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A Perspective on Mathematical Problem-Solving Expertise Based on the Performances of Male Ph.D. Mathematicians

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"The major part of every meaningful life is the solution of problems: a considerable part of the professional life of technicians, engineers and scientists, etc. is the solution of mathematical problems." (Halmos [6, p. 523])

As noted by mathematician and expositor Paul Halmos, learning to solve problems is an essential tool for life and has become an important component of the K-16 mathematics curriculum. Over the years mathematicians, psychologists, and educators have embraced various theories on problem solving in order to understand ways to teach students to become better problem solvers. In order to accomplish this goal researchers have examined differences between expert and novice problem solvers with respect to problem-solving behavior on various tasks.

Problem Solving Expertise

What constitutes problem solving expertise? The answer to this question has been evolving over the past 25 years. During the formative stages, information processing served as a framework in understanding how humans solve problems. This process, based on production rules and computer simulations, identified patterns of expert and novice problem solving behavior on well-defined problems. (For a comprehensive description of information processing theory see Newell and Simon [7].) As a result, the term "expert" became synonymous with someone who has accumulated a substantial amount of knowledge in a particular domain,

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that is, a content knowledge specialist. Schoenfeld [21] believed that the domain specific definition of expert appeared narrow and limiting since it eliminated the clumsy and unstructured problem-solving performances of individuals struggling and making progress while solving problems in unfamiliar domains.

In a series of research studies and summary articles Schoenfeld [21, 23] recast the definition of expert. His research indicated that problem-solving experts possessed a wide range of attributes that include: domain knowledge (the tools a problem solver brings to bear on a problem), problem-solving strategies (Polya-like heuristics), metacognitive skills (issues of control—selecting strategies and solution paths to explore or abandon, appropriate allocation of one's resources, etc.), and a certain set of beliefs (a particular world view of mathematics).

A pilot study was conducted to replicate Schoenfeld's work on problem-solving expertise. The eight subjects in the study (six males, two females) were Ph.D. mathematicians. As was common at the time, the study presumed expertise on the part of the experts—most people would consider a Ph.D. an expert mathematician, especially with regard to problems that had rather elementary content demands. Seven subjects were university faculty while the other taught high school. The subjects were asked to think aloud while solving four problems. The problems chosen for the study required a background in high school mathematics and perseverance and creativity in order to solve them. The results of the pilot study were quite disturbing. In many cases the experts could not solve the problems and exhibited few of the skills Schoenfeld attributed to experts. Why? How could this be?

The data indicated that the pilot study experts may not have been experts at all, that is, acquiring a Ph.D. in mathematics (a domain knowledge expert) does not guarantee an individual is an expert in solving problems. To remedy this situation a new study was conducted involving two groups of Ph.D. mathematicians.

The Study

Subjects.

The qualitative nature of this study demanded a small number of subjects be examined in depth. In addition, in order to control for gender differences, all males were selected to participate. Two groups were formed: a) group A—eight men who earned a doctorate or its equivalent in mathematics and have achieved national or international recognition in the mathematics community, and b) group B—eight men who earned a doctorate in mathematics but have not achieved such recognition. In the end, the original 16 contacted agreed to participate in the study.

Subjects in group A were chosen by professional and peer recommendations and according to the medals and honors awarded then by the mathematics community. Subjects in group B were chosen by professional and peer recommendations only. At the time of the study, 12 participants worked in an academic institution, three in industry, one was retired, and all lived in the Northeast. For more detailed information about the subjects (e.g. awards, medals, publications, etc.) see Appendix A.

Data collection.

Each subject was asked to solve four problems and to answer a questionnaire concerning his beliefs about mathematics and mathematical problem solving.

Problem-Solving Booklet. The problems chosen for the study were "ill- structured" problems, that is, no known algorithm or procedure is immediately available to the problem solver while solving the problem (Frederiksen [4]).

A search through problem-solving books (Ball [1]; Bryant [2]; Schoenfeld [15]; Polya [9, 10, 11]; Rapaport [12]; Shlarsky [24]; Trigg [26]; Wickelgren [27]) and journals (Schoenfeld [13, 14, 16, 18, 19]) produced a list of mathematics problems from which four were selected for the study. The problems can be found in Appendix B and the reader is invited to solve them.

