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THE BI-FACTOR METHOD

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The Bi-factor Method of factor analysis is described and illustrated with a small group of fourteen tests. A detailed illustration is given of how the method may be modified to the case of overlapping group factors. It is advocated that the Bi-factor pattern in unmodified form be used to determine the adequacy of tests for the measurement of unitary traits.

I. *Introduction.* In the present paper we shall give a brief description of the Bi-factor Method of factor analysis introduced in the Preliminary Reports on the Spearman-Holzinger Unitary Trait Study.* We shall also illustrate how the method may be modified for the analysis of variables of greater complexity than assumed in the original theory.

The simplest form of the Bi-factor pattern is merely an extension of Spearman's Two-factor pattern to the case of group factors. The Spearman pattern is a theoretical frame of reference consisting of a general factor running through all variables and uncorrelated factors present in each variable. The Bi-factor pattern is also a theoretical frame of reference in which a general factor is assumed to run through all variables with specific factors in each variable, but in addition a number of uncorrelated group factors, each through two or more variables, are also included. The minimum number of factors of these three types for n variables may then be briefly summarized as follows: one general factor, n specific factors, and q group factors where q is usually much smaller than n. In the modified pattern some of the group factors may overlap.

The general plan of analysis is to re-sort all of the n tests by a combination of methods, so as to bring into small groups those tests which correlate higher amongst themselves than they do with the remaining tests.† When this is accomplished, the large table of intercorrelations will show triangles of relatively high correlations along the diagonal of the whole table. The factor weights of the general

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†It is assumed that all intercorrelations are positive or insignificantly negative for mental variables. In case most of the correlations of a variable x_i with the other variables are negative, the variable $-x_i$ may be used to yield positive values.

factor are computed by selecting only one test within each sub-group (in all possible combinations). The general factor is next removed. The main body of residuals is then numerically small, but the triangles of high correlations for the group factors remain on the diagonal. The weights of these group factors may next be computed and the factors removed to furnish final residuals for study. If the latter show that no greater complexity of group factors is required, the two specific factors in each test (specificity and unreliability) may be readily calculated and the final factor pattern set up.

In case the first residuals show a complexity greater than the simple Bi-factor form, the extra factor may be introduced into what may be called the Modified Bi-factor pattern. After their proper allocation (to be illustrated below), the whole analysis is repeated in terms of the new frame of reference and new final residuals computed. These are again examined for any significant factor overlap to determine the final goodness of fit of the original correlations and modified factor pattern.

The arbitrary nature of the Bi-factor frame of reference has been commented on elsewhere as a distinct limitation of the method. By the above modification this objection would seem to disappear, but we should like to emphasize the view that without modification of the pattern, the limitation may be regarded as a defect of the tests used rather than of the method. We should go so far as to argue that no modification should be necessary if the tests are properly made to measure single group factors. If some tests do reveal two or more group factors, then they are poor tests for the purpose of factor appraisal, and should be improved or discarded. In short, the Bi-factor frame of reference may serve as a guide to the construction of tests as measures of factor ability, as well as a very simple and easy basis for analysis.

II. *The Bi-factor Pattern.* In the present analysis it is assumed that all variables are represented in standard form and that all factors are uncorrelated. The chief advantages arising from assuming uncorrelated factors are in the simplicity of analysis and economy of measurement. If two factors are uncorrelated, a measure of one does not involve the other, a difficulty which has made present-day testing confusing and well-nigh hopeless. Thus an ordinary reading comprehension test and a verbal intelligence test will be highly correlated, and these two labels almost useless as indexes of these traits considered as two abilities. For economical measurement, simplicity, and parsimony, uncorrelated factors are indispensable.

