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COMPARISON OF TWO FACTORIAL ANALYSES

KARL J. HOLZINGER AND HARRY H. HARMAN
University of Chicago

A Bi-factor analysis is made of Professor Thurstone's battery of fifty-seven tests employing his tetrachoric correlations. Although this analysis is made entirely independent of his multiple factor analysis, a very close agreement is found between the group factors obtained here and Thurstone's verbal descriptions previously published.

1. *Introduction*

Professor Thurstone has recently described[1] some of his preliminary analyses of a battery of fifty-seven tests given to 240 students. Before this description appeared, he was kind enough to furnish us with his tetrachoric correlations for independent analysis by the Bi-factor method. The striking agreement between our pattern and the verbal description of the factor allocations by Professor Thurstone makes a more complete numerical comparison interesting and significant. We therefore propose to present our analysis to be compared later with one or more of his factorizations.

The correlations employed are of the tetrachoric form. Strictly speaking, the factorial algebra does not apply to such coefficients, since it has been worked out in terms of product-moment correlations. The tetrachoric values, however, may be regarded as rough approximations to the product-moment values, and we shall proceed with the analysis as if they were such coefficients. The sampling error for the tetrachoric values is of course larger than for the Pearson coefficients, and this will be allowed for in testing the final residuals.

In Table I we have presented the complete set of intercorrelations. These have been given to two decimals which is adequate for the size of the sample and sampling error of the coefficients. All subsequent work will be carried to two places, the decimal point being omitted throughout to save space in the tables when there is no ambiguity.

[1]L. L. Thurstone, "The Factorial Isolation of Primary Abilities," *Psychometrika*, 1936, **1**, No. 3, pp. 175-182. The present analysis was made immediately after the appearance of this article. The publication of our article was properly postponed until the appearance of Professor Thurstone's numerical solution.

TABLE I Intercorrelations of Fifty-Seven Variables

	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
4																													
5	83																												
6	54	63																											
7	49	61	76																										
8	19	15	46	31																									
9	47	42	37	37	10																								
10	47	61	52	57	05	49																							
11	58	64	55	62	15	43	73																						
12	35	42	29	31	30	16	51	57																					
13	27	34	31	42	31	37	52	46	48																				
14	37	42	64	59	33	28	60	60	37	28																			
15	14	28	24	22	22	37	37	35	46	56	22																		
16	46	54	48	48	19	37	72	70	59	46	46	48																	
17	13	15	36	34	31	06	26	32	16	21	25	06	09																
18	10	25	48	34	59	—10	26	43	40	28	34	19	18	61															
19	28	46	38	25	40	—06	28	33	37	25	31	07	25	51	47														
20	00	12	37	21	42	—16	08	20	22	21	28	12	06	48	68	62													
21	32	46	58	46	51	21	43	62	43	37	37	28	37	61	61	67	59												
22	20	34	56	31	54	—07	22	29	31	30	40	15	18	40	56	64	61	58											
23	22	27	42	30	42	15	32	32	35	23	31	22	20	42	56	54	63	58	57										
24	28	40	47	33	51	05	23	33	37	33	15	15	20	50	42	53	53	74	62	55									
25	28	34	44	30	40	21	32	47	37	40	18	18	34	45	45	43	33	61	42	36	61								
26	25	33	71	42	22	15	22	46	25	12	40	10	37	40	45	29	21	40	25	43	30	21							
27	17	09	33	25	40	09	13	22	06	25	37	25	00	46	61	38	57	49	49	43	40	24	22						
28	21	35	58	40	53	12	35	38	31	15	40	22	24	45	47	43	34	70	42	50	42	48	42	42					
29	43	32	44	20	31	05	30	30	34	18	28	19	18	40	42	35	33	55	42	41	52	42	00	43	49				
30	19	27	45	43	40	07	42	30	30	25	40	28	46	58	48	39	58	48	48	45	47	37	22	40	40	45			
31	01	11	18	18	19	06	10	—06	12	31	21	16	12	27	21	17	28	06	19	14	11	11	00	34	30	18	64		
32	26	25	25	28	10	18	40	07	28	28	37	25	26	12	14	17	20	24	21	16	15	07	02	21	19	19	57	62	
33	—03	08	28	17	10	17	18	15	35	22	32	26	22	26	26	15	23	23	22	11	21	04	03	29	26	29	62	62	67
34	18	16	40	15	12	06	29	26	25	25	28	16	30	22	37	22	27	24	30	27	24	15	19	40	28	42	51	54	53
35	40	43	41	27	37	11	30	30	29	27	25	22	37	18	36	36	20	40	33	30	35	42	16	22	43	52	52	40	31
36	26	31	20	24	—03	06	30	30	11	06	34	—10	28	40	26	33	27	30	33	11	20	17	00	20	18	30	36	06	15
37	40	38	46	43	52	10	37	18	29	26	44	26	35	26	36	42	17	33	41	33	48	30	16	34	50	42	40	35	38
38	32	34	31	33	16	09	20	35	22	00	25	16	18	28	27	22	23	30	18	34	30	28	10	28	30	43	43	28	37
39	34	31	50	37	30	16	42	42	34	16	35	31	29	50	39	34	34	57	52	42	53	52	16	25	55	56	66	29	38
40	59	66	42	48	28	21	43	50	34	40	42	30	56	11	27	45	30	42	30	33	48	51	10	22	33	39	40	21	33
41	57	64	64	64	46	40	60	60	40	46	59	28	61	46	46	49	42	54	40	46	49	46	37	29	52	43	60	25	34
42	58	54	48	43	43	22	43	42	31	40	38	22	52	13	29	29	16	26	29	32	39	35	22	16	20	41	39	18	22
43	52	60	68	58	53	18	46	50	53	34	46	28	54	45	40	55	50	56	53	50	72	48	40	34	49	47	66	30	28
44	51	59	60	59	51	21	48	62	45	37	46	31	47	37	52	49	28	62	28	43	50	57	34	46	57	57	47	16	16
45	52	51	66	48	52	28	44	39	35	43	31	19	40	37	51	47	48	50	54	52	55	40	35	43	51	54	50	28	29
46	43	25	30	18	—02	28	22	32	29	18	28	19	27	10	12	10	—04	32	33	30	20	20	21	02	24	20	26	14	12
47	43	53	31	31	06	42	57	40	29	48	31	45	54	03	—03	07	03	37	19	30	33	28	28	16	21	28	35	18	21
48	25	38	32	10	21	13	28	21	07	13	29	22	17	04	20	12	16	20	32	27	20	10	10	10	39	36	41	38	07
49	27	48	40	30	19	11	32	35	38	24	30	31	40	18	15	31	05	42	33	18	36	30	16	03	28	42	30	05	12
50	22	28	33	31	22	08	30	36	25	19	21	15	21	08	20	32	15	42	23	20	29	23	20	—10	23	35	17	—01	09
51	07	12	29	28	02	16	25	28	29	03	48	04	21	25	26	03	06	19	10	12	10	16	28	12	21	10	32	16	10
52	51	62	51	45	15	43	69	46	25	31	37	16	49	03	16	34	—04	37	25	25	25	19	25	12	29	22	17	12	40
53	—05	02	25	09	21	00	13	13	21	21	34	22	09	48	42	34	59	37	53	40	33	15	03	40	33	42	50	24	22
54	35	31	36	09	25	21	35	23	25	21	31	34	30	05	31	32	11	23	20	30	35	02	06	29	12	29	05	05	24
55	40	46	61	59	51	25	49	40	38	37	51	31	51	18	36	49	37	37	42	33	33	15	25	46	28	33	49	22	40
56	40	42	48	45	16	31	57	43	46	40	34	37	46	08	18	16	02	33	18	15	05	24	19	07	18	11	18	09	37
57	42	53	48	46	30	28	69	49	46	48	45	53	61	11	29	37	21	33	28	28	28	28	12	16	31	32	32	19	42
58	57	75	38	46	00	50	66	76	35	31	30	20	46	11	05	35	12	42	03	29	30	15	20	10	10	09	16	14	19
59	15	26	38	29	12	29	19	25	22	10	31	07	34	07	04	10	00	—06	10	—02	08	10	22	00	10	17	17	26	10
60	72	76	67	67	28	25	61	77	62	48	43	40	68	28	35	37	27	46	22	21	37	60	40	11	46	40	35	12	12

