concrete phenomena it does is that it has many of the same structural features as those phenomena" (p. 451); and Narens and Luce (1990) put it clearest of all: "*In many empirical situations considered in science . . . there is a good deal of mathematical structure* already present *in the empirical situation. Measurement produces numerical correlates of that structure*" (p. 133).

However, this implication is the thin end of the realist wedge: if mathematical structure is present in empirical situations, there is no ontological divide between mathematical structure and empirical situations. This consequence applies with equal force to both Stevens and Suppes, and not only to the more sophisticated versions of the representational theory of Swoyer, Mundy, and Hölder (interpreted as a representational theorist) because, as we have seen, Stevens and Suppes argue that measurement depends upon isomorphisms between empirical and numerical structures as much as these other theorists do. Isomorphism means literally "having the same form as" and if the form of an empirical structure is the same as that of a numerical structure and the form of that numerical structure is mathematical (as it must be), it follows with the iron logic of the syllogism that the form of the relevant empirical structure is also mathematical. Contradicting its foundational presupposition (viz., that mathematical structures are not present in the empirical context of measurement), representational theory entails that the empirical structures numerically represented in measurement possess mathematical form. That is, mathematical structure is present in empirical situations. Might not numbers have the same real presence?

Consider real numbers: "The real numbers form a complete ordered field" and "Any two complete ordered fields are isomorphic" (Birkhoff & MacLane, 1965, pp. 86, 92). This means, "Every such system can be related by a one-to-one, similar and isomorphic correspondence to the system of real numbers. Hence, there is essentially only one such system" (Waismann, 1951, p. 210). Thus, wherever there is a complete ordered field, there are real numbers. If our physical theories are correct and unbounded continuous quantities exist in nature, instantiated in, for example, distance (i.e., space) and time, real numbers, as ratios of magnitudes of such quantities, not only exist in the real world, they infuse it utterly. Most importantly in the present context, when we measure, we do not assign numbers to represent ratios; rather, we estimate real numbers already present as ratios. The representational theory, itself, entails that the logic of measurement is instantiation, not representation as it asserts.

Hölder's (1901) axioms capture the structure of the kinds of attributes appearing in the theories of physics.[29] Hölder proved that structures of that kind instantiate the positive real numbers. That which has the structure of the real numbers instantiates real numbers, for they are defined by their structure. Representational measurement theory is based upon these presuppositions: (a) numbers are not intrinsic to the structures numerically represented and (b) measurement requires isomorphisms between the system of numbers and the empirical systems represented. But if physical quantities possess the structure Hölder described, numbers are intrinsic to them. The presuppositions of representational measurement theory are contradictories. Thus, representational theory refutes itself.

In proposing his axioms of quantity, Hölder's (1901) concern was to demonstrate how our experience of real numbers emerges from our experience of quantitative attributes

(Cantù, 2013; Radu, 2003). His aim was to establish an epistemological thesis. However, he also demonstrated an ontological one, namely, the system of real numbers is intrinsic to the spatiotemporal world. This was the view also argued for by Frege (1903) and Whitehead (Whitehead & Russell, 1913). James Franklin (2014) calls it *Aristotelian realism* and provides an extensive defence of the position that mathematics is the science of structure with many of the structures investigated by mathematicians, including number systems, being instantiated in the world around us.

How could Suppes miss this apparently obvious flaw in representational theory? It is not obvious to anyone already committed to the conventional view that numbers are abstract entities, because to such a person numbers seem to be entities in their own right. David Bostock (2009) observed:

> The numbers, whether natural or rational or real or complex, exemplify certain kinds of structures, and from the point of view of pure arithmetic they do not seem to have any other properties than those that the structure assigns them. (pp. 306–307)