The present notation is as follows:

$N \equiv$ number of individuals in the sample
$n \equiv$ total number of tests
$x_i \equiv$ standard score for test i
$r_{ij} \equiv$ product-moment correlation between tests i and j.
$\alpha \equiv$ general factor (designated as u_1 in the Reports* and in the numerical example)
β, γ, δ, etc. $\equiv$ group factors
$a_i \equiv$ weight of general factor for test i
b_i, c_i, d_i, etc. $\equiv$ weights for group factors β, γ, δ, etc.
s_i and $t_i \equiv$ weights of the specific and unreliability factors, respectively
$\underline{r}_{ij} = r_{ij} - a_i a_j \equiv$ a residual correlation with factor α removed from tests i and j.
$r_{11} \equiv$ reliability coefficient

A hypothetical factor pattern written in tabular form is as follows:

TABLE I

HYPOTHETICAL BI-FACTOR PATTERN

Variable	α	β	γ	δ	Specificity s_i	Unreliability $t_i = \sqrt{1-r_{11}}$
1	a_1	b_1			s_1	t_1
2	a_2	b_2			s_2	t_2
3	a_3	b_3			s_3	t_3
4	a_4	b_4			s_4	t_4
5	a_5		c_5		s_5	t_5
6	a_6		c_6		s_6	t_6
7	a_7		c_7		s_7	t_7
8	a_8		c_8		s_8	t_8
9	a_9			d_9	s_9	t_9
10	a_{10}			d_{10}	s_{10}	t_{10}
11	a_{11}			d_{11}	s_{11}	t_{11}
12	a_{12}			d_{12}	s_{12}	t_{12}
	Σa^2_i	Σb^2_i	Σc^2_i	Σd^2_i	Σs^2_i	Σt^2_i

The weights for the specificity and unreliability factors should be staggered in the pattern, so that no two s's or t's appear in the same column, but they are written here in more compact form to save space.

All of the weights represent the loadings of the test with the fac-

**op. cit.*

tor in question, or the correlation of the factor with the variable they resolve. Thus b_3 is the correlation of the β factor with test 3, etc.

From the above assumptions, the following relations due to Spearman may be noted:

$$r_{ij} = a_i a_j + b_i b_j \ , \tag{1}$$

$$1 = a_i^2 + b_i^2 + s_i^2 + t_i^2 \ , \tag{2}$$

$$t_i^2 = 1 - r_{11} \text{ where } r_{11} = \text{reliability} \ , \tag{3}$$

$$n = \text{sum of squares of all factor weights} \ , \tag{4}$$

$$\sum_1^n a_i^2 = \text{portion of total variance due to } a \ . \tag{5}$$

In the original formulation of the Bi-factor theory, weights such as b_1, b_2, and b_3 were taken proportional to the corresponding weights a_1, a_2, and a_3, but this is not necessary for analysis as pointed out in *Preliminary Report* No. 7.

If we assume that the factors are allocated as in the hypothetical example above, the a factor may next be removed by taking only one test from each sub-group in all possible combinations to compute the necessary weights. Thus to compute a_1 we use tests 1, 5, and 9; 1, 6, and 9; etc., to give,

$$r_{1a}^2 = a_1^2 = \frac{r_{15}r_{19} + r_{16}r_{19} + \text{etc.}}{r_{59} + r_{69} + \text{etc.}} \ . \tag{6}$$

A complete outline for performing these calculations from simple sums has been described in *Preliminary Report* No. 7.

When the weights a_i are thus found the products $a_i a_j$ are formed and then the residuals $\underline{r}_{ij} = r_{ij} - a_i a_j$ determined. The group factors may next be removed in any order by the direct use of Spearman's formula

$$r_{i\beta}^2 = b_i^2 = \frac{\left[\left(\sum_1^{m-1} r_{c_x} \right)^2 - \sum_1^{m-1} r_{c_x}^2 \right]}{2 \left[\sum_1^{C_2^m} r - \sum_1^{m-1} r_{c_x} \right]} , \tag{7}$$

where m is the number of tests in the sub-group (here four), r_{c_x} the correlation in a column of these correlations for the group factor, and r a correlation in the set of C_2^m values (here six).

The weights t_i are obtained directly from the reliability coefficients r_{1I}. Thus $t_1 = \sqrt{1 - r_{1I}}$ where r_{1I} = reliability of test 1. The values for s_i are found from the relation $s_i^2 = 1 - a_i^2 - b_i^2 - t_i^2$.