TABLE I (continued) Intercorrelations of Fifty-Seven Variables

	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
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33																												
34	62																											
35	47	56																										
36	14	11	13																									
37	30	41	68	23																								
38	52	57	48	09	43																							
39	48	62	58	30	72	64																						
40	10	27	45	28	55	37	50																					
41	16	28	40	40	54	16	48	54																				
42	11	29	36	20	42	32	46	78	40																			
43	33	48	36	33	58	42	62	48	72	47																		
44	10	28	26	31	43	46	54	49	67	48	69																	
45	38	45	60	27	46	33	50	42	50	51	66	43																
46	11	14	30	04	24	06	32	23	19	32	17	18	23															
47	20	19	23	12	30	10	22	50	31	43	48	31	37	43														
48	30	26	39	04	46	40	37	20	16	37	44	22	20	42	43													
49	04	09	23	08	20	18	23	33	25	38	27	31	30	36	48	20												
50	02	—01	13	11	18	15	17	33	31	23	42	29	13	26	42	36	40											
51	16	10	10	19	10	22	19	—07	25	06	28	34	03	13	19	25	19	19										
52	29	28	22	06	34	37	34	46	43	48	40	34	38	19	48	34	38	25	31									
53	30	30	27	19	17	22	29	21	28	07	37	16	39	15	03	26	20	24	12	—03								
54	04	20	32	04	27	18	13	23	34	13	23	28	33	07	28	08	20	06	12	34	21							
55	22	40	39	11	38	28	29	33	59	41	62	59	54	12	25	13	40	23	28	43	37	56						
56	26	30	23	20	39	40	46	45	28	52	24	31	33	27	33	13	33	14	10	46	—08	30	46					
57	26	43	33	28	39	37	40	62	46	64	43	53	43	18	43	25	28	28	00	46	28	37	61	63				
58	12	13	16	—03	18	30	38	27	49	24	31	22	33	18	32	—10	28	19	12	56	—22	38	38	60	36			
59	24	22	17	—06	21	25	06	07	16	16	20	19	11	13	03	16	13	—12	46	31	—10	06	10	17	07	32		
60	10	30	37	24	31	46	52	58	62	62	62	63	43	28	45	38	37	37	37	57	09	26	59	62	62	82	10	

2. *Properties of the B-Coefficient*

Before proceeding further with the analysis we may define and note some characteristics of the B-coefficient.[1] Briefly, this coefficient is the average of the intercorrelations of a certain group of tests divided by their average correlation with all remaining tests. It gives a measure of the extent to which this group of tests belong together in ascertaining an underlying factor.

We shall define the B-coefficient more rigorously now, and in so doing shall use the following notation:

$k \equiv$ number of tests in the argument of B;

$n \equiv$ total number of tests;

Roman subscripts run over the range $1, 2, \cdots, k$;

Greek subscripts run over the range $1, 2, \cdots, n$;

$x_j \equiv j^{\text{th}}$ test in the argument of B (not test number j);

$x_\alpha \equiv \alpha^{\text{th}}$ test in the total ordered group of tests (not test number α);

$a \equiv \sum_{i \neq j} r_{x_i x_j} \equiv$ sum of intercorrelations of tests in B;

$c \equiv \sum_{i \neq \alpha} r_{x_i x_\alpha} - 2 \sum_{i \neq j} r_{x_i x_j} \equiv$ sum of remaining correlations of tests in argument of B with all other tests.

