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Factors and primes: a specific numerical ability

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SYNOPSIS An autistic young man and a normal control were asked to factorize numbers and to recognize and generate primes. Both subjects made a similar number of errors and employed similar strategies, but they differed markedly in the speeds at which the arithmetical operations were carried out.

INTRODUCTION

In the third century BC the Greek astronomer and mathematician, Eratosthenes, devised an algorithm to determine which numbers were primes, i.e. could not be divided by any whole number except themselves and one. The method consists of identifying all primes smaller than the square root of N, and then eliminating all those numbers which are multiples of these primes. So, if N were for instance 64, all primes below 8 would be identified (2, 3, 5 and 7). Then in addition to even numbers, all multiples of 3 (9, 15, 21, etc.), all multiples of 5 (25, 35, etc.) and all multiples of 7 (49, 63, etc.) would be discarded. The remaining numbers (11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61) are prime numbers. The method is reliable, but becomes slow and laborious as N gets large.

Mathematicians still debate whether there is a rule, as yet undiscovered, which would enable one to predict which numbers are primes. However, at present it appears as if there is a wilful randomness in the occurrence of these numbers, so that their identification remains dependent on trial and error methods, though shortcuts that suggest which numbers may be primes do of course exist.

Factorizing is the process by which a given number is divided into its composites until primes are reached and it can be divided no further, or until all the composites of which the number consists are determined without a remainder. Smith (1983, Ch. 22) relates how tables devised by Colburn in the early 19th century, enabling one to factorize any number, came into being. Colburn, who was one of the most celebrated calculating child prodigies could, at the age of six, rapidly find the factors of any number up to a million or more. However, he could not tell people how he achieved this feat, and relates how he was sometimes reduced to tears by the persistence of his questioners. But he later reported that at the age of nine, he woke up one night and said to his father, 'I can tell you how I find out the factors.' His father began to write down what the child told him, and from this account the rules governing Colburn's tables for identifying factors were derived. The tables enable one to factorize, for example, 1401 by consulting combinations of numbers yielding products ending in 01. In fact, 3×467 yields the desired result. As the tables show, 467 is a prime and, therefore, the only factors of 1401 are 3 and 467. Colburn's tables are extensive and their consultation takes up a great deal of time. Moreover, memorizing them in order to carry out the necessary mental calculations would represent a considerable feat of memory. Thus, as with identifying prime numbers, factorizing proceeds usually by trial and error, using available short cuts.

In the present context, what makes Colburn's account specifically relevant and interesting is the fact that, according to his own report, his ability to carry out a complex cognitive operation preceded by several years his capacity to account for the strategy which he employed. Thus, conscious access to the rules was not necessary for their effective use. This may be relevant for understanding why many numerical calculators are childhood prodigies, and also why the ability can be found in people whose

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general intelligence can range from the severely mentally handicapped to the intellectually brilliant.

That high level calculating skills may be based on rules which, though they are used effectively, can nevertheless not be verbally stated, was illustrated by us in studies with idiot-savant calendrical calculators (O'Connor & Hermelin, 1984; Hermelin & O'Connor, 1986). These subjects in spite of below average IQ's could, nevertheless, use strategies which enabled them to name with speed any given day on which a particular date in the past or the future would fall. The Gregorian calendar is subject to certain rules and internal consistencies. One example of this, which we used in an experiment, is that the calendar repeats itself exactly every twenty-eight years. We found that idiot-savant calendrical calculators used this rule, although without in most cases being able to state it verbally.

With great mental calculators a knowledge of, and familiarity with, the relationships between numbers and the structure of the number system, at whatever level of conscious accessibility, must be assumed. Smith (1983, Ch. 34,) quotes Wim Klein, perhaps one of the greatest contemporary calculators as saying to him, 'Numbers are friends to me, more or less. It doesn't mean the same for you, does it, 3844? For you it's just a three and an eight and four and a four. But I say, "Hi, 62 squared"'.

Such an ability to immediately and directly perceive relationships between numbers does not however seem to be related to the level of general intelligence. It may be a specific, intelligence-independent innate ability, or it could be a function of a particular interest in and a preoccupation with numbers, which would presumably also be innate. Such a special interest would motivate an individual to familiarize himself so thoroughly with the number system that any regularities and structures would become apparent and could be extracted and stored.

