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ON THE RELATIONSHIP BETWEEN THE HIGHER-ORDER FACTOR MODEL AND THE HIERARCHICAL FACTOR MODEL

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The relationship between the higher-order factor model and the hierarchical factor model is explored formally. We show that the Schmid-Leiman transformation produces constrained hierarchical factor solutions. Using a generalized Schmid-Leiman transformation and its inverse, we show that for any unconstrained hierarchical factor model there is an equivalent higher-order factor model with direct effects (loadings) on the manifest variables from the higher-order factors. Therefore, the class of higher-order factor models (without direct effects of higher-order factors) is nested within the class of unconstrained hierarchical factor models. In light of these formal results, we discuss some implications for testing the higher-order factor model and the issue of general factor. An interesting aspect concerning the efficient fitting of the higher-order factor model with direct effects is noted.

Key words: factor analysis, higher-order factor models, hierarchical factor models, bi-factor solutions, general factor, model equivalence.

1. Introduction

Schmid and Leiman (1957) propose a transformation for deriving hierarchical factor solutions from higher-order factor solutions with simple factor clusters structure. The result of the so-called Schmid-Leiman transformation is the derivation of a single order of hierarchical factors. Using the fictitious example in Schmid and Leiman, the factor structure of a hierarchical factor model is illustrated in Table 1. Wherry (1959) provides a different method that produces hierarchical factor solutions equivalent to that of Schmid and Leiman. Because of the orthogonality of the hierarchical factors, Schmid and Leiman argue that hierarchical factor models are more interpretable than the corresponding higher-order factor models with oblique factors.

The translation between the hierarchical factor model and the higher-order factor model using the Schmid-Leiman transformation bears some important implications for the psychometric interpretation of the so-called higher-order factors. For example, Humphreys (1981) wrote,

> The Schmid-Leiman (1957) transformation of oblique factors in several orders into a single order of orthogonal factors defined by the original variables shows very clearly that the only difference between a first order or so-called primary factor and a higher order factor lies in the number of variables which define it. Breadth is the key concept, not superordination, yet factor theorists continue to discuss factors in two orders as if they belonged to different species of abilities and as if their factors had completely independent existences. (pp. 90–91)

In addition, Wherry (1959) claims that because the hierarchical factor models and the higher-order factor models "are mathematical equivalent, the question of whether factors are 'really' oblique or orthogonal is unanswered" (p. 50). Regardless of the truthfulness of Humphreys' or Wherry's conclusions, the basis for their claims is not well-founded. That is, in general the class of hierarchical factor models is not equivalent to the class of higher-order factor

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TABLE 1.
An example of hierarchical factor solutions derived from the Schmid-Leiman transformation (Source: Schmid and Leiman, 1957)

	Layer 3	Layer 2			Layer 1					
	I	II	III	IV	V	VI	VII	VIII	IX	X
1	.5120	.3840			.4800					
2	.5760	.4320			.5400					
3	.3920	.2940				.4999				
4	.3360	.2520				.4285				
5	.1920		.2560				.7332			
6	.0960		.1280				.3666			
7	.2520		.3360					.5600		
8	.0720		.0960					.1600		
9	.5670			.5784					.3923	
10	.3150			.3214					.2180	
11	.1260			.1285						.5723
12	.1470			.1499						.6677

models. This has been pointed out independently by McDonald (in press), McLeod and Thissen (1997), and Mulaik and Quartetti (1997). Using some illustrative examples, Mulaik and Quartetti show that the Schmid-Leiman transformation actually yields more restricted hierarchical factor solutions than would be needed. McLeod and Thissen describe the hidden constraints of the Schmid-Leiman hierarchical factor solutions. If one were to fit directly a hierarchical factor model to the data using a structural equation modeling software (e.g., EQS by Bentler, 1990; LISREL by Joreskög & Sörbom, 1993), a different, as well as less restricted, hierarchical factor solution than that derived from the Schmid-Leiman transformation is obtained. Mulaik and Quartetti conclude that the classes of higher-order factor models and hierarchical factor models are just *near* equivalent in some cases. (Mulaik & Quartetti use terminology differently than the present authors. Because they tried to make terminology consistent with an earlier article by Gustafsson and Balke (1993), hierarchical factor models in their paper refer to the usual higher-order factor models. However, we follow the more traditional terminology adopted by Gorsuch, 1983, McDonald, 1985, Schmid & Leiman, 1957, Tucker 1940, and Wherry, 1959.)

We will demonstrate formally in this article the nonequivalence of the two classes of factor models. More importantly, we will show that the class of higher-order factor models is nested within the class of hierarchical factor models. The two classes of models will be equivalent when certain free parameters are added to the higher-order factor models. In this regard, a "relaxed" version of the higher-order factor model, which is formally equivalent to the hierarchical factor model, is derived. The relaxed higher-order factor model includes direct effects of the higher-order factors on the manifest variables. As we shall see, the derivation has some important implications for testing, as well as the fitting, of higher-order factor models. In addition, because the bi-factor (general-plus-specific) model is a special case of the hierarchical factor model, we will also discuss the implications of our results for the bi-factor model.