The B-coefficient is then defined as

$$B(x_1, x_2, \cdots, x_j, \cdots, x_k) \equiv \frac{\frac{a}{C_2^k}}{\frac{c}{k(n-k)}} = \frac{2(n-k) \sum_{i \neq j} r_{x_i x_j}}{(k-1)\left[\sum_{i \neq \alpha} x_i x_\alpha - 2 \sum_{i \neq j} x_i x_j\right]}.$$

Since the B-coefficient is the ratio of two averages its properties may be studied by means of them. The average of the intercorrelations tends to decrease as the number of tests in B increases since the tests are added on the basis of highest correlation with tests already in the argument of B. Similarly, the average of the remaining correlations tends to decrease with an increase in k. The decrease in the average of intercorrelations, however, is relatively greater than that of the remaining correlations, and hence the B-coefficient decreases in general.

An exception to this may occur with the addition of a test to the argument of B which has relatively high intercorrelations with the

[1] First introduced in *Preliminary Report on Spearman-Holzinger Unitary Trait Study*, No. 7. Prepared at the Statistical Laboratory, Department of Education, University of Chicago, 1936.

TABLE II

ALLOCATION OF TESTS INTO GROUPS

$B(x_1, x_2, \cdots, x_j, \cdots, x_k)$	$100B$	Notes
B(4,5)	232	
B(4,5,60)	206	
B(4,5,60,58)	219	
B(4,5,60,58,11)	211	
B(4,5,60,58,11,10)	201	
B(4,5,60,58,11,10,16)	197	
B(4,5,60,58,11,10,16,52)	194	
B(4,5,60,58,11,10,16,52,57)	188	
B(4,5,60,58,11,10,16,52,57,7)	184	
B(4,5,60,58,11,10,16,52,57,7,6)	179	
B(4,5,60,58,11,10,16,52,57,7,6,26)	172	(1)
B(4,5,60,58,11,10,16,52,57,7,6,45)	171	(2)
B(4,5,60,58,11,10,16,52,57,7,6,56)	180	
B(4,5,60,58,11,10,16,52,57,7,6,56,55)	178	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,59)	168	(1)
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14)	175	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,54)	170	(2)
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,45)	169	(2)
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12)	171	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13)	168	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13,15)	165	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13,15,9)	166	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13,15,9,26)	161	(1)
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13,15,9,54)	163	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13,15,9,54,45)	159	
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13,15,9,54,45,26)	155	(3)
B(4,5,60,58,11,10,16,52,57,7,6,56,55,14,12,13,15,9,54,45,59)	154	(3)
B(40,42)	222	
B(40,42,37)	167	(4)
B(40,42,41)	152	(4)
B(21,24)	194	
B(21,24,19)	189	
B(21,24,19,28)	170	(2)
B(21,24,19,22)	186	
B(21,24,19,22,18)	175	(2)
B(21,24,19,22,23)	182	
B(21,24,19,22,23,20)	193	
B(21,24,19,22,23,20,18)	190	
B(21,24,19,22,23,20,18,17)	191	
B(21,24,19,22,23,20,18,17,8)	188	
B(21,24,19,22,23,20,18,17,8,53)	186	
B(21,24,19,22,23,20,18,17,8,53,27)	170	
B(21,24,19,22,23,20,18,17,8,53,27,28)	186	
B(21,24,19,22,23,20,18,17,8,53,27,28,25)	181	
B(21,24,19,22,23,20,18,17,8,53,27,28,25,43)	177	(1)
B(21,24,19,22,23,20,18,17,8,53,27,28,25,29)	178	(5)
B(21,24,19,22,23,20,18,17,8,53,27,28,25,29,43)	174	(6)
B(41,43)	164	
B(41,43,44)	164	
B(41,43,44,30)	155	(7)
B(41,43,44,39)	150	(7)

TABLE II (continued)

ALLOCATION OF TESTS INTO GROUPS

$B(x_1, x_2, \cdots, x_j, \cdots, x_k)$	$100B$	Notes
B(37,39)	196	
B(37,39,30)	159	(2)
B(37,39,35)	189	
B(37,39,35,30)	167	(2)
B(37,39,35,38)	179	
B(37,39,35,38,34)	181	
B(37,39,35,38,34,30)	170	(2)
B(37,39,35,38,34,33)	183	
B(37,39,35,38,34,33,32)	180	
B(37,39,35,38,34,33,32,31)	186	
B(37,39,35,38,34,33,32,31,30)	186	
B(37,39,35,38,34,33,32,31,30,36)	165	(8)
B(47,49)	173	
B(47,49,46)	169	
B(47,49,46,48)	159	(2)
B(47,49,46,50)	165	
B(47,49,46,50,51)	127	(1)
B(47,49,46,50,48)	161	
B(47,49,46,50,48,51)	140	(9)

NOTES ON TABLE II

(1) Rejected because of large drop in B.
(2) Test omitted temporarily; it reappears in group later.
(3) Tests 26 and 59 cause a sufficient drop in B for their rejection from this group. Furthermore these tests are not of the same general character as those in the "verbal" group, namely, tests 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 45, 52, 54, 55, 56, 57, 58, and 60.
(4) "Logical reasoning" group composed of doublet, 40 and 42. Tests 37 and 41 rejected because of great difference in B.
(5) Test 29 retained because of its spatial character which is in harmony with the remaining tests in the group.
(6) Test 43 rejected because of drop in B and its composition. The "spatial" group consists of tests 8, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29 and 53.
(7) Tests 30 and 39 omitted because of drop in B and their numerical character. The "analogies" group consists of tests 41, 43, 44.
(8) "Arithmetical" group composed of tests 30, 31, 32, 33, 34, 35, 37, 38 and 39. Test 36 rejected because of wide difference in B.
(9) Although test 51 seems to be of the same general nature as the other tests in the group, the sudden drop in B does not warrant its retention within the group. Hence, the "memory" group consists of tests 46, 47, 48, 49 and 50.

preceding tests, but a low total of all correlations. In this case the decrease in the average of the intercorrelations is relatively smaller than that of the remaining correlations, and B increases.[1] Similar

[1] A good example of this phenomenon is found in Section 3 where the addition of test 20 to the "spatial" group increases B. From Table II it will be observed that the B-coefficient rises from 182 to 193 upon the addition of test 20.

reasoning accounts for the fact that a test can be rejected from a group temporarily and then appear in the group later.[2]

As the number of tests in B increases, the decrease in the above averages becomes less and these averages tend toward stability. A consequence of this is that an actual difference between two successive B values has a greater relative importance as the number of tests in B increases.