Smith (1983) states that many calculators do indeed think that their ability is not so much a function of a special talent, but is rather due to an interest in and preoccupation with numbers and their internal relationships. For instance, Aitken (1954), one of the great mathematicians of this century and a formidable mental calculator, was of the opinion that this ability was not really different in kind from that possessed by other people, but was simply due to a different degree of facility. However, there is no evidence given in Aitken's (1954) account that he carried out any searching enquiry on this issue. Rather, his view might be due to either modesty regarding his own outstanding ability, or perhaps it represents an incomprehension of other people's problems with mathematics.

That Aitken was not very like other people can be deduced from the fact that he could dictate π to over 100 places to his secretary and could repeat it again a few minutes later for her to verify.

This kind of statement, though not uncommon, presents some problems of interpretation. Apart from the fact that there might be some difficulty in determining which came first, the talent or the preoccupation, many calculators can be induced to offer a 'theory' concerning the origins of their talent. These range from the extremely simple, 'Daddy taught me', offered by one of our idiot-savant subjects up to Aitken's view. But sometimes these 'theories' can be shown to be unjustified or misleading. For example, another idiot-savant calendrical calculator, when asked how he performed his surprising feats, replied, 'I make all sorts of mathematical calculations'. However, tests showed that his best efforts in arithmetic could not justify his statement.

The second fact which bears on the interpretation of such statements as Aitken's is that child prodigies can carry out complex calculations at speeds equal to those of adult calculators. In other words, before familiarity or practice could have had much opportunity to produce what Aitken calls 'facility', the skill is apparently fully developed. For instance, we have investigated a nine-year-old autistic boy who could carry out calendrical calculations as fast as subjects in their twenties. This child also used the same rules and regularities of the calendar as much older subjects. If such abilities depended upon constant practice, one might expect that speed of calculation would increase with age.

In any case, it appears that special talents for, and outstanding interests in, particular subject matters are relatively intelligence-independent, so that they may occur in people with very low, with average or with very high IQ's. Case studies of outstanding mental calculating ability in those who were otherwise mentally handicapped have repeatedly appeared in the literature. In the context of the present investigation, a report by Horwitz *et al.* (1965) is of special relevance. It concerns a pair of twenty-six-year-old mentally retarded male twins, diagnosed as autistic, who had an astounding facility in generating large prime numbers. The twins were observed to exchange prime numbers with each other, as in a game, and when presented with primes of a magnitude up to ten digits, they would respond by supplying the next prime. The present study with an autistic young man also investigates the ability to recognize and generate prime numbers. One special feature of this case is that in our study the individual completely lacked any language ability and was unable to speak, comprehend or communicate through signs. It is, therefore, relevant to consider how numerical calculations may be coded. According to Binet (1894) there are two types of calculators, those who mentally 'see' numbers and those who 'hear' them. But Aitken has stated that when he thought of numbers it was neither through the visual nor the auditory medium, but what was involved was a 'compound faculty' for which he could not provide an adequate description. He points out, however, that the qualities of the representations of musical memory and composition have as yet also not been adequately described. What may be the case is that, as Morton (1968) proposed in regard to language, numerical systems or music are stored in the form of abstract representations which refer to, but do not consist of, the diverse sensory modes in which these systems are perceived and expressed. Our previous findings have made it clear that such high level representations underlie the domain specific outstanding performances achieved by musical idiots-savants (Hermelin, *et al.* 1989). In the following investigation an autistic young man and a normal control subject were compared for their ability to recognize and to generate prime numbers. An attempt was also made to compare the strategies which the two subjects used in these tasks.