2. The Form of Hierarchical Factor Models and its Relationships with Higher-order Factor Models

The basic structure of the hierarchical factor model is illustrated in Table 1 (taken from Schmid & Leiman, 1957). For a typical hierarchical factor model, there are several *layers* of factors. In each layer, each manifest variable loads on exactly one of the factors in that layer. The numbering of the layers is determined by the number of factors in the layer, starting from the layer with the largest number of factors. All factors in a hierarchical factor model are said to be in the same order, in the sense that they are orthogonal to each other. When there are just two layers of factors, with a single "general" factor in the second layer and several "group" factors in the first layer, the hierarchical factor model reduces to the idealized pattern of the bi-factor model (Holzinger & Swineford, 1937; see an example in Table 2 of the Appendix).

Using the numerical results in Table 1, the hidden constraints in the Schmid-Leiman hierarchical factor model are clear. For example, when comparing the non-zero loadings between columns 1 and 2, it is observed that all the first four pairs of loadings are of a 4:3 ratio. When comparing column 2 with column 5, the ratios for the first two pairs of non-zero loadings are both 4 to 5. Similar observations can be drawn for other pairs of non-zero loadings on factors between layers. These hidden proportionality constraints in the Schmid-Leiman transformation may be removed for a general hierarchical factor solution. This is illustrated in the right panel of Figure 1, where the Schmid-Leiman hierarchical models form a sub-class of general hierarchical factor models. For simplicity, the unique variance for the factors and the manifest variables are not shown. In the top row of the two-by-two array of models in Figure 1, an equivalence relationship between the class of higher-order factor models and the class of Schmid-Leiman hierarchical factor models is established by the Schmid-Leiman transformation and its inverse transformation, the existence of which will be established later.

A higher-order factor model with direct effects from the general factor to the manifest variables is shown in the lower left corner of Figure 1. Clearly, the class of higher-order factor models is nested within this class of models, as shown in the left column of Figure 1. More importantly, as will be shown, this class of models is also equivalent to the class of general hierarchical factor models. Thus, it bridges the relationship between the models in the main diagonal of Figure 1. To show the equivalence of the models in the bottom row, a generalized Schmid-Leiman transformation and its inverse will be established. Once this general relationship is shown, the model equivalence for the upper row models will follow as a special case. Before deriving such a transformation, it is useful to review the original Schmid-Leiman transformation from a new perspective.

3. A Different Look at the Schmid-Leiman Transformation

As described by Schmid and Leiman (1957), the higher-order factor solution is obtained by repeated factoring of the correlation matrices for the (higher-order) factors. Starting with a correlation matrix $\mathbf{R}_o(n_o \times n_o)$ of n_o manifest variables, the fundamental equation for the factor analysis model is

$$\mathbf{R}_o = \mathbf{P}_1\mathbf{R}_1\mathbf{P}_1' + \mathbf{U}_o^2, \tag{1}$$

where $\mathbf{P}_1$ is an $n_o \times n_1$ matrix for factor loadings ($n_o > n_1$), $\mathbf{R}_1$ is an $n_1 \times n_1$ correlation matrix for (the first order) factors, and $\mathbf{U}_o^2 = \mathbf{I}_{n_o} - \text{diag}(\mathbf{P}_1\mathbf{R}_1\mathbf{P}_1')$ is an $n_o \times n_o$ diagonal matrix of unique variance for variables; $\mathbf{I}_{n_o}$ is an $n_o \times n_o$ identity matrix, and diag($\mathbf{A}$) is a diagonal matrix formed by retaining only the diagonal elements of $\mathbf{A}$ and with all other elements in diag($\mathbf{A}$) identically zero. If $\mathbf{R}_1$ is not an identity matrix, it can further be factored to yield second-order factors. In general, the factor model for the i-th level factors is

$$\mathbf{R}_i = \mathbf{P}_{i+1}\mathbf{R}_{i+1}\mathbf{P}_{i+1}' + \mathbf{U}_i^2, \tag{2}$$

where $\mathbf{P}_{i+1}$ is an $n_i \times n_{i+1}$ matrix for factor loadings ($n_i > n_{i+1}$), $\mathbf{R}_{i+1}$ is an $n_{i+1} \times n_{i+1}$ correlation matrix for the $i+1$-th level factors, and $\mathbf{U}_i^2 = \mathbf{I}_{n_i} - \text{diag}(\mathbf{P}_{i+1}\mathbf{R}_{i+1}\mathbf{P}_{i+1}')$ is an $n_i \times n_i$ matrix of unique variance for the i-th level factors. We further assume that all unique variances in $\mathbf{U}_i^2$ are bigger than zero so that $\mathbf{U}_i$, obtained by taking the positive square root of the diagonal elements of $\mathbf{U}_i^2$, is invertible. This assumption is mild when discussing model relationships, although practical model fitting may not always result in an invertible $\mathbf{U}_i$, or positive unique variance estimates.