3. *The Preliminary Allocation of Tests to Groups*

The Bi-factor analysis is begun by computing the B-coefficients. In this analysis we have used $100B$ to avoid decimals. The values of these coefficients with notes are presented in Table II.

We begin the computation of B-coefficients by selecting the largest correlation from Table I. This yields $100B(4, 5) = 232$. Next test 60 is selected because it has a higher correlation with test 4 or 5 than any other in the table. The work is continued in this manner until test 26 is added. A drop of seven points in the coefficient is considered sufficient reason for dropping this test, and similarly in the case of test 45, although the latter reappears in the group near the end. The group is closed with the rejection of tests 26 and 59 as explained in the note because of the drop in B and the nature of these two tests. The first group is tentatively described as "verbal".

A new group is now formed using tests 40 and 42. When other tests are added to this group the drop in B is so great that we regard the "logical reasoning" factor as a "doublet" and proceed to another group. This is begun with tests 21 and 24 and continued as described in the notes until test 43 is rejected. The group appears to be "spatial".

The next group starts with tests 41 and 43, but other tests beyond 44 are rejected because of their nature and the drop in B values. The name "analogies" has been temporarily used here. The remaining two groups are identified as shown by the table and notes. They may be called "arithmetical" and "memory" respectively.

From the preliminary analysis of the B-coefficients and reference to the nature of the tests themselves, all tests have been allocated to one of six groups with the exception of tests 26, 36, 51 and 59.

4. *The Modified Bi-Factor Pattern*

After the preliminary groups of tests have been determined, the next step in the analysis is the calculation of the weights for the gen-

[2]The rejection of test 28 as the fourth test in the "spatial" group and its retention later as the twelfth test is an example to be found in Section 3.

TABLE III

General Factor Loadings by Preliminary Analysis

Test	u_1	Test	u_1	Test	u_1
4	54	23	52	42	64
5	65	24	58	43	87
6	81	25	53	44	77
7	63	26	44	45	77
8	45	27	38	46	36
9	25	28	59	47	50
10	60	29	59	48	43
11	64	30	69	49	46
12	57	31	31	50	37
13	48	32	38	51	33
14	66	33	35	52	53
15	39	34	46	53	32
16	58	35	57	54	34
17	41	36	34	55	64
18	51	37	64	56	44
19	54	38	49	57	61
20	36	39	69	58	36
21	67	40	68	59	26
22	54	41	81	60	70

eral factor u_1. This we have done as described in Report 7, and their values are given in Table III. We then computed the residual correlations,

$$\bar{r}_{x_i x_j} = r_{x_i x_j} - r_{x_i u_1} r_{x_j u_1}.$$

In order to save space the table of these residuals has been omitted here, and instead, portions of this table will be presented when necessary. An examination of the residual correlations shows the necessity for modifying the original Bi-factor Pattern.

Most of the groups appear to have been properly selected because of the small residuals with other tests and larger clusters among themselves. The residuals for the "verbal" group, however, indicate that a re-allocation of tests is necessary. In Table IV we present the residual intercorrelations among the "verbal" group.

First, tests 14 and 45 have negligible residuals with the "verbal" group and are therefore omitted. Next we note that tests 54 and 55 have a high residual intercorrelation, but small irregular correlations with the remaining tests in the "verbal" group. Hence, we assume that the "doublet" is measuring some other factor such as "rhythm". Finally we observe that tests 9, 10, 11, 12, 13 and 15 have high intercorrelations. Tests 9, 10 and 11 also have appreciably high intercorrelations with the remaining tests in the "verbal" group, while tests 12, 13 and 15 have low intercorrelations with this group. We assume,

therefore, that tests 9, 10, 11, 12, 13 and 15 form another group, say "completion", and that tests 9, 10 and 11 measure the "verbal" factor also.

The second factor plan as far as the "verbal" group is concerned may then be written in the form as shown in Table V. The crosses indicate appreciable factor weightings.

The only other necessary revision in the factor pattern arises in the introduction of a new group factor which we may call "imagination". The residual correlations among tests 6, 14, 26, 51 and 59 have been reproduced in the small Table VI. These values are all positive and are relatively high as compared with their correlations with the remaining tests.

None of these tests except 6 has been allocated at this stage to another group by the B-coefficients. We observe again the residual correlations of test 6 with the tests in the "verbal" group and find that they are negligible and so consider test 6 to measure u_1 and "imagination".

The final factor plan thus includes seven group factors which we shall designate as follows:

$v =$ "verbal",
$i =$ "imagination",
$s =$ "spatial",
$c =$ "completion",
$m =$ "arithmetical",
$a =$ "analogies",
$o =$ "memory".

In addition to these groups we have two "doublets":

$l =$ "logical reasoning",
$r =$ "rhythm".

Under the new hypothesis every test has been assigned to some group except tests 36 and 45. The assumption on these tests is that they measure only u_1 and specifics.

From this new allocation of tests we proceed to recalculate the weights of the general factor u_1. The values are given in the final factor plan of Table VII. It will be observed that the values from the first column of this table are in close agreement with those of Table III.

Residuals with the general factor removed are given in Table VIII. The tests have been arranged so that the groups may be identified more conveniently.