Subjects were instructed to take as much time as necessary to solve each problem. In addition to writing down the solution, subjects were asked to think aloud while solving each problem. The researcher was present and took notes during the session. The notes were used to pose questions regarding the solution to the problem just completed. The entire session was audio taped. The audio tapes were transcribed, coded, and analyzed [17] and served as a means of understanding the solution process and the strategies used to solve each problem.

Mathematics Beliefs Questionnaire (MBQ). Beliefs, and in particular mathematics beliefs, play a significant role in an individual's effort and performance while solving problems (Schoenfeld [20, 21]; Silver [25]). A 10-item questionnaire was designed to elicit responses concerning an individual's beliefs about mathematics and mathematical problem solving. The MBQ can be found in Appendix C. The questionnaire and a blank tape were sent to each subject. Audio-taped responses were transcribed, coded and analyzed using procedures developed by the researcher [3].

Results and discussion.

The results of this study indicate clearly that the problem-solving performance of the group A mathematicians was that of experts (as defined by Schoenfeld [21, 23]) while the group B mathematicians performed like novices. (See Table 1 for a summary of the results.) Evidence will be demonstrated by examining the differences in the knowledge base, problem-solving strategies, metacognitive skills, and the beliefs of the mathematicians in groups A and B.

Knowledge base.

The knowledge base represents the "tools" a problem solver brings to bear on a problem. "It is intended as an inventory of all the facts, procedures, and skill—in short, the mathematical knowledge—that the individual is capable of bringing to bear on a particular problem" (Schoenfeld [21, p. 17]). Although all the subjects have acquired the necessary knowledge to solve the problems

TABLE 1. Summary of results

	Group A	Group B
Knowledge Base	Sufficient	Sufficient
Strategies	Productive	Minimal
Metacognitive Skills	Productive	Minimal
Beliefs	Productive	Counterproductive

a significant difference between the subjects in the two groups can be found in the depth and breadth of their research interests, and the number of articles, reviews, and books they've published. (For further information see Appendix A.) While solving the problems, in some instances, subjects in both groups could not recall the necessary mathematical facts/information needed to solve a problem in a particular way. In the end, lacking certain mathematical facts appeared to hinder the performance of the group B mathematicians while this did not effect the preformance of the group A mathematicians. For example, in responding to question 3 on the MBQ, subject B-2 stated,

> "OK, if I don't have enough information, facts or theorems whatever at my disposal, it may handicap me seriously and then I may not be able to make the right connections, that is, find similar problems. . . . For example, on the . . . first part of this interview there was a problem where I couldn't remember the law of cosines. Actually it was the law of sines that I should have remembered. I felt it was intimately related to the problem and yet I couldn't recall what it was or the exact structure of the result, even enough to kind of reconstruct it. So I couldn't recover the theorem and I felt that it was necessary to solve the problem. So to the extent that I was correct then a lack of this knowledge made the solution impossible."

Overall subjects in group B failed to solve or complete 12 (and possibly more) problems because they could not recall the necessary information needed to solve them.

Problem-solving strategies.

Each problem was rated and categorized as correct (correct approach/correct implementation) or incorrect (correct approach/incomplete implementation, correct approach/incorrect implementation, incorrect approach, no solution). In particular, problem 1 was rated correct if a subject reported 49 different ways to change one-half dollar (i.e., using quarters, dimes, nickels, and pennies). Problem 2 was rated correct if a response fell within the range of 10^{11} to 10^{16} cells. Solutions to problems 3 and 4 can be found in Polya [11] and Schoenfeld [17] respectively. These, along with other solutions generated by mathematicians not participating in the study, were used as a basis for rating problems 3 and 4. Otherwise a problem was rated incorrect and placed in one of the appropriate categories.

Of the 32 problems attempted by each group, subjects in group A solved 29 correctly while subjects in group B solved 7 correctly. (See Table 2.) In general, problems 2, 3, and 4 seemed unfamiliar to subjects in both groups. Of the 4 problems: problem 1 seemed easy but tedious for the subjects in group A while easy but unmanageable for subjects in group B; problem 2 was solvable for subjects in group A and difficult for subjects in group B; problem 3 seemed interesting and relatively challenging for subjects in group A while very difficult for subjects in group B; and problem 4 seemed very easy for subjects in group A and relatively difficult for subjects in group B. A brief summary of the approaches/strategies implemented by the subjects is discussed below.