However, during the 20th century, philosophers of mathematics saw them as possessing surplus content: Russell (1903) defined numbers as classes; the formalists, as uninterpreted symbols in a formal, axiomatic system; and the intuitionists, as mental constructions. By the mid-20th century, the consensus was that numbers are sets, which came about because, as Suppes (1972) once remarked, "It is, it seems to me, one of the great intellectual surprises about the structure of mathematics that all of the standard mathematical notions can be reduced by explicit definition just to the simple concept of set membership" (p. 12). Suppes (1960, p. 129) favoured von Neumann's (1923) construction of the natural numbers from the empty set, where $1 = \{0\}, 2 = \{0, \{0\}\}, 3 = \{0, \{0\}, \{0, \{0\}\}\}$ and so on ("*0*" symbolising the empty set). This process of construction was extended to include the real numbers: integers were seen as ordered pairs of natural numbers; rational numbers, as ordered pairs of integers; and real numbers, infinite sets of rational numbers. These constructions were thought to guarantee the "existence" of numbers as abstract entities. They were seen as independent entities capable of standing in relation to empirical structures and while, in the case of real numbers, they were seen to possess the structure of a complete ordered field, it was thought that there was more to them than that structure alone.

However, Suppes (2012) recently expressed reservations about this view:

> We have the structure of the natural numbers; we have them satisfied by many different kinds of entities; there isn't any magic answer "what really is the number 2 as a separate independent something or other in the universe of abstract things?" That's not an important question. It's like letting go of some mistaken theological view, like the number of angels that can sit on the end of a pin. (24:00–24:38)

Indeed, it would seem that the project of trying to specify the supposedly surplus content of numbers was destined to grind to a standstill.[30] If, as Bostock said, "from the point of view of pure arithmetic they do not seem to have any other properties than those that the structure assigns them" (2009, p. 307), there is no rational basis for choosing between the various alternatives offered, only extraneous bases. The important thing is the structure of the system of numbers. In the above quotation, Suppes was giving free rein to his “Aristotelian instincts for the central place of form” (Suppes, 2009, p. 161). Earlier, he had noted, “what Aristotle meant by form was structure” (Suppes, 2012, 18:23–18:26), but he did not follow these instincts to where they inevitably lead, namely, to the consequences that numbers are structures per se and the logic of numerical application is instantiation, not representation.

Aristotelian instincts

Thus far, I have argued top-down that numbers are located in the world, reasoning from the structure of the number system and the structure of quantitative attributes, but numbers can also be found bottom-up, by reflecting upon contexts in which we directly experience them. Aristotle said, “a number, whatever number it is, is always a number of certain things” (as cited in McKeon, 1941, p. 923). He was noting that natural numbers occur only in contexts of things. He noted further, “number is a plurality of *units*” (as cited in McKeon, 1941, p. 837), which means when we identify specific numbers, we do so by observing a relation between, on the one hand, a plurality (say, books on my desk) and, on the other hand, the relevant unit (say, *being 1 book*). The *number*, as something general, is the relation of relative magnitude (i.e., the *ratio*) between these. A specific natural number, say 10, is not an object; it is a relation between magnitudes (Michell, 1994). It is present wherever there are 10 things of a kind, for in that situation, the relation between the two magnitudes, being 10 things (of the relevant kind) and the unit, being 1 thing (of that kind) is present. The fact that we experience this relation does not mean that, at the same time, we register the fact of its being a relation.

Numbers are part of the form of situations, not their matter. To this extent, the formalists were correct, but they went astray in thinking of a formal system as a symbolic structure. Formal languages in mathematics are theories (about numbers or sets or other relational structure). Numbers do not emerge from formal languages in mathematics any more than physical phenomena emerge from physical theories. To think so is to adopt a version of philosophical idealism, a position at odds with the empirical realism implicit in science.[31] Jonathan Lear (1982) summarised: “For Aristotle, mathematics is true, not in virtue of the existence of separated mathematical objects to which its terms refer, but because it accurately describes the structural properties and relations which actual physical objects do have” (p. 191).

Because the ancient Greek concept of number encompassed only the natural numbers, Aristotle did not see ratios of magnitudes as numbers. However, real numbers are also relations, as Hölder (1901) showed, namely, they are ratios between magnitudes of unbounded continuous quantitative attributes. If Smith stands 2 metres tall, the ratio of his height to the height of the standard metre is the real number 2 and since these heights are spatiotemporally located instances of the attribute length, any relation holding between them will likewise be located. There are no mysteries here, no abstract entities, nothing otherworldly. When we measure, we estimate real numbers; we do not conjure them from the realm of abstract entities and assign them to the attributes we measure, for numbers are already there in the midst of those empirical situations. Measurement is our attempt to know them where they are.