TABLE IV
Residual Intercorrelations of Preliminary "Verbal" Group

	4	5	6	7	9	10	11	12	13	14	15	16	45	52	54	55	56	57	58	60
4																				
5	48																			
6	10	10																		
7	15	20	25																	
9	33	26	17	21																
10	15	22	03	19	34															
11	23	22	03	22	27	35														
12	04	05	—17	—05	02	17	21													
13	01	03	—08	12	25	23	15	21												
14	01	—01	11	17	11	20	18	—01	—04											
15	—07	03	—08	—03	27	14	10	24	37	—04										
16	15	16	01	11	22	37	33	26	18	08	25									
45	10	01	03	—01	09	—02	—10	—09	06	—20	—11	—05								
52	22	28	08	12	30	37	12	—05	06	02	—05	18	—03							
54	17	09	08	—12	12	15	01	06	05	09	21	10	07	16						
55	05	04	09	19	09	11	—01	02	06	09	06	14	05	09	34					
56	16	13	12	17	20	31	15	21	19	05	20	20	—01	23	15	18				
57	09	13	—01	08	13	32	10	11	19	05	29	26	—04	14	16	22	36			
58	38	52	09	23	41	44	53	14	14	06	06	25	05	37	26	15	44	14		
60	34	30	10	23	07	19	32	22	14	—03	13	27	—11	20	02	14	31	19	57	

TABLE V
New Hypothesis on Original "Verbal" Group

Tests	u_1	v	c	r
4	x	x		
5	x	x		
6	x	x		
7	x	x		
9	x	x	x	
10	x	x	x	
11	x	x	x	
12	x		x	
13	x		x	
14	x			
15	x		x	
16	x	x		
45	x			
52	x	x		
54	x			x
55	x			x
56	x	x		
57	x	x		
58	x	x		
60	x	x		

TABLE VI
Residual Intercorrelations Among Tests in "Imagination" Group

	6	14	25	51	59
6					
14	11				
26	35	11			
51	02	26	13		
59	17	14	11	37	

TABLE VII

FACTOR PATTERN

Test	Factors with Coefficients										Specifics	Total Variance
	u_1	v	i	s	c	m	l	a	o	r		
4	.56	.47									.68	1.00
5	.66	.54									.52	1.00
6	.82		.35								.45	1.00
7	.65	.28									.71	1.01
8	.45			.42							.79	1.00
9	.27	.50			.22						.79	1.00
10	.62	.54			.28						.50	1.00
11	.64	.51			.22						.53	1.00
12	.56				.41						.72	1.00
13	.48				.68						.55	1.00
14	.66		.35								.66	.99
15	.39				.60						.70	1.00
16	.59	.38									.71	1.00
17	.40			.56							.73	1.01
18	.51			.58							.64	1.01
19	.54			.47							.70	1.00
20	.36			.72							.59	1.00
21	.67			.55							.50	1.00
22	.53			.54							.65	1.00
23	.52			.48							.71	1.00
24	.57			.50							.65	1.00
25	.52			.31							.80	1.00
26	.41		.48								.80	.99
27	.38			.52							.76	.99
28	.58			.36							.73	1.00
29	.58			.27							.77	1.00
30	.68					.41					.61	1.00
31	.30					.62					.72	.99
32	.39					.54					.75	1.01
33	.35					.74					.57	1.00
34	.46					.64					.62	1.01
35	.57					.42					.71	1.01
36	.34										.94	1.00
37	.63					.27					.73	1.00
38	.48					.43					.76	.99
39	.68					.42					.60	1.00
40	.68						.58				.45	1.00
41	.81							.18			.56	1.00
42	.64						.58				.50	1.00
43	.86							.11			.50	1.00
44	.77							.27			.58	1.00
45	.75										.66	1.00
46	.36								.48		.80	1.00
47	.51								.57		.64	.99
48	.42								.36		.83	.99
49	.47								.33		.82	1.00
50	.36								.46		.81	1.00
51	.30		.48								.82	.99
52	.54	.42									.73	1.00
53	.32			.45							.83	.99
54	.40									.53	.75	1.00
55	.70									.53	.48	1.00
56	.46	.49									.74	1.00
57	.63	.30									.72	1.01
58	.39	.76									.52	1.00
59	.23		.49								.84	1.00
60	.71	.62									.33	1.00
Total Variance	17.18	3.01	.90	3.41	1.17	2.41	.67	.12	1.01	.56	26.55	56.99
Per Cent Variance	30.15	5.28	1.58	5.98	2.05	4.23	1.18	.21	1.77	.98	46.59	100.00