TABLE 2. The number of correct and incorrect solutions exhibited by subjects in both groups on problems 1–4

Group	Pr.	Correct	Incorrect			
			Correct approach/ incomplete	Correct approach/ incorrect	Incorrect approach	No solution
A						
	1	7	1			
	2	6		2		
	3	8				
	4	8				
Total		29	1	2		
B						
	1	2	3	2	1	
	2	1	4	2	1	
	3		4	1	1	2
	4	4	1	2	1	
Total		7	12	7	4	2

> *Problem 1:* In how many ways can you change one-half dollar? (Note: The way of changing is determined if it is known how many coins of each kind—cents, nickels, dimes, quarters are used.)

Seven subjects in group A and two in group B solved this problem correctly by fixing the number of quarters (2, 1, and 0) and then listing and enumerating the remaining possibilities working from the larger to the smaller denominations. Six subjects in group A stated other approaches (using generating functions, equations involving integer solutions) to the problem but realized a simple listing-and-counting approach was a more efficient way to proceed.

The remaining subjects in group B reported various approaches to the problem (using generating functions, the inclusion-exclusion principle, integer solutions to a Diophantine equation, enumerating all possibilities using 1 coin at a time, using 2 coins at a time, using 3 coins at a time and finally 4 coins) but either implemented the plan incorrectly or did not complete the solution.

Subjects in both groups were able to generate a variety of appropriate strategies to solve this problem. Since the answer is a relatively small number, the most reasonable approach is to list a few cases, look for a pattern and count the total number, that is a "brute force" approach. All subjects in group A had prior experience solving this problem (all remembered seeing or doing this or a similar problem before). Therefore, they had a ballpark estimate of the answer and used this brute force method to solve the problem. In contrast, only two in group B decided to use this method while the others used more sophisticated, generalizable, unreasonable approaches (e.g., using generating functions to solve this problem is analogous to using a canon to kill a fly) that ultimately led to an incorrect solution. These results indicate that being able to choose a reasonable approach to a problem from among a variety of viable approaches is an important aspect in solving problems and maybe a trait of expert problem solvers. Further research needs to be conducted on how one acquires an ability to distinguish between reasonable and unreasonable approaches to a problem.

> *Problem 2:* Estimate, as accurately as you can, how many cells might be in an average-sized adult human body? What is a reasonable upper estimate? A reasonable lower estimate? How much faith do you have in your figures?

The solution to this problem requires little technical information but does require the problem solver to make "good estimates" of average cell volume and average body volume. Six subjects in group A and one subject in group B solved this problem correctly. Of these subjects five in group A and one in group B implemented a plan involving estimates of the volume of a cell and an average-sized adult human body. The remaining subjects in group A implemented plans involving estimates of the weight of a cell and an average-sized adult human body (two subjects were correct) and estimating the number of atoms in a cell and the weight of an average-sized adult human body. Of the six subjects in group

A who solved this problem correctly, five based their estimates of average cell volume on the magnification needed to see a cell through a microscope. Answers reported by subjects in group A were: 10^9, 10^{11}, 10^{12}, 10^{13}, 10^{14}, 10^{15}, 10^{16}, and 10^{20}.

The remaining subjects in group B offered various approaches to the problem—estimating the volume of a cell and an average-sized adult human body (three subjects), working either by weight or volume (one subject), working with the weights of the quantities (one subject), but in each case did not implement the plan. One subject estimated the volume of a cell and an average-sized adult human body but incorrectly estimated the volume of the cell. Another subject estimated the number of atoms in a cell and the volume of an average-sized adult human body while the remaining subject offered no plan or approach to the problem, reported some quick estimates and quickly (after 50 seconds) gave up on the problem. Answers given by subjects in group B were 10^{10}, 10^{13}, 10^{25}, while the remaining five subjects reported no solution.

> *Problem 3:* Prove the following proposition: If a side of a triangle is less than the average of the other two sides, then the opposite angle is less than the average of the other two angles.

Many of the subjects commented they could not recall ever seeing this problem before and some in group A were impressed with the simplicity and elegance of the proposition. All subjects in group A solved this problem correctly while no one in group B was able to do so. Six subjects (three in group A, three in group B) used the law of cosines while five subjects (three in group A, two in group B) used the law of sines to solve it. Of the remaining subjects in group A, one worked with a special case of the problem while the other used a string of trigonometric identities involving the sine and cosine of the angles.