If the residuals with a removed reveal any extra overlap, it is desirable to revise the pattern plan before computing the group factors. A frequency distribution of final residuals is made for the modified pattern and the value .6745 σ of this distribution compared with the probable error of a zero correlation for a sample of N. This will serve as an approximate standard for judging the goodness of fit of the Bi-factor pattern to the observed correlations. If the value .6745 σ of the residuals is appreciably less than this standard it is probable that insignificant group factors have been introduced into the pattern.

III. *Allocation of Tests to Groups.* We may now turn to the general problem of analyzing a set of observed variables by the Bi-factor method. In allocating the tests for the determination of group factors, three procedures have been followed:

(a) The use of a B-coefficient which is merely the average of all intercorrelations of tests 1, 2, $\cdots$, m, divided by the average of all correlations of tests 1, 2, $\cdots$, m with the remaining tests not in this group. The quotient expresses the extent to which the tests 1, 2, $\cdots$, m belong together in the sense that they have high correlations amongst themselves and relatively low correlations with other tests in the whole set.

The work is begun by selecting the two tests with the highest correlation in the whole large table, and then the test which correlates most highly with either of these, and proceeding likewise until a drop in B value is obtained.

(b) The nature of the tests themselves is carefully studied as the B-coefficients are computed. If the coefficients fail to show appreciable drop, tests are sometimes included or rejected upon examination of their content.

(c) The residuals are always reproduced in full to reveal overlap, extra factors, or the wrong allocations of tests. If a test has been omitted from a group to which it belongs, consistently positive residuals will occur between this test and those of the group. If, on the other hand, a test has been included which does not belong in the group, a negligible loading for the group factor usually is found.

These three methods have been used successfully on several sets of correlations described in our recent *Preliminary Reports*.

TABLE II

PARTIAL CORRELATIONS WITH AGE ELIMINATED

(FOURTEEN TESTS; 355 CASES)

Test	1	2	3-4	6	28	29	32	34	35	36a	13	18	25b	77
1														
2	.514													
3-4	.477	.662												
6	.433	.497	.415											
28	.424	.397	.319	.444										
29	.350	.427	.376	.530	.437									
32	.083	.152	.173	.064	.027	.018								
34	.239	.254	.172	.371	.211	.224	.264							
35	.140	.083	.137	.214	.139	.066	.203	.334						
36a	.286	.368	.229	.394	.267	.340	.191	.442	.234					
13	.305	.545	.482	.354	.262	.349	.166	.202	.007	.360				
18	.260	.526	.373	.348	.193	.358	.115	.159	—.014	.372	.677			
25b	.231	.437	.424	.310	.160	.245	.129	.058	—.030	.235	.603	.596		
77	.250	.426	.368	.279	.189	.273	.133	.039	—.037	.241	.586	.613	.559	

TABLE III

B-COEFFICIENTS

B(13, 18)	2.00
B(13, 18, 77)	2.14
B(13, 18, 77, 25b)	2.40
B(13, 18, 77, 25b, 2)	2.18
B(13, 18, 77, 25b, 2, 3-4)	2.20
B(2, 3-4)	1.85
B(2, 3-4, 1)	1.72
B(2, 3-4, 1, 6)	1.59
B(2, 3-4, 1, 6, 29)	1.60
B(6, 29)	1.68
B(6, 29, 28)	1.67
B(6, 29, 28, 36a)	1.43
B(6, 29, 28, 36a, 34)	1.40
B(34, 36a)	1.75
B(34, 36a, 35)	1.75
B(34, 36a, 35, 32)	1.64
B(34, 36a, 35, 32, 6)	1.30
B(34, 36a, 35, 32, 6, 29)	1.13

IV. *A Bi-factor Analysis of Fourteen Tests.* Our most recent analysis, now completed, includes the factorization of forty-four tests for 355 cases. We have used this material to select a small group of fourteen tests to illustrate the modified Bi-factor pattern. This subgroup of tests contains the only portion of the total battery presenting necessity for modification. Brief descriptions of the fourteen tests are given in *Preliminary Reports* 1 and 3.