TABLE VIII

FINAL RESIDUAL CORRELATIONS

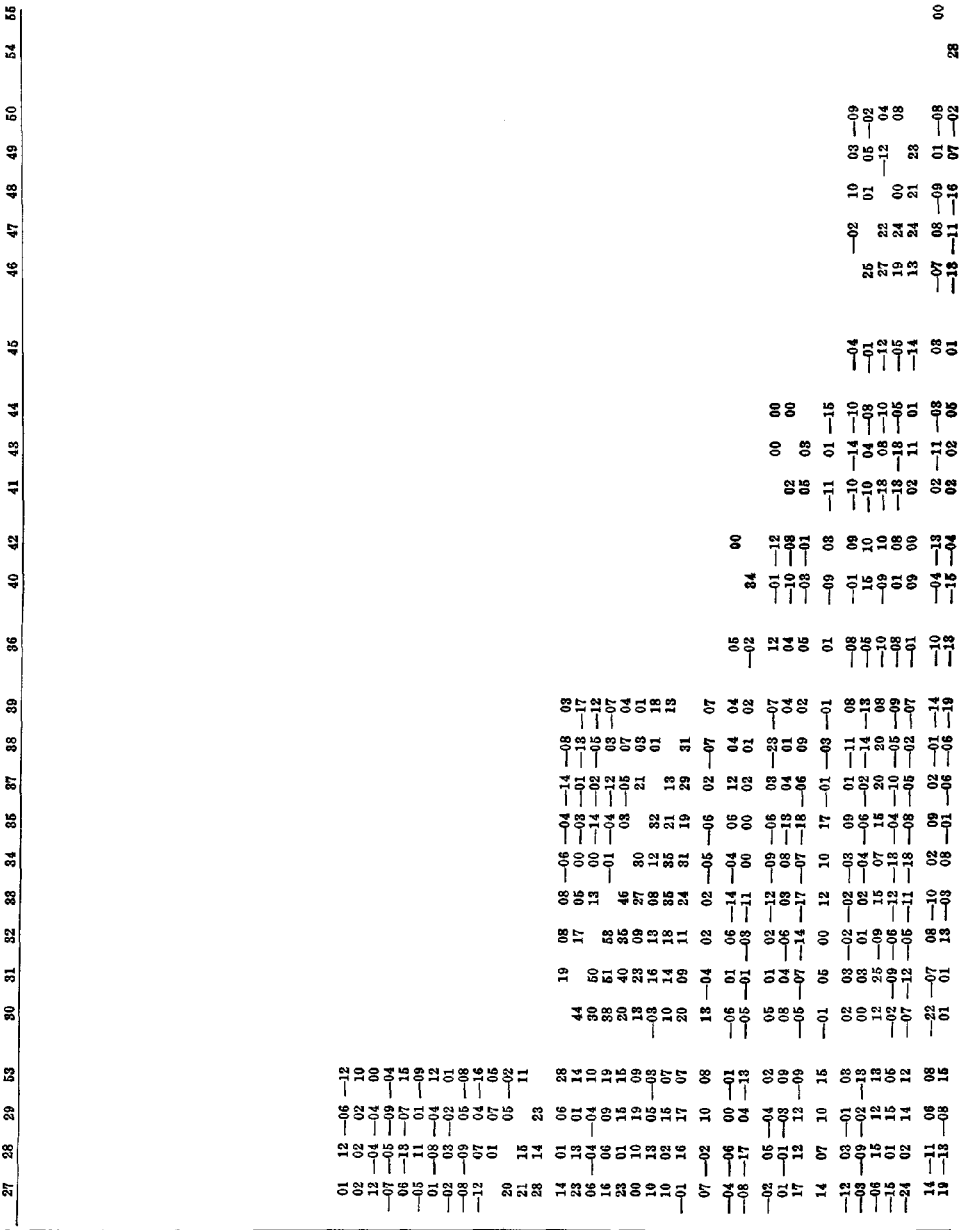
Below diagonal: residual correlations with u_1 eliminated.
Above diagonal: residual correlations with respective factors eliminated

	4	5	7	9	10	11	12	13	15	16	52	56	57	58	60
4		21	00	08	—13	—02				—05	01	—09	—07	—01	03
5	46		03	—03	—09	—06				—06	03	—14	—05	08	—04
7	13	18		05	02	06				—01	—02	01	—03	00	04
9	32	24	19		—01	—05	—08	09	13	02	07	—06	—04	01	—25
10	12	20	17	32		—01	06	03	—04	14	13	02	14	01	—16
11	22	22	20	26	33		12	00	—03	13	—10	—11	—06	12	00
12	04	05	—05	01	16	21		—07	—01						
13	00	02	11	24	22	15	21		—04						
15	—08	02	—03	26	13	10	24	37							
16	13	15	10	21	35	32	26	18	25		01	00	13	—06	02
52	21	26	10	28	36	11	—05	05	—05	17		00	—01	03	—07
56	14	12	15	19	28	14	20	18	19	19	21		19	05	—01
57	07	11	05	11	30	09	11	18	28	24	12	34		—12	—02
58	35	49	21	39	42	51	13	12	05	23	35	42	11		07
60	32	29	21	06	17	32	22	14	12	26	19	29	17	54	
6	08	09	23	15	01	03	—17	—08	—08	00	07	10	—04	06	09
14	00	—02	16	10	19	18	00	—04	—04	07	01	04	03	04	—04
26	02	06	15	04	—03	20	02	—08	—06	13	03	00	—14	04	11
51	—10	—08	08	08	06	09	12	—11	—08	03	15	—04	—19	00	16
59	02	11	14	23	05	10	09	—01	—02	20	19	06	—07	23	—06
8	—06	—15	02	—02	—23	—14	05	09	04	—08	—09	—05	02	—18	—04
17	—09	—11	08	—05	01	06	—06	02	—10	—15	—19	—10	—14	—05	00
18	—19	—09	01	—24	—06	10	11	04	—01	—12	—12	—05	—03	—15	—01
19	—02	10	—10	—21	—05	—02	07	—01	—14	—07	05	—09	03	14	—01
20	—20	—12	—02	—26	—14	—03	02	04	—02	—15	—23	—15	—02	—02	01
21	—06	02	02	03	01	19	05	05	02	—03	01	02	—09	16	—02
22	—10	—01	—03	—21	—11	—05	01	05	—06	—13	—04	—06	—05	—18	—16
23	—07	—07	—04	01	00	—01	06	—02	02	—11	—03	—09	—05	09	—16
24	—04	02	—04	—10	—12	—03	05	06	—07	—14	—06	—21	—08	08	—03
25	—01	00	—04	07	00	14	08	15	—02	03	—09	00	—05	—05	23
27	—04	—16	00	—01	—11	—02	—15	07	10	—22	—09	—10	—08	—05	—16
28	—11	—03	02	—04	—01	01	—01	—13	—01	—10	—02	—09	—06	—13	05
29	11	—06	—18	—11	—06	—07	02	—10	—04	—16	—09	—16	—05	—14	—01
53	—23	—19	—12	—09	—07	—07	03	06	10	—10	—20	—23	08	—34	—14
30	—19	—18	—01	—11	00	—14	—08	—08	01	06	—20	—18	—11	—11	—13
31	—16	—09	—02	—02	09	—25	—05	17	04	—06	—04	—05	00	02	—09
32	04	—01	03	07	16	—18	06	09	10	02	19	19	17	04	—16
33	—23	—15	—06	08	—04	—07	15	05	12	01	10	10	04	—02	—15
34	—08	—14	—15	—06	00	—03	—01	03	—02	03	03	09	14	—05	—03
35	08	05	—10	—04	—05	—06	—03	00	00	03	—09	—03	—03	—06	—03
37	05	—04	02	—07	—02	—22	—06	—04	01	—02	00	10	—01	—07	—14
38	05	02	02	—04	—10	04	—05	—23	—03	—10	11	18	07	11	12
39	—04	—14	—07	—02	00	—02	—04	—17	04	—11	—03	15	—03	11	04
36	07	09	02	—03	09	08	—08	—10	—23	08	—12	04	07	—16	00
40	21	21	04	03	01	06	—04	07	03	16	09	14	19	00	10
42	22	12	01	05	03	01	—05	09	—03	14	13	23	24	—01	17
41	12	11	11	18	10	08	—05	07	—04	13	—01	—09	—05	17	04
43	04	03	02	—05	—07	—05	05	—07	—06	03	—06	—16	—11	—03	01
44	08	08	09	00	00	13	02	00	01	02	—08	—04	04	—08	08
45	10	01	—01	08	—03	—09	—07	07	—10	—04	—03	—02	—04	04	—10
46	23	01	—05	18	00	09	09	01	05	06	00	10	—05	04	02
47	14	19	—02	28	25	07	00	24	25	24	20	10	11	12	09
48	01	10	—17	02	02	—06	—17	—07	06	—08	11	—06	—01	—26	08
49	01	17	—01	—02	03	05	12	01	13	12	13	11	—02	10	04
50	02	04	08	—02	08	13	05	02	01	00	06	—03	05	05	11
54	13	05	—17	10	10	—03	03	02	18	06	12	12	12	22	—02
55	01	00	13	06	06	—05	—01	03	04	10	05	14	17	11	09