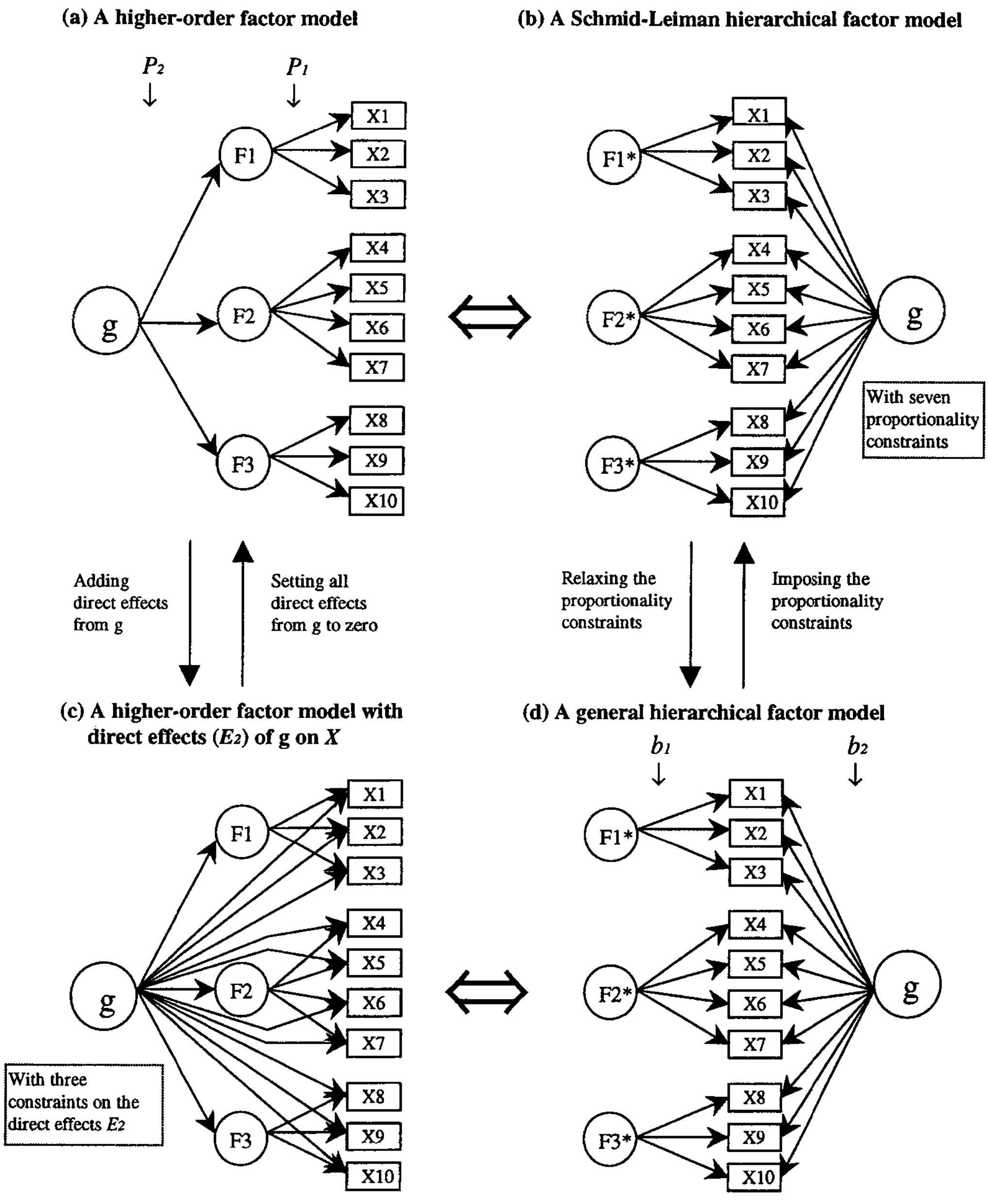

FIGURE 1.
Higher-order factor models and hierarchical factor models.