SUBJECTS

There were two subjects. One was a male psychologist who also holds a mathematics degree. The other was a young man aged twenty, who at the age of three years had been diagnosed as typically autistic, showing the classic signs of Kanner's syndrome. He is reported to have appeared to be a normal baby during his first few months, but he had a rather large head, and at ten months he had convulsions. He sat up at seven months and walked at fifteen months. However, he never imitated gestures, such as pointing or waving goodbye. He never began to speak and did not respond to language. He took very little interest in adults and did not try to communicate in any way. When aged between two and four years he was very destructive and oblivious to danger. During this period he suffered further convulsions. He was good at games involving shapes, colours and construction and could do 100-piece jigsaws when four-years-old, which like many autistic children he could do just as well with the reverse side showing. An EEG at this stage was regarded as 'inconclusive'. He began attending a special school for autistic children at age six. He learned to 'write' with a pencil, i.e. he learned to copy letters and numbers. But he has not improved in this skill since his schooldays and his written numbers are often difficult to make out. He also learned a few elementary Paget Gorman signs, though he never used them spontaneously. He was very good with money, time, calendars, maps and numbers. He can add, subtract, multiply and divide large numbers, and was known to be able to factorize.

The intellectual status of the subject is not easy to determine. He did not obtain any score on the Peabody Picture Vocabulary test, and on the non-verbal Columbia Mental Maturity Scale his IQ is 67. On the other hand when tested on the Raven's progressive matrices his performance is equivalent to an IQ of 128. In order to investigate further the reasons for these two discrepant intelligence test scores, we divided the material of the Columbia test into two categories. In some displays the items are non-representational shapes from among which 'the odd one out' has to be selected. In other items several objects are depicted, from which the correct one has to be chosen. Responses to shape problems were correct 73% of the time whereas responses to pictures of objects were right only 23% of the time. Thus, it seems that the subject's high performance IQ is confined to non-representational material.

Both the subject's parents have degrees in mathematics although neither works as a professional mathematician. When they told us of their son's numerical ability we asked whether he could identify prime numbers. As he had not been previously confronted with such a task, some primes were slipped into a list of numbers which he was asked to factorize. In his mother's words, 'When he came to these he looked at me as if I were mad'.

METHOD

Both subjects were individually presented with three tasks, each at three levels of difficulty. The tasks required them to (*a*) factorize three, four and five figure numbers, (*b*) to recognize prime numbers of the same three magnitudes from among non-primes, and (*c*) to produce three, four and five figure primes between stated limits. Performance on all tasks was timed. As verbal communication of the task requirements was not possible with the autistic subject he was always shown two examples on each task and at each level of difficulty. The factorizing tasks took the form of the investigators writing down a number and then, after an equal sign writing down the factors. As stated, according to verbal reports by his parents he had not been presented with such a task before, so that no light can be shed on the process by which he understood the significance of the required operation. For recognition the experimenters wrote down some previously selected numbers, including primes and ringing some of the primes. Finally, for the generating task two prime numbers for each of the different magnitudes used were written by the investigators. After each such demonstration the subject was given the pencil, the appropriate task sheet, and simply proceeded with the required operations. As he carried these out appropriately and without hesitation there was obviously no difficulty for him in understanding what he was asked to do.

For factorizing, subjects were presented with ten numbers between 212 and 221, between 1001 and 1011, and between 10002 and 10013. The numbers were written, one under the other, on lined sheets of paper, and a different sheet was used for each magnitude.

In the recognition tasks ten prime numbers had to be identified from among twenty other numbers. The subjects were required to ring the primes with a pencil. The numbers ranged from 301 to 393 for the hundreds, from 1201 to 1309 for the thousands, and from 10301 to 10427 for the ten thousands.

For the generating tasks, in which ten primes at each level of magnitude had to be produced, the hundred's range was from 227 to 281. In the thousand's ten prime numbers between 1019 and 1091 had to be produced, and the last task was the generation of ten successive prime numbers between 10037 and 10133. In the first range the subject saw '227' then ten blank lines, then '281' as a concluding number. He was required to fill in the blanks. The same procedure was repeated for the higher ranges.

RESULTS

The number of correct responses and the average time needed for factorizing, recognizing and generating primes of a given magnitude are shown in Table 1.

Overall, 150 numbers had to be dealt with, 90 in the three recognition tasks, where decisions had to be made about each number, and 30 each for factorizing and generating. Out of this total 97 numbers were correctly identified by the control and 109 by the autistic subject.

In the factorizing tasks the control subject mistook five divisible numbers as primes, and the autistic subject made four such errors. The remaining factorizing errors, one from the autistic and four from the control subject consisted of giving arithmetically incorrect factors.