	6	14	26	51	59
6		—02	22	—13	02
14	10		—02	11	—01
26	37	13		—05	—08
51	04	28	16		15
59	19	16	13	39	
8	09	03	04	—12	02
17	03	—01	24	13	—02
18	06	00	24	11	—08
19	—06	—05	07	—13	—02
20	07	04	06	—05	—08
21	03	—07	13	—01	—21
22	13	05	03	—06	—02
23	—01	—03	22	—04	—14
24	00	—28	07	—07	—05
25	01	—16	00	00	—02
27	02	12	06	01	—09
28	10	02	18	04	—03
29	—04	—10	—24	—07	04
53	—01	13	—10	02	—17
30	—11	—05	—06	12	01
31	—07	01	—12	07	19
32	—07	11	—14	—02	01
33	—01	09	—11	05	16
34	02	—02	00	—04	11
35	—06	—13	—07	—07	04
37	—07	02	—10	—09	07
38	—08	—07	—10	08	14
39	—06	—10	—12	—01	—10
36	—08	12	—14	09	—14
40	—14	—03	—18	—27	—09
42	—04	—04	—04	—13	01
41	—02	06	04	01	—03
43	—03	—11	05	02	00
44	—03	—05	02	11	01
45	03	—19	04	—20	—06
46	00	04	06	02	05
47	—11	—08	07	04	—09
48	—02	01	—07	12	06
49	01	—01	—03	05	02
50	03	—03	05	08	—20
54	03	05	—10	00	—03
55	04	05	—04	07	—06

	8	17	18	19	20	21	22	23	24	25
8		—11	12	—04	—04	—02	07	—01	04	04
17	13		09	03	—06	03	—11	—06	—01	07
18	36	41		—08	08	—05	—02	01	—16	00
19	16	29	19		09	05	10	03	08	00
20	26	34	50	43		—05	03	09	—04	—08
21	21	34	27	31	35		—08	—03	08	09
22	30	19	29	35	42	22		03	05	—03
23	19	21	29	26	44	23	29		01	—06
24	25	27	13	32	32	36	32	25		15
25	17	24	18	15	14	26	14	09	31	
27	23	31	42	17	43	24	29	23	18	04
28	27	22	17	12	13	31	11	20	09	18
29	05	17	12	04	12	16	11	11	19	12
53	07	35	26	17	47	16	36	23	15	—02
30	09	31	13	02	34	02	12	10	08	02
31	05	15	06	01	17	—14	03	—02	—06	—05
32	—08	—04	—06	—04	06	—02	00	—04	—07	—13
33	—06	12	08	—04	10	00	03	—07	01	—14
34	—09	04	14	—03	10	—07	06	03	—02	—09
35	11	—05	07	05	—01	02	03	00	03	12
37	24	01	04	08	—06	—09	08	00	12	—03
38	—06	09	03	—04	06	—02	—07	09	03	03
39	—01	23	04	—03	10	11	16	07	14	17
36	—18	26	09	15	15	07	15	—07	01	—01
40	—03	—16	—08	09	06	—04	—06	—02	09	16
42	14	—13	—04	—06	—07	—17	—05	—01	03	02
41	10	14	05	05	13	00	—03	04	03	04
43	14	11	—04	09	19	—02	07	05	23	03
44	16	06	13	07	00	10	—13	03	06	17
45	18	07	13	06	21	00	24	13	12	01
46	—18	—04	—06	—09	—17	08	14	11	—01	01
47	—17	—17	—29	—21	—15	03	—08	03	04	01
48	02	—13	—01	—11	01	—08	10	05	—04	—12
49	—02	—01	—09	06	—12	11	08	—06	09	06
50	06	—06	02	13	02	18	04	01	08	04
54	07	—11	11	10	—03	—04	—01	09	12	—19
55	19	—10	00	11	12	—10	05	—03	—07	—21

TABLE VIII (continued)
FINAL RESIDUAL CORRELATIONS
Below diagonal: residual correlations with u_1 eliminated.
Above diagonal: residual correlations with respective factors eliminated



The weights of the group factors are determined by the use of Professor Spearman's 1914 formula,[1] and the factors then removed in any order by means of the formula

$$\bar{r}_{x_i x_j} \equiv r_{x_i\ x_j} - r_{x_i \tau}\, r_{x_j \tau},$$

where τ is any one of the factors. Residuals with the group factors eliminated have been printed above the main diagonal opposite the corresponding residuals with only u_1 removed in Table VIII.