Two of the three remaining subjects in group B used techniques from analytic geometry while the remain subject explored three different approaches before reporting that the statement of the problem was incorrect.

> *Problem 4:* You are given a fixed triangle T with base B. Show that it is always possible to construct, with ruler and compass, a straight line parallel to B such that the line divides T into two parts of equal area.

All subjects in group A and four subjects in group B solved this problem correctly. Seven subjects in group A implemented a plan which involved: a) recognizing the upper triangle is similar to the given triangle T, and b) calculating the distance from the upper vertex of T to the line parallel to the base of T such that the line divides T into two parts of equal area. The remaining subject in A began with a right triangle and proceeded in a way similar to the other subjects. Six subjects did the construction during the solution of the problem.

In group B, two subjects implemented a plan similar to the one used by the seven subjects in group A, one used Galois Theory and one used analytic geometry to solve this problem. The remaining subjects implemented plans involving: a) ideas from analytic geometry (one subject),b) the center of gravity of the figure (one subject), and c) the similarity property between the triangles (two subjects). In each case the subject did not complete the solution. Two subjects reported how to do the construction during the solution of the problem.

Metacognitive skills-control.

The term " ... metacognition has two separate but related aspects: a) *knowledge and beliefs* about cognitive phenomena, and b) the regulation and *control* of cognitive actions" (Garofalo [5, p. 163]). The regulation and control component of metacognition suggests an active monitoring system that helps keep a solution on a correct path. Control decisions in problem solving have a major impact on a solution path and actions taken at this level typically include decisions regarding: a) solution paths to explore (or not to explore), b) abandoning, pursuing, changing or selecting approaches/strategies on a problem and c) the allocation of resources at a problem solver's disposal (Schoenfeld [21, p. 27]).

After examining the problem-solving behavior of experts (Ph.D. mathematicians) on problems similar to the ones in this study, Schoenfeld [21] categorized four types of control decisions. The results of this study suggest that a fifth category (Type A) should be added. The five categories observed in the data are given in Table 3.

The following protocol presents the problem-solving performance of John (a pseudonym for subject A-1) on problem 3 and is an example of Type D control behavior.

Problem-solving protocol.

(1) Prove the following proposition: If a side of a triangle is less than the average of the other two sides, then the opposite angle is less than the average of the other two angles.
(2) The side of the triangle is less than the average, OK ...
(3) The second statement, the conclusion is equivalent to the opposite angle is less than 60° ...
(4) Because they ... they add up to 180° and if it's less than the average then it's less than 60° ...
(5) What can you get from that ...
(6) The law of cosines tells you ... $x^2 - xy(\cos 60°) + y^2 = z^2$ ...
(7) And this is really something ... less than $\cos 60°$ ...
(8) $\cos 60°$ is 1/2 and there must be a 2 in there to make it come out right ... OK, so $x^2 - xy + y^2$ ...
(9) I don't think this is true ...
(10) $x^2 - xy + y^2$, let's try a few examples here ...
(11) So when, so when it's equal to the average, the easy case is 4, 3, 5 right triangles ...
(12) Four is the one in question ...

TABLE 3. The effects of different types of control decisions on problem-solving success: a spectrum of impact

Type A.	There is (virtually) no need for control behavior. The appropriate facts and procedures for problem solution can not be accessed from long-term memory. For all practical purposes, the problem solver gives up and resources are not exploited at all.
Type B.	Bad decisions guarantee failure: Wild goose chases waste resources, and potentially useful directions are ignored.
Type C.	Executive behavior is neutral: Wild goose chases are curtailed before they cause disasters, but resources are not exploited as they might be.
Type D.	Control decisions are a positive force in a solution: Resources are chosen carefully and exploited or abandoned appropriately as a result of careful monitoring.
Type E.	There is (virtually) no need for control behavior: The appropriate facts and procedures for problem solution are accessed in long-term memory.