In the recognition tasks, leaving aside for the moment the recognition of five digit primes, the control made fourteen omissions and nine wrong inclusions, while the autistic subject made one omission and eight inclusion errors.

The autistic subject's co-operation on the recognition of five digit primes could not be obtained during the first testing session. When on a later occasion another attempt was made to collect this data, he gave only fifteen correct responses, errors being mainly due to wrong inclusions.

With generating, the control made fourteen and the autistic subject eleven omission errors. The control made twelve incorrect inclusions, while the autistic young man gave eleven numbers incorrectly as primes. Thus, there was

Table 1. *Number of correct responses and mean decision times*

Task	Control		Subject	
	Correct	Av time (sec)	Correct	Av time (sec)
Recognizing				
301–393	20/30	11·46	29/30	1·16
1201–1309	18/30	12·90	22/30	2·90
10307–10427	23/30	10·73	15/30	2·00
Generating				
227–281	8/10	12·9	9/10	6·20
1019–1091	5/10	25·6	5/10	6·00
10037–10133	4/10	50·0	5/10	10·00
Factorizing				
212–221	8/10	22·6	9/10	8·8
1001–1011	7/10	25·5	8/10	20·8
10002–10013	4/10	48·0	7/10	38·2

some tendency for more consistent correct responding by the autistic subject.

One interesting feature about the pattern of errors is that, excepting again the five digit recognition tasks, twelve errors out of thirty-six by the control and twenty-six by the autistic subject, were shared by the two calculators. Thus, both often thought that the same numbers were primes when in fact they were not, and omitted to recognize some of the same prime numbers. However, these shared errors did occur almost exclusively with those four digit numbers which were not divisible by either three or eleven. This suggests that both subjects may often have employed similar strategies. The control subject did indeed report that his strategy had been to divide the numbers into three categories. The first of these categories consisted of numbers divisible by 3 or 11, as there are simple rules determining which these are. In the second list were numbers thought to be divisible by factors other than 3 or 11, and the final group contained primes.

If one assumes that such a procedure was indeed followed by both subjects, then no number chosen incorrectly by the control, and only one false number selected by the autistic subject as a prime, was divisible by 3 or 11. On the other hand, of seven other non-primes in this range, six were incorrectly identified as primes by the autistic and four by the control subject. The same error pattern is evident in most of the other tasks, and thus the shared errors result from both subjects eliminating those numbers divisible by 3 and 11. However, the rule for excluding such numbers was not as consistently used, and presumably less explicitly formulated, by the autistic as compared with the control subject. Thus, in the generating tasks, numbers divisible by 3 and 11 were never erroneously produced as primes by the control, while the autistic subject gave two such wrong responses. Furthermore, for five digit prime recognition, when the autistic subject finally did carry out this task, he discarded these rules altogether. Six of his errors consisted of identifying as primes numbers divisible by 3 or 11, and his abandoning of the previously used rule is the reason for his high error score on this particular task. Nevertheless, apart from this instance, the similar error pattern over the rest of the responses is accounted for by both the subjects' use of the 3 and 11 rules, thus implying that both employed similar strategies.

While the control showed an overall tendency to omission errors in the recognition tasks, the autistic subject was biased towards wrong inclusions. Thus, taking again the four figure recognition task, of seven non-primes which were not divisible by either 3 or 11, six were wrongly chosen by the autistic subject and four by the control. On the other hand, of the ten primes seven were rejected by the control and only one by the subject.

If the total number and the pattern of errors for the two subjects were on the whole not dissimilar, the speeds at which they achieved these similar performances differed markedly.

With all tasks, and with all number magnitudes the autistic subject was much faster than the control ($P = 0{\cdot}002$, Sign Test). As Table 1 shows, the most marked inter-subject speed differences were evident in the tasks in which prime numbers had to be recognized, and were least marked when factorizing was required.