The set of ninety-one intercorrelations is presented as Table II. The tests have been grouped, for convenience, in accordance with the interpretation of the B-coefficients of Table III.*

The B-coefficients suggest the following pattern, in which x represents a factor weight different from zero (specific factors not indicated):

TABLE IV

FIRST PATTERN PLAN

Test	Factors				
	u General	*e* Spatial	*a* Mental Speed	*m* Motor Speed	*v* Verbal
1	x	x			
2	x	x			
3-4	x	x			
6	x		x		
28	x		x		
29	x		x		
32	x			x	
34	x			x	
35	x			x	
36a	x			x	
13	x				x
18	x				x
25b	x				x
77	x				x

The values of r_{iu} for this pattern are next found by means of formula (6)†, and are recorded in Table V. The ninety-one products $r_{iu}r_{ju}$ are subtracted from the corresponding entries of Table II to give the residual correlations of Table VI. If the pattern plan is a reasonable one, the values in bold-face type will be used for the calculation of the group factors and the remaining values (in roman type) should be negligible.

*Computed by method of *Report* 7, pp. 3-5.
†See *Report* 7, pp. 8-10, for detailed outline of procedure.

TABLE V

VALUES OF r_{iu}

Test i	r_{iu}
1	.615+
2	.828
3-4	.697
6	.703
28	.489
29	.570
32	.168
34	.323
35	.090
36a	.522
13	.599
18	.532
25b	.424
77	.417

TABLE VI

RESIDUAL CORRELATIONS WITH u ELIMINATED

Test	1	2	3-4	6	28	29	32	34	35	36a	13	18	25b	77
1														
2	**.005**													
3-4	**.048**	**.085**												
6	.001	—.085	—.075											
28	.123	—.008	—.022	**.100**										
29	—.001	—.045	—.021	**.129**	**.158**									
32	—.020	.013	.056	—.054	—.055	—.078								
34	.040	—.013	—.053	.144	.053	.040	**.210**							
35	.085	.008	.074	.151	.095	.015	**.188**	**.305**						
36a	—.035	—.064	—.135	.027	.012	.042	**.103**	**.273**	**.187**					
13	—.063	.049	.064	—.067	—.031	.008	.065	.009	—.047	.047				
18	—.067	.086	.002	—.026	—.067	.055	.026	—.013	—.062	.094	**.358**			
25b	—.030	.086	.128	.012	—.047	.003	.058	—.079	—.068	.014	**.349**	**.370**		
77	—.006	.081	.077	—.014	—.015	.035	.063	—.096	—.075	.023	**.336**	**.391**	**.382**	

Examination of the residual correlations reveals a tendency toward an overlap between the mental-speed tests and the motor-speed tests with the exception of Test 32. Accordingly, a second pattern plan (Table VII) is set up. The new plan differs from the original one only through the addition of three factor weights, namely, $r_{(34)a}$, $r_{(35)a}$, and $r_{(36a)\,a}$.

TABLE VII

SECOND PATTERN PLAN

Test	Factor				
	u'	e	a	m	v
1	x	x			
2	x	x			
3-4	x	x			
6	x		x		
28	x		x		
29	x		x		
32	x			x	
34	x		x	x	
35	x		x	x	
36a	x		x	x	
13	x				x
18	x				x
25b	x				x
77	x				x

In order to compute the revised values, it must be remembered that in the Bi-factor method each r_{iu} is evaluated from triplets of tests no two of which are assumed to contain the same group factor. While the Bi-factor method is one which most readily lends itself to routine calculation, the same principle of selecting tests in threes having no common group factor may be applied to more complicated types of pattern. In the present example, it now becomes necessary

TABLE VIII

VALUES OF $r_{iu'}$

Test i	$r_{iu'}$
1	.621
2	.891
3-4	.751
6	.584
28	.417
29	.511
32	.168
34	.257
35	.076i
36a	.480
13	.632
18	.557
25b	.463
77	.457

to eliminate from the original formulae for r_{iu} each set of three tests which contains any of the following pairs:

6 - 34	28 - 34	29 - 34
6 - 35	28 - 35	29 - 35
6 - 36a	28 - 36a	29 - 36a

The adjustment may readily be made on the work-sheet which was used to calculate the coefficients, r_{iu}. The new coefficients, $r_{iu'}$, are given in Table VIII. Products and residual correlations are computed as before, the latter appearing in Table IX.