(13) And you ask is that less than $60°$ ...
(14) Well, it is ... OK ...
(15) So z^2 ...
(16) OK, the angle is less than $60°$ ...
(17) So the cosine is more than the $\cos 60°$ ...
(18) So you've got $x^2 - xy + y^2$ ...
(19) And that falls with one relation or another to ...
(20) z^2 ... I'll fill in the greater than or less than sign later to see which one it's supposed to be ...
(21) z^2, now what we're concerned about is z^2, z ... compared to the average is $(x + y)/2$ ...
(22) So we're clearing the fractions, we got $4x^2 - 4xy + 4y^2$ versus $x^2 + 2xy + y^2$ ...
(23) Does that look right ...
(24) $3x^2 - 6xy + 3y^2$ which is $3(x - y)^2$ which is positive ...
(25) So now I believe it's true ...
(26) Because you've got something that's a square. Now, you've just got to work out the signs to make it come out right ...
(27) $3(x - y)^2$ is positive, that means that $4x^2 - 4xy + 4y^2 > (x + y)^2$ ... and that means that $x^2 - xy + y^2$ is bigger than ... bigger than or equal to

... the average squared ...

(28) This isn't working ...

(29) The average is less than ... that, which is equal to, that is $x^2 - xy + y^2$ ... which is equal to $x^2 - 2xy(\cos\theta) + y^2$ and now which way ... which one ...

(30) θ is less ... $\cos\theta$ is ... more ... $-2xy(\cos\theta)$ is less ...

(31) So that $x^2 - 2xy(\cos\theta)$ ... this isn't working ...

(32) Is less than some quantity that the average is also less than ...

(33) That doesn't make any sense ...

(34) If the side, if the side is less than the average of the sides ...

(35) Well, let me read the problem over ...

(36) If the side is less than the average of the sides then the angle is less than the average of the angles ...

(37) OK ... so, the side z is ... how embarrassing, I have to go to a second sheet on my own subject ...

(38) Now $z < (x+y)/2$, square both sides ...

(39) So that $4z^2$ is less than, not less than or equal to, less than $(x+y)^2$ ...

(40) And $4z^2$ is in turn $4x^2 - 4xy(\cos\theta) \ldots 8xy(\cos\theta) + 4y^2$ ...

(41) Expand both sides ... $4x^2 - 8xy(\cos\theta) + 4y^2 < x^2 + 2xy + y^2$ ...

(42) So in other words ... $3x^2 - 6xy + 3y^2$ ... I have to get the $(x-y)^2$ in there somehow and I'll work out the other things later ...

(43) Is less than $8xy(\cos\theta) - 4xy$ ...

(44) So ... and zero is less than the whole smear ...

(45) Because $0 < 3(x-y)^2 < 4xy(2\cos\theta - 1)$ ...

(46) So $2\cos\theta - 1$ is positive ...

(47) $\cos\theta > 1/2$ ...

(48) That tells us that $\theta < 60°$.

Problem-solving analysis.

John read the problem and identified the conditions and the goal of the problem (Statements: 1-2). He decided the law of cosines would be a plausible way to approach the problem (6) but realized inconsistencies in his work (9) and felt compelled to check whether the statement of the problem was true.

At this point, John decided to examine a special case of the problem (a Polya-like strategy), that is, a 3-4-5 right triangle. By examing and learning from this example (11-24), John was convinced the statement of the problem was true (25). Once again he returned to his original plan (the law of cosines) but soon recognized contradictions in his work (33). This led him to reread the problem (34).

Immediately he realized he misinterpreted the statement of the problem. Working from the conditions of the problem and using the law of cosines, he worked in an efficient and accurate manner and completed a correct solution to the problem (36-48).

This protocol is an example of control at its best and the role it can play in solving a problem. Schoenfeld [21] discovered that individuals exhibiting efficient control behavior maintain an internal dialogue and argue with themselves while solving a problem. " ... Plans are not made, they are evaluated and contrasted with other possible plans ... Solutions are monitored and assessed 'on line,' and signs of trouble suggest that current approaches might be terminated and others considered ... " (p. 140)

It is apparent that John monitored, assessed, and evaluated his work effectively throughout the entire process. In short, control became a positive force in helping him solve this problem. His performance is indicative of Schoenfeld's notion of what it means to be an expert problem solver--behavior that takes into account individuals who solve problems but navigate the solution space making progress in a meandering but meaningful manner.