DISCUSSION

The level and quality of the autistic subject's performance in this study is similar to that of the mathematically trained control of above average intelligence, but is at the same time much faster. If we were to accept Aitken's view of his own calculating skills, we would have to conclude that such a speed resulted from a facility based on practice in the use of algorithms, not in themselves qualitatively different from those employed by people with some mathematical training but no special calculating skill. We can perhaps assume that the present autistic subject, like Aitken, is also interested in, and pre-occupied with, certain aspects of the relationships between numbers, although we can also be fairly certain that he does not have Aitken's mathematical knowledge. The speed which he has shown both in factorizing numbers, and in the recognizing and generating of primes cannot therefore be due primarily to mathematical knowledge, but must be dependent either on some superior innate calculating talent or on a continuous pre-occupation and practice with the number system and numerical calculation or on both.

These two alternatives, a superior talent or continuous practice, may not be mutually exclusive but may be associated. As the tendency of autistic people to become pre-occupied with repetitive actions or ideas has been frequently observed, this may account for autistic idiots-savants becoming pre-occupied with a specific domain such as calculation, or music or drawing. However, not every autistic person is an idiot-savant and therefore one must assume that some additional quality other than a penchant for repetitive behaviour is contributing to what Aitken has called 'facility', or speed of operation. It would seem that we can only describe this additional quality as a talent. Thus, many autistic people, and a lesser percentage of the mentally handicapped, may have a tendency to engage in repetitive behaviour, but very few are gifted artists, musicians or calculators.

If in the present case, therefore, it is not pre-occupations alone which can explain the subject's speed of arithmetical calculation, how are we to account for this speed?

It has been stated frequently that the speed of information processing as measured for instance by reaction time studies, is closely associated with level of general intelligence. Jensen (1979) for example has demonstrated that those with mental handicap also have long reaction times as compared with those of average intelligence. However, it may well be that general speed of information processing is unrelated to those operations in which a subject has a specific ability. Thus, not only are the levels of those cognitive operations which are relevant to such special domains not necessarily related to general cognitive efficiency (Anderson, 1986), but the operational speeds in this area are also independent of the individual's general speed of information processing. We have some indication of this possibility from the results of a visual reaction time (RT) study with idiot-savant calendrical calculators (Hermelin & O'Connor, 1983). The simple as well as the complex visual RT's of these subjects were in accordance with those expected from the levels of their IQ's, whereas their speed of calendrical calculation was much faster than that usually obtained from people with much higher IQ's.

It will be recalled that we had concluded that the present subject's good and also very fast performance on the matrices test might not have been a pure measure of Spearman's (1927) 'g', but an indication of a special spatial-numerical ability in a person who was perhaps in other areas of below average intelligence.

Turning to the qualitative characteristics of the calculating ability manifested in the present study, the results suggest that the mental operations and strategies used by the autistic subject and by the control were similar. In the main, both subjects used rules in which numbers divisible by 3 and 11 were eliminated from consideration, though this rule was more consistently used and probably also more explicitly formulated by the control. This is particularly evident in one of the recognition tasks where it was completely abandoned by the autistic subject. One should consider whether such less consistent rule use may be a consequence of the complete absence of verbal formulation in the autistic subject. We obtained a related finding in an experiment which tested the musical improvisation ability of an idiot-savant with a very low level of language development (Hermelin *et al.* 1989). When improvising on a whole tone composition this subject remained in the whole tone scale for 96 bars, but then, unlike a normal control musician he abandoned this mode, and reverted to the key of F Major for the last four bars of his improvisation.

There is another finding which we obtained from a study with idiot-savant graphic artists, which may be of relevance to present results (O'Connor & Hermelin, 1987). In this experiment, it was found that gifted artists were no better than IQ-matched controls in recognizing a previously presented shape from among other similar shapes. However, when asked to draw the same shape from memory, their performance was more accurate than that of controls, and also more successful than their own recognition attempts. This finding is reminiscent of the present results, where the autistic subject could produce five figure primes more successfully than he could recognize them. Perhaps a procedure in which a correct item has to be distinguished from a number of possible alternatives, provides special difficulties for those with below average intelligence levels.

Overall the results of the present experiment suggest that the operation of simple arithmetical procedures can be performed with considerable speed by a speechless, autistic person whose measured intelligence on tests other than those involving purely spatial reasoning is well below average. The speed of his processing of such procedures was greater than that of an intellectually superior adult while the strategies he employed seemed to resemble closely those used by his control.

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