TABLE IX

Residual Correlations with u' Eliminated

Test	1	2	3-4	6	28	29	32	34	35	36a	13	18	25b	77
1														
2	-.039													
3-4	.011	-.007												
6	.070	-.023	-.024											
28	.165	.025	.006	.200										
29	.033	-.028	-.008	.232	.224									
32	-.021	.002	.047	-.034	-.043	-.068								
34	.079	.025	-.021	.221	.104	.093	.221							
35	.140	.083	.137	.214	.139	.066	.203	.334						
36a	-.012	-.060	-.131	.114	.067	.095	.110	.319	.234					
13	-.087	-.018	.007	-.015	-.002	.026	.060	.040	.007	.057				
18	-.086	.030	-.045	.023	-.039	.073	.021	.016	-.014	.105	.325			
25b	-.057	.024	.076	.040	-.033	.008	.051	-.061	-.030	.013	.310	.338		
77	-.034	.019	.025	.012	-.002	.039	.056	-.078	-.037	.022	.297	.358	.347	

In Table IX the two largest positive residual correlations among those expected to approximate zero are $\underline{r}_{(1)(28)}$ and $\underline{r}_{(1)(35)}$. It is not unlikely that Test 1, a fairly easy, timed test, should measure the mental-speed factor, a. Table IX indicates further that the factor e probably is not significant for this pattern. Two of the three residual correlations assumed to contain this factor are negative. These two revisions are made in the third pattern plan (Table X). It should be noted in passing that all the revising was done at once in the larger battery of forty-four tests. No third plan was necessary. The fact that not all the overlapping was apparent in Table VI may be attributed to the unreliability of pattern weights which are estimated from only a small number of tests.

TABLE X

THIRD PATTERN PLAN

Test	Factor u''	a	m	v
1	x	x		
2	x			
3-4	x			
6	x	x		
28	x	x		
29	x	x		
32	x		x	
34	x	x	x	
35	x	x	x	
36a	x	x	x	
13	x			x
18	x			x
25b	x			x
77	x			x

The Bi-factor method, together with certain modifications, is again employed for the computation of the new general factor weights, $r_{iu''}$. A work-sheet corresponding to Tables 4 and 5 on page 18 of *Report 7* may be used, with the tests arranged in the following four groups.

	Tests
1.	2
2.	3-4
3.	$a+m = 1, 6, 28, 29, 32, 34, 35, 36a$
4.	$v = 13, 18, 25b, 77$

Since $r_{(32)a}$ is assumed to be zero, a number of additional sets of three tests are available:

1 — 32 — 2	28 — 32 — 2
1 — 32 — 3-4	28 — 32 — 3-4
1 — 32 — v	28 — 32 — v
6 — 32 — 2	29 — 32 — 2
6 — 32 — 3-4	29 — 32 — 3-4
6 — 32 — v	29 — 32 — v

The resulting coefficients appear in Table XI. Products and residual correlations are computed for the third time. In table XII there are no further overlaps or adjustments apparent.

TABLE XI

VALUES OF $r_{iu''}$

Test i	$r_{iu''}$
1	.533
2	.896
3-4	.768
6	.546
28	.381
29	.481
32	.175+
34	.237
35	.033i
36a	.436
13	.658
18	.576
25b	.500
77	.485—

TABLE XII

RESIDUAL CORRELATIONS WITH u'' ELIMINATED

Test	1	2	3-4	6	28	29	32	34	35	36a	13	18	25b	77
1														
2	.036													
3-4	.068	—.026												
6	**.142**	.008	—.004											
28	**.221**	.056	.026	**.236**										
29	**.094**	—.004	.007	**.267**	**.254**									
32	—.010	—.005	.039	—.032	—.040	—.066								
34	**.113**	.042	—.010	**.242**	**.121**	**.110**	**.223**							
35	**.140**	.083	.137	**.214**	**.139**	**.066**	**.203**	**.334**						
36a	**.054**	—.023	—.106	**.156**	**.101**	**.130**	**.115**	**.339**	**.234**					
13	—.046	—.045	—.023	—.005	.011	.033	.051	.046	.007	.073				
18	—.047	.010	—.069	.034	—.026	.081	.014	.022	—.014	.121	**.298**			
25b	—.035	—.011	.040	.037	—.030	.005	.041	—.060	—.030	.017	**.274**	**.308**		
77	—.009	—.009	—.004	.014	.004	.040	.048	—.076	—.037	.030	**.267**	**.334**	**.317**	