Using Schoenfeld's work [21] on control as a guide the problem-solving protocols were analyzed and categorized into the various types of control behavior (see Table 4). An examination of Table 4 illustrates quite dramatically the difference between the subjects in the two groups regarding control behavior during the solution of the problems.

TABLE 4. Types of problem-solving behavior exhibited by subjects in both groups on problems 1-4

	Type				
Groups	A	B	C	D	E
A			2	22	8
B	5	3	17	7	

Beliefs.

Schoenfeld [21] recognized that mathematical behavior, which appears to be solely cognitive in nature, may in fact be influenced by affective components (p. 155). For example, beliefs regarding problem solving (e.g., perseverance, confidence, motivation, interest, etc.) contribute significantly to an individual's performance on a problem. The questionnaire responses indicated clearly that the beliefs of subjects in group A are different than those in group B with respect to mathematics and problem solving. Responses to the questionnaire are briefly discussed next.

On question 1 (see Appendix C) subjects in both groups indicated the most important characteristics or qualities of an expert problem solver include: experience, knowledge of mathematics, the use of analogies, confidence, perseverance and motivation. In addition, subjects in group A believed successful problem-solving experiences contribute significantly to becoming an expert while subjects in group B seemed to place more emphasis on motivation and perseverance.

According to subject A-4,

> "First, I say a deep knowledge of mathematics itself ... and a real understanding of interrelations between the different parts of mathematics and of the relationships of mathematical models to real world problems ... Beyond that there is the issue of confidence based on past experience of success ... "

Subject B-3's response was representative of group B. He stated,

> " ... you have to be well motivated to do it, in that you think the problem is worth solving, that there is a value in ... in doing it in some sense ... Another thing that will make for decent problem solving ability is your depth in that field ... Third, it's pertinacity. It's related to the ... you've really got to stick to significant problems for many, many, many years. Somehow, not necessarily years but for a lot of time."

On question 2, 12 subjects (seven in group A, five in group B) noted the importance of a "good memory" in solving problems while 11 subjects (seven in group A, four in group B) understood the significance of recalling analogous problems for successful problem solving.

On the role memory plays in relation to problem solving subject A-4 stated,

> " ... One does not call upon facts, information, and theorems as if they are simply part of one's library. There is an interaction between the progress one makes in attempting to understand the problem and the recall of facts and information. But it is very much a matter of conducting an efficient search within a relatively small area for relevant facts, information, and theorems and that of having a vast stock of such facts available. It is by one's ability to localize the search that one renders oneself efficient as a problem solver."

According to subject B-2,

> "OK, if I don't have enough information, facts or theorems whatever at my disposal, it may handicap me seriously and then I may not be able to make the right connections, that is, find similar problems ... to ... get a handle on solving the problem ... "

Subject A-3 theorized several kinds of memory may exist—a "memory for analogies or associations" and a "memory for discrete facts" and indicated both are necessary while solving problems. He stated,

> "It's my feeling there are several kinds of memory. One is the memory of a specific theorem, a result that might be useful in solving the problem or a specific technique. But there may be a deeper, intuitive memory in which you see a problem of a certain type and you somehow feel that some approaches will be more successful than others. Even though you can't put your finger on any specific technique or theorem which will apply but you may recast the problem into ... something where you feel more comfortable or more likely to come up with the appropriate solution."

On question 3, subjects were asked to comment on the issue of not having enough knowledge to solve a problem. Nine subjects (four in group A, five in group B) responded they would not attempt the problem, while four subjects (three in group A, one in group B) stated they would use external sources in order to solve the problem. The remaining responses included that this fact would: 1) have no effect at all, and 2) lead to an incorrect solution to the problem.

While solving the problems subjects in group A exuded confidence and possessed a self-assured, almost cocky attitude in their ability. They were aware of their reputation and enjoyed the challenge of not only finding a correct solution but solving the problem as quickly as possible. The same attitude was echoed in their responses to question 4—seven out of eight considered themselves to be expert problem solvers. On explaining why he felt this way, subject A-4 stated,

> " ... Yes I do consider myself to be fairly good at problem solving and that is because I believe, first, I do understand the role of mathematics in problem solving and, second, because I have had a sufficient history of success to make me rather confident in my ability to solve problems."

In contrast only one subject in group B believed he was an expert problem solver. The responses indicated a significant difference in the level of confidence between the groups.