In order to test the goodness of fit of the modified pattern to the whole set of correlations, a frequency distribution of the final residuals has been made as shown in Table IX. The standard deviation of these is .098 and $.6745\sigma = .066$. The probable error of a zero tetrachoric correlation is .072. These two values agree to two decimal places and hence the factor pattern may be regarded as a satisfactory fit.

5. *Comparison of the Bi-Factor Pattern with a Multiple Factor Analysis*

From Professor Thurstone's preliminary analysis cited above, we may make a comparison of the corresponding factor loadings by the two methods. In Table X the factors have been arranged in the order of significance as stated by Thurstone; a single cross indicating what he calls an "appreciable" loading and a double cross designating a "high" factor loading. We have also included the names and symbols employed in both analyses.

In the case of the "number" or "arithmetical" factor the agreement is perfect and almost so in the case of the "spatial" factor. The "memory" factor also reveals remarkable agreement. When we come to the "verbal" factors, the agreement although not perfect is remarkably close.

We do not find such perfect results on comparing the less prominent factors. Our "imagination" factor is quite comparable to the "perceptual speed" of Professor Thurstone's analysis. The "induction" factor has no counterpart in the Bi-factor analysis while the "analogies" and "rhythm" factors are not represented in the Multiple Factor analysis. Finally, the "deduction" factor, although minor in significance, agrees perfectly in its conspicuous loadings with our "logical reasoning."

A formal difference in the two analyses occurs in the case of the

[1]See *Preliminary Report on Spearman-Holzinger Unitary Trait Study*, No. 2, equation (6).

TABLE IX
FREQUENCY DISTRIBUTIONS OF FINAL RESIDUAL CORRELATIONS

Value of Residual	Frequency
.325 — .345	1
.305 — .325	1
.285 — .305	–
.265 — .285	3
.245 — .265	6
.225 — .245	17
.205 — .225	10
.185 — .205	22
.165 — .185	25
.145 — .165	34
.125 — .145	54
.105 — .125	69
.085 — .105	84
.065 — .085	88
.045 — .065	99
.025 — .045	123
.005 — .025	155
—.015 — .005	133
—.035 — —.015	124
—.055 — —.035	124
—.075 — —.055	107
—.095 — —.075	82
—.115 — —.095	63
—.135 — —.115	50
—.155 — —.135	41
—.175 — —.155	29
—.195 — —.175	19
—.215 — —.195	11
—.235 — —.215	12
—.255 — —.235	5
—.275 — —.255	3
—.295 — —.275	1
—.315 — —.295	–
—.335 — —.315	–
—.355 — —.335	1
Total	1596

Mean	=	.004
Standard Deviation	=	.098
.6745 × S.D.	=	.066
Probable Error of Zero Correlation	=	.072
Q_3	=	.068
Q_1	=	—.060
Quartile Deviation	=	.064

general factor which we obtain and which Professor Thurstone apparently does not. The presence of this factor in our pattern is due to our hypothesis of its existence and the essentially positive correlations throughout, which afford a basis for the evaluation of u_1. It can be shown that each of the group factors in the multiple factor analysis can be expressed as a linear function of the corresponding group factor and the general factor of the Bi-factor analysis. We have shown elsewhere[1] how to obtain the exact mathematical relationships between the factors of various multiple factor solutions and those of

[1]Holzinger, K. J., and Harman, H. H., "Relationship between Factors obtained from Certain Analyses," *The Journal of Educational Psychology*, May, 1937, pp. 321-346.

TABLE X
COMPARISON OF FACTOR LOADINGS
xx — high factor loading
x — appreciable loading

Test	Number *N*	Arithmetical *m*	Visualizing *V*	Spatial *s*	Memory *M*	Memory *o*	Word Fluency W_1	Completion *c*	Verbal Relations W_2	Verbal *v*	Perceptual Speed *P*	Imagination *i*	Induction *I*	Deduction *D*	Logical Reasoning *l*	Analogies *a*	Rhythm *r*
4									xx .47								
5									xx .54								
6											x .35						
7									xx .28		x						
8			x .42										x	x			
9							.22		xx .50								
10							x .28		xx .54								
11							.22		.51								
12							xx .41										
13							x .68		x								
14									x			.35					
15							xx .60										
16							x		xx .38								
17			x .56														
18			xx .58														
19			x .47														
20			xx .72														
21			x .55														
22			xx .54														
23			xx .48														
24				.50													
25				.31										x			
26											xx .43						
27			xx .52														
28				.36													
29				.27									x				
30		xx .41															
31		xx .62															
32		xx .54															
33		xx .74															
34		x .64															
35		x .42											x				
36																	
37		x .27											x				
38		x .43															
39		x .42															
40									x					xx .58			
41									xx							.18	
42									x					xx .58			
43																.11	
44																.27	
45			x														
46					xx .48												
47					xx .57												
48					xx .36												
49					x .33		x										
50					x .46												
51					x						x .48						
52										.42							
53			x .45														.53
54																	.53
55							x		x								
56							xx			.49							
57							xx		xx .30								
58									x .76								
59												.49					
60							x		x .62								

the Bi-factor solution. We plan to show these algebraic relationships between the factors of the present study as soon as the numerical solution of the multiple factor analysis is available.