Consider again Suppes' claim that ancient Egyptians "could not think of 2 + 3, but only of 2 bushels of wheat plus 3 bushels of wheat" (Suppes & Zinnes, 1963, p. 4). What is involved in seeing 2 bushels of wheat as *2 bushels of wheat*? To be able to discern 2 bushels of wheat from whatever else is in one's visual field, one must know what kind of thing a single bushel of wheat is. Then, to recognise that there are two of these present one must cognise that each of them is a thing of that kind, and that this bushel here with that bushel there is 2 bushels. It requires seeing a relation between the plurality of bushels and a single bushel of wheat and this relation is just what the number two is. To think of 2 bushels of wheat involves thinking of 2; to think of 3 bushels of wheat likewise involves thinking of 3; and, so, to think of 2 bushels of wheat plus 3 bushels of wheat is to think of 2 + 3. Suppes misrepresented the ancient Egyptians: neither they, nor anyone else for that matter, has ever thought of 2 bushels of wheat plus 3 bushels of wheat without thinking of 2 + 3 because that would be a cognitive impossibility, like, say, trying to think of a blue shirt without thinking of blue.

None of this is to say that the ancient Egyptians realised that in thinking of 2 bushels of wheat they also thought of 2. While we cognise relations constantly, awareness of the fact that we cognise *relations* and not only *properties* and *objects* can be difficult and to understand *relations*, as distinct from *objects* and *properties*, even more so. Aristotle said relations are "least of all things a kind of entity or substance" (as cited in McKeon, 1941, p. 914) and for centuries philosophers tended to neglect them. "It is not until the late nineteenth century with C. S. Peirce, William James, and Bertrand Russell that relations begin (no more than begin) to come into focus" (Armstrong, 1989, p. 29). Thus, when asked what the number 2 is, the average person will not readily volunteer "a relation between a plurality and a unit." Such a person would probably agree with Graham Flegg (1983) that "we can be aware of this number in at least four different ways—as a numeral, as a number-word, as a concept in our minds, and as a property possessed by every collection of two objects" (p. 3). But numerals such as "2" and number-words such as "two" are not numbers (despite common parlance) any more than the expression *filet mignon* is an actual filet mignon: "2" and "two" are symbols for or names of the number 2. And a person's concept of two (as a mental construction) is no more the actual number 2 than my *concept of filet mignon* is an actual, edible filet mignon. It is only with the "property possessed by every collection of two objects" (Flegg, 1983, p. 3) that we come tantalisingly close to the real thing, which still eludes us. Every collection of *two* objects is a collection of *one* pair; hence, numbers are not *properties* of collections, for the same thing cannot be both 1 and 2 at the same time.[32] It is only *in relation to* a definite unit that the number of a collection is uniquely fixed. That is, while superficially it may look like a property, a plurality's being two in number is really a relation between it and a unit.

Once the hurdle of recognising relations as features of real situations is surmounted, numbers can be accepted as features of such situations and any view of them as nonnatural entities can be rejected. Recognising that numbers are relations present in the world does not detract from the idea that mathematicians investigate a distinctive subject matter: mathematics is the science of structure (quantitative structure being simply one variety). Since structure is ubiquitous, mathematics is not confined to any one kind of application (say, its application in physics), but is applicable to all sciences.[33] In terms of the distinction made at the outset, this conclusion implies that the logic of measurement is not representation; it is instantiation.

What would the consequences be for psychological measurement if psychologists rejected the representational view of measurement and returned to the naturalistic, realist view? A naturalistic, realist account of measurement is given in a number of places, for example, Michell (1994, 2003, 2005). Adopting it would entail that anyone aspiring to measure psychological attributes would need to consider evidence that the relevant attributes actually possess quantitative structure, since real numbers are instantiated in psychological attributes only if those attributes possess quantitative structure. If evidence of quantitative structure is lacking, claiming to measure is scientifically premature at best and false at worst. As this is a matter that I have dealt with at length elsewhere, beginning with Michell (1990) and up to Michell (2019), I will not elaborate further upon the matter here, except to point out that much that is routinely called "psychological measurement" might not be measurement at all. It could mean that psychologists need to consider the possibility that at least some of the attributes they aspire to measure possess a nonquantitative structure and are therefore not measurable. This does not mean that such attributes cannot be investigated scientifically, only that they cannot be measured. On the positive side, however, where it can be shown that psychological attributes possess quantitative structure, it would allow psychology to reattach itself to the rich traditions of quantitative science characterising the physical sciences and to claim to be able to measure in the same sense of that term as used in physics, which has been, since Galton and Fechner, an enduring aspiration of many who have sought to investigate psychological phenomena. However, experience teaches us that, in general, psychologists have no appetite to proceed down the scientific path of considering evidence for quantitative structure.