Question 5 asked the subjects to describe general strategies that would be useful in solving a problem. The strategies cited as most helpful are recalling and using analogous problems (five in group A, three in group B) and examining individual or special cases of a problem (three in each group).

On the issue of using alternative methods to solve a problem (question 6), five subjects in group A indicated they almost always rework problems while the remaining three do so under certain conditions. Four subjects in group B stated almost never do they look for alternative ways to solve problems while only three stated almost always and one under certain conditions.

On this issue subject A-6 stated,

> "Well the answer is always. I never let anything go to bed. I never consider it done and always look for an alternative method ... [it isn't] quite always ... but usually I always go over and look for another solution and play with it. You see that's kind of a nice thing for me, at any rate because the painful job has been done and now I'm reaping the harvest. Playing with it and redoing it and seeing how it fits into other places is just fun, it's easy."

It seems, according to the phrase in the last sentence, "seeing how it fits into other places," this subject has acquired the mathematician's aesthetic, that is " ... a predilection to analyze and understand, to perceive structure and structural relationships, to see how things fit together" (Schoenfeld [22, p. 87]). According to Schoenfeld, developing this aesthetic is a core aspect of learning to think mathematically. Arguably, the subjects in group B may not have acquired this aesthetic.

Group A subjects stated more mathematical areas they enjoyed and usually worked on than the subjects in group B. In addition, group A subjects stated fewer mathematical areas they did not enjoy and usually did not work on than the subjects in group B (questions 7 and 8).

Subjects in both groups believed that having confidence in an area of mathematics would enhence their ability to solve problems in that area (question 9). Similarly, subjects in group A believed that a lack of confidence would not deter them from solving the problem while subjects in group B felt a lack of confidence would affect their ability to solve problems. Subject B-2's response is typical of group B. He stated,

> "I think it would ... have a great effect. I think I would be demoralized. I would be disinclined to expend a serious effort on this problem. I would doubt my abilities to do so. I would give up the ghost, you know, very easily. I would ... be anxious to be rid of having to grapple with this problem and I'd quit as soon as possible. I may make a show of looking at it but I'd be pretty quick to throw up my hands."

In short, the responses to the questionnaire reveal that the beliefs about mathematics and problem solving held by subjects in group A are dissimilar to those held by the subjects in group B. To the extent that beliefs impact problem solving performance, it would appear the beliefs acquired by group A (group B) subjects would positively (negatively) influence their performance on the problems.

Summary and Final Remarks

Expert-novice paradigms have served as a means of studying individual differences in various problem-solving contexts. (A brief overview of the expert-novice research can be found in Owen [8].) As a result of the research in this area the term expert came to be known as someone who knows the domain cold and can solve problems in a nearly automatic and routine fashion. Schoenfeld [21] argued that this type of behavior represented a certain type of competency and failed to reflect the notion that *problem solving experts*, as opposed to *content experts*, are people who manage to solve problems even when the solution is not readily apparent to them. He believed that in addition to knowing the domain, problem solving expertise involves other attributes such as, problem-solving skills, metacognitive skills, and a certain set of mathematical beliefs. A pilot study was conducted to replicate his work on expertise. The results were disappointing and indicated that the ostensible experts in the study, eight Ph.D. mathematicians, may not have been problem solving experts at all. To test this idea a new study was conducted involving two groups of male Ph.D. mathematicians. Group A consisted of eight Ph.D. mathematicians with a national or international reputation in the mathematics community while group B consisted of eight Ph.D. mathematicians without such a reputation.

In the study, subjects were asked to think aloud while solving four mathematics problems (Appendix B) and answer a questionnaire regarding their beliefs about mathematics and problem solving (Appendix C). In solving the problems, the group A mathematicians outperformed the group B mathematicians by a wide margin. Further analysis revealed differences between subjects in the two groups in: a) their knowledge base (see Appendix A), b) their problem-solving skills (in many instances, the strategies used by the group A subjects were helpful in solving the problems while those used by the group B subjects were less helpful and sometimes counterproductive), c) their metacognitive skills (while solving the problems control was a positive force for the group A subjects and control was either missing or a negative force for the group B subjects), and d) their beliefs about mathematics and problem solving (the mathematical belief system of subjects in group A was different than that of the subjects in group B).