The third pattern plan having been accepted as final, the group factor loadings are computed from the bold-face residual correlations of Table XII. In the case of the a factor, care must be taken not to include any set of three tests which includes two of Tests 34, 35, and 36a. The m factor weights are computed from the residual correlations with u'' and a eliminated. The v factor weights are computed from the residual correlations of Table XII. The final group factor residual correlations are given in Table XIII.

TABLE XIII

RESIDUAL CORRELATIONS WITH u''

AND GROUP FACTORS ELIMINATED

Tests	1	6	28	29	32	34	35	36a
1								
6	—.044							
28	.063	—.055						
29	—.034	.031	.054					
32								
34	.012	.056	—.037	—.018	—.004			
35	.044	.036	—.011	—.056	.035	—.031		
36a	—.022	.015	—.018	.033	—.024	.040	—.004	

Tests	13	18	25b	77
13				
18	.009			
25b	.004	—.013		
77	—.012	.003	.008	

Table XIV is the completed factor pattern, describing the fourteen tests in terms of one general factor, three group factors, and specifics. The general factor turns out here to be the function meas-

TABLE XIV

FACTOR PATTERN

Test	Factor					
	u''	a	m	v	s_i	t_i
1	.533	.318			.710	.332
2	.896				.265i	.517
3-4	.768				.530	.360
6	.546	.586			.461	.382
28	.381	.496			.695+	.354
29	.481	.403			.738	.249
32	.175+		.377		.794	.444
34	.237	.318	.603		.570	.393
35	.033i	.303	.446		.709	.455+
36a	.436	.240	.370		.660	.424
13	.658			.494	.411	.393
18	.576			.586	.296	.487
25b	.500			.547	.407	.534
77	.485—			.564	.548	.382
Total Variance	3.877	1.103	.842	1.205—	4.571	2.401
Per cent Variance	27.69	7.88	6.01	8.61	32.65	17.15+

ured by Professor Spearman's Visual Perception Tests, two of which measure none of the group factors by this analysis.

The frequency distribution of the ninety-one final residual correlations is presented in the last column of Table XV in order that the value .6745 σ might be compared with the probable error of a zero correlation for 355 cases, as noted in Section II. The roman entries of Tables VI and IX are also included to show the extent to which proper adjustment of the pattern plan reduces the variability of the residual correlations. The factorization of the present small set of tests may have been over-refined in our effort to illustrate methods of modifying the Bi-factor pattern, as .6745 $\sigma = .0285 < .0358$.

TABLE XV

FREQUENCY DISTRIBUTIONS OF FINAL RESIDUAL CORRELATIONS

Value	Tables VI	IX	XII	XIII	XII + XIII
.150 - .169	1	1			
.130 - .149	1	2	1		1
.110 - .129	2	....	1		1
.090 - .109	2	1	....		....
.070 - .089	6	5	3		3
.050 - .069	7	4	3	3	6
.030 - .049	6	6	12	6	18
.010 - .029	8	12	7	2	9
—.010 - .009	9	8	14	6	20
—.030 - —.011	9	10	8	7	15
—.050 - —.031	5	7	7	4	11
—.070 - —.051	10	4	3	2	5
—.090 - —.071	5	3	1		1
—.110 - —.091	1	....	1		1
—.130 - —.111	....	....			
—.150 - —.131	1	1			
Total	73	64	61	30	91
Mean	.0053	.0082	.0077	.0008	.0054
S.D.	.0616	.0558	.0450	.0354	.0422
.6745(S.D.)	.0416	.0376	.0304	.0239	.0285
PE of zero r	.0358	.0358	.0358	.0358	.0358

The Bi-factor analysis illustrated above is not only very simple, but the calculation is relatively easy as compared with other methods. The total time for computation, done by one person, was less than ten hours for the present example.