This is because representational measurement theory misrepresents measurement in ways that appear to protect psychology's disciplinary aspirations without the need to accrue evidence of quantitative structure. That is why it endures. Representational theory was not an inconsequential 20th-century metaphysical detour simply sidetracking psychology for 70 years to no other effect. Its consequences are significant and on present indications, irreversible. The reception (or lack of it) of Suppes' version within psychology was a litmus test, showing that psychologists do not abandon confused methodological conceptions for more rigorous ones when the pay-off is potentially detrimental to the self-serving image they want to project, whatever the scientific merits of the more rigorous alternative.[34] Thus, we can anticipate that Stevens' legacies will linger for the long term. These include the use of "measurement" to refer to nothing more than numerical data (e.g., frequencies, codings, etc.); claims to measure attributes without even bothering to look for hard evidence of quantitative structure; the entrenched practice of misrepresenting numerical coding as nominal or ordinal scales of measurement; and the conviction that interval scales are a distinct type of measurement when they are simply measures (in the traditional sense) of quantitative differences. These confusions are consequential because they serve psychology's enduring fantasy of being a quantitative science.

When the scientific methods of experiment and measurement became the defining methods the new psychology of the late 19th century aspired to, psychologists distanced themselves from the "armchair" methods of the older mental philosophy and shunned conceptual analysis, which left them vulnerable to confused metaphysical ideas, especially when these were packaged as "science," not philosophy. However, this historical episode shows that inquiring into the mind's ways of working requires the investigator to be "at once scientific and philosophic" (Anderson, 1962, p. 183), and in this context, being philosophic means finding a metaphysics grounded in this world (i.e., in situations located in space and time)[35] and not one given to the excesses of philosophical idealism, like abstract entities.

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Notes

1. The number of angels that can "dance on a needle's point" is said to have preoccupied scholastic metaphysicians and, since the 17th century, has been employed as a metaphor for any vacuous topic or, as the pun was first intended, any "needless point" (Harrison, 2016) despite the fact that this issue appears not to have been discussed at all in medieval times, although Aquinas is reported to have asked, "whether a multiplicity of angels can co-exist in the same place?" (Ross, 1985, p. 496).
2. It is, of course, the same position adopted in all of my writings on measurement, from Michell (1990) onwards.
3. See Tal (2017) for a review.
4. Readers interested in how the concept of number was treated in debates over the foundations of mathematics during the late 19th and early 20th centuries should consult Grattan-Guinness (2000) and Gray (2008), both of whom give detailed histories.
5. For an English translation see Michell and Ernst (1996, 1997).
6. "Incommensurable magnitudes do not have to one another a ratio which a number has to a number" (Euclid, *Elements*, Book X, Proposition 7).
7. Otto Ludwig Hölder (1859–1937) was professor of mathematics at the University of Leipzig when he published this paper.
8. Hölder's axioms define an *unbounded continuous quantity* because that was the structure then presumed in quantitative physical theories (Cantù, 2013).
9. Note that Newton says, "abstracted" not "abstract" that is, he did not see numbers as abstract entities, only as ratios between quantities considered independently of extraneous matters.
10. Campbell (1920) appears to be an exception, but in writing that book he donned his philosopher's cap.
11. For example, Wildhack's recent statement, "Any measured quantity may thus be expressed by a number (the magnitude ratio) and the name of the unit" (2005, p. 483) aligns closely with Clifford's (1882) in the 19th century: "Every quantity is measured by the ratio which it bears to some fixed quantity, called the unit" (p. 525).