Although all the mathematicians in group B possess a strong content knowledge base and have produced some research, they are not problem-solving experts. The results of this study indicate clearly that it is possible for people to be content experts while possessing only modest problem solving skills—that problem solving expertise is a property separable from content expertise. In addition, the evidence from this study begins to focus the picture of what it means to be a problem solving expert and lends strong support to Schoenfeld's research [21, 23] in this area.

The second implication of this study raises the difficult issue of how we train undergraduate and graduate mathematicians at our universities. It is apparent that university mathematics departments train students in subject matter but not in problem solving skills. To the extent that solving problems is important (see the opening quote) and to the extent that training students in problem-solving skills is possible (and I believe it is) then the mathematics community needs to rethink the culture in which students are trained to be mathematicians.

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Appendix A

Subjects	[a]Publications	
	[b]TA	[c]Reviews
A1	28	—
A2	43	38
A3	143	45
A4	274	169
A5	164	60
A6	135	—
A7	33	29
A8	16	190
Total	836	531
B1	9	15
B2	15	5
B3	33	100
B4	6	—
B5	10	—
B6	52	59
B7	4	—
B8	3	—
Total	132	179

Awards–Group A[d]: Honorary Degrees Awarded–12, Silver Medal (University Helsinki), Centenary Medal (John Carroll University), Polya Prize in Combinatorics, Newcomb Cleveland Prize, National Medal of Science, Wolf Prize, Ford Prize, Numerous Visiting Professor Appointments.

Elected Positions–Group A[d]: Co-Chairman, Cambridge Conference on School Mathematics; Chairman, US Comm. on Math Inst.; President–AMS; Vice President–AMS; President–MAA; Vice President–MAA; Member–National Science Board.

No distinguished awards or elected positions were found for subjects in group B.

[a]Available information found in the Math/Sci Database (1940–1992)

[b]TA=Total author-published articles, contributions to articles/books, editor of books

[c]Reviews—Reviews of articles

[d]Available information in American Men and Women in Science–1992

Appendix B

Problem 1

In how many ways can you change one-half dollar? (Note: The way of changing is determined if it is known how many of each kind–cents, nickels, dimes, quarters are used.)

Problem 2

Estimate, as accurately as you can, how many cells might be in an average-sized adult human body? What is a reasonable upper estimate? A reasonable lower estimate? How much faith do you have in your figures?

Problem 3

Prove the following proposition: If a side of a triangle is less than the average of the other two sides, then the opposite angle is less than the average of the two other angles.

Problem 4

You are given a fixed triangle T with base B. Show that it is always possible to construct, with ruler and compass, a straight line parallel to B such that the line divides T into two parts of equal area.

Appendix C

1. Please describe the qualities, characteristics or factors that you think make an individual an expert problem solver in mathematics.

2a. Suppose you are asked to solve a mathematics problem (i.e., either a research problem or a textbook problem and one that you do not recall doing before). How does your memory for facts, information, theorems, etc., affect your problem solving?

2b. What effect do you think this fact (i.e., your answer to part a) may have upon your ability to solve the problem?

2c. Why?

3a. Suppose you are asked to solve a mathematical problem and immediately after reading the problem you realize that you do not think you have enough knowledge to solve the problem. What effect do you think this fact might have upon your ability to solve the problem?

3b. Why?

4a. Do you consider yourself to be an expert problem solver in mathematics?

4b. Why?

5. Suppose you are asked to solve a mathematics problem (i.e., either a research problem or a textbook problem and one that you do not recall doing before). What general strategies or techniques do you think you would use to help you toward the solution of the problem?

6a. After solving a mathematics problem when do you rework and use or not use alternative methods to solve the problem?

6b. Why?

7. Please describe the type[s] of mathematics problem[s] you enjoy and usually work on.
8. Please describe the type[s] of mathematics problem[s] you do *not* enjoy and do *not* usually work on.

9a. Which areas or branches of mathematics do you feel most confident working in?

9b. Suppose that a mathematics problem you are working on falls in one of the areas or branches of mathematics you feel most confident working in. What effect do you think this would have upon your ability to solve the problem?

9c. Why?

10a. Which areas or branches of mathematics do you feel least confident working in?

10b. Suppose that a mathematics problem you are working on falls in one of the areas or branches of mathematics you feel least confident working in. What effect do you think this would have upon your ability to solve the problem?

10c. Why?

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