12. See Gray (2008).
13. Discussions of these schools of thought can be found in any good text on the philosophy of mathematics, such as Bostock (2009).
14. Campbell's theory of measurement and his critique of psychological measurement are discussed at length in Michell (1999).
15. For example, Bridgman, in a conversation with a metaphysician, claimed that his difficulty with understanding metaphysics "is that I do not know, and I have never found anyone who could tell me, what the nature of the checks is to which you may subject these metaphysical facts or truths after you have got them, to find whether you have got what you think you have" (cited in Burchard, 1950, p. 245) and in his intellectual autobiography, Carnap (1963) wrote, "I came to hold the view that many theses of traditional metaphysics are not only useless, but even devoid of cognitive content" (p. 45).
16. Green (1992) and Rogers (1989) discuss the place of operationism in psychology and Bickhard (2001) details the metaphysics of operationism. Passmore (1943, 1944, 1948) exposes the metaphysical content of logical positivism.
17. Stevens preferred the term "numeral" to that of "number" because he thought of the former as the less ambiguous of the two. However, he admitted, "'numeral' has the defect that it sometimes means the physical ink mark on a piece of paper and it sometimes means the essentially *logical* relation that a numeral may stand for. This second meaning is in line with the formalist's view of mathematics, according to which arithmetic is regarded as the rules of a game played with numerical symbols" (1951, p. 22). So, by "numeral," it seems, he meant the same as the formalists meant by "number."
18. Bridgman would not endorse Stevens' interpretation of operationism, writing, "I have rather washed my hands of him" (in a 1936 letter to the philosopher, Arthur Bentley, another interpreter, as quoted in Walter, 1990, p. 184).
19. This is what Stevens did with his psychophysical method of magnitude estimation, stipulating that it delivers ratio scale measurement and what psychometricians did with mental tests, stipulating that they deliver interval scale measurement.
20. This was important in the era of "Big Science" when research grants were more likely to be awarded to sciences claiming quantitative status (Schorske, 1997).
21. Suppes (1978) reported that his "interest in the theory of measurement was generated during [his] graduate student years by listening to lectures of Ernest Nagel in the philosophy of science" (p. 264).
22. Like Nagel, Suppes (1951) attempted to improve on Hölder's axioms.
23. It is Suppes' version in the sense that it is his style, first introduced in his 1958 paper with Dana Scott (Scott & Suppes, 1958), which remained invariant throughout his publications on measurement with other colleagues. According to Duncan Luce (1979), "More than any other living person, Suppes has affected contemporary presentations of theories of measurement" (p. 93). This is not to downplay the original and important contributions of Luce, Krantz, and Tversky to the content of *Foundations of Measurement*, but simply to note that this content was always clothed in the style Suppes advocated.
24. "Qualitative" in the sense of not already explicitly involving numbers.
25. For more on this point see the discussion in Swoyer (1987).
26. Louis Narens (1985, 2002) with his isomorphism theory seems also to shift in the same direction.
27. Not so, however, Duncan Luce. As Suppes (1997) noted, "Duncan has been very much more attracted to the continuum as a framework for theories of measurement than I have" (p. 107). For example, see Luce and Narens (1992).
28. For a discussion of continuity see my comments in this journal (Michell, 2017, p. 422).

29. I have not burdened the reader by including Hölder's axioms in this paper. The interested reader may find it useful to consult the translation by Catherine Ernst and myself (Michell & Ernst, 1996, 1997) or my exposition in Michell (1999).
30. Suppes (2012) intimated that it had stagnated, saying, "foundations of mathematics has gone out of style—it is simply not a central topic. If you ask any outstanding mathematics department in the United States 'are you going to support somebody in foundations' they'll say 'well not now—we're going to wait until they get some new kinds of results'" (13:39–14:00).
31. For arguments against philosophical idealism see Anderson (1962) and against formalism in philosophy of mathematics see Bostock (2009) and Franklin (2014).
32. Frege (1884).
33. Readers interested in the deeper issues raised by this philosophical view of number and mathematics will find it instructive to read the detailed defence of it given by the mathematician, James Franklin (2014).
34. Psychologists' unwillingness to reject operational definitions and null hypothesis significance testing are other instances of the same phenomenon.
35. For an indication of what such a metaphysics looks like in the context of psychology, see Hibberd (2009) and Petocz and Mackay (2013) and in the context of measurement, Michell (2005).

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