Repeated factoring is done until at the $k - 1$ level the factor model becomes

$$\mathbf{R}_{k-1} = \mathbf{P}_k\mathbf{P}_k' + \mathbf{U}_{k-1}^2, \tag{3}$$

indicating that the k-th level factors are orthogonal or there is just one factor at the k-th level. Schmid and Leiman (1957) assume that all factor loading matrices $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k$ have simple cluster structures. That is, only a single non-zero factor loading is permissible in each row of $\mathbf{P}_i$. This structure of factor loading matrices implies that

$$\mathbf{U}_i^2 = \mathbf{I}_{n_i} - \operatorname{diag}(\mathbf{P}_{i+1}\mathbf{P}_{i+1}'), \tag{4}$$

for all $i = 0$ to $k - 1$. Equation (4) also implies that none of the elements in $\mathbf{U}_i^2$ needs to be treated as free parameters in the factor model, as they are expressed as functions of factor loading matrices.

The repeated factoring procedure described above is obscure with respect to the transformation of the factors during the Schmid-Leiman transformation. A more transparent description is through the specification of the model equation among factors and variables. First, the factor model for the manifest variables can be written as

$$\mathbf{z} = \mathbf{P}_1\mathbf{f}_1 + \mathbf{U}_o\mathbf{u}_o, \tag{1a}$$

where $\mathbf{z}$ is an $n_o \times 1$ random vector for standardized manifest variables, $\mathbf{f}_1$ is an $n_1 \times 1$ random vector for the first-order factors, u_o is an $n_o \times 1$ random vector for unique factors, and $\mathbf{U}_o$ is an $n_o \times n_o$ diagonal matrix for unique factor loadings. As usual, both $\mathbf{f}_1$ and $\mathbf{u}_o$ are assumed to have zero mean vectors and are uncorrelated with each other. The correlation matrix, as well as the covariance matrix, for $\mathbf{f}_1$ is $\mathbf{R}_1$. The correlation matrix, or the covariance matrix, for $\mathbf{u}_o$ is an identity matrix. Equation (1a) implies a correlation structure $\mathbf{R}_o$ of $\mathbf{z}$ specified exactly in (1). With similar assumptions and notation as in above, factoring of the i-th level factors is described by the equation

$$\mathbf{f}_i = \mathbf{P}_{i+1}\mathbf{f}_{i+1} + \mathbf{U}_i\mathbf{u}_i, \tag{2a}$$

which leads to the correlation structure in (2). Finally, factoring stops at the $k - 1$ level when $\mathbf{f}_k$ in the equation

$$\mathbf{f}_{k-1} = \mathbf{P}_k\mathbf{f}_k + \mathbf{U}_{k-1}\mathbf{u}_{k-1} \tag{3a}$$

is either a single factor or a set of orthogonal factors. This leads to the correlation structure for $\mathbf{f}_{k-1}$ in (3).

Utilizing the equations described in (1a) through (3a), the model equation under the Schmid-Leiman transformation is

$$\begin{aligned}
\mathbf{z} &= \mathbf{P}_1(\mathbf{P}_2\mathbf{f}_2 + \mathbf{U}_1\mathbf{u}_1) + \mathbf{U}_o\mathbf{u}_o \\
&= \mathbf{P}_1(\mathbf{P}_2(\mathbf{P}_3\mathbf{f}_3 + \mathbf{U}_2\mathbf{u}_2) + \mathbf{U}_1\mathbf{u}_1) + \mathbf{U}_o\mathbf{u}_o \\
&= \cdots\cdots \\
&= \mathbf{P}_1(\mathbf{P}_2(\cdots(\mathbf{P}_{k-1}(\mathbf{P}_k\mathbf{f}_k + \mathbf{U}_{k-1}\mathbf{u}_{k-1}) + \mathbf{U}_{k-2}\mathbf{u}_{k-2}) \cdots + \mathbf{U}_1\mathbf{u}_1)) + \mathbf{U}_o\mathbf{u}_o.
\end{aligned} \tag{5}$$

Upon expansion and rearranging the terms in (5), the final form of the Schmid-Leiman hierarchical factor model is

$$\mathbf{z} = \mathbf{B}_o\mathbf{h} + \mathbf{U}_o\mathbf{u}_o, \tag{6}$$

where

$$\mathbf{h} = \{\mathbf{f}'_k, \mathbf{u}'_{k-1}, \mathbf{u}'_{k-2}, \cdots\cdots, \mathbf{u}'_1\}' \tag{7}$$

is a vector of $\sum_{i=1}^{k} n_i$ hierarchical factors, and

$$\begin{aligned}
\mathbf{B}_o = \{&\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 \cdots \mathbf{P}_{k-1}\mathbf{P}_k\mathbf{U}_k \vdots \\
&\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 \cdots \mathbf{P}_{k-1}\mathbf{U}_{k-1} \vdots \\
&\vdots
\end{aligned}$$

$$\mathbf{P}_1\mathbf{P}_2\mathbf{U}_2 \vdots$$
$$\mathbf{P}_1\mathbf{U}_1\} \tag{8}$$

is an $n_o \times \sum_{i=1}^{k} n_i$ factor loading matrix. In (8), $\mathbf{U}_k = \mathbf{I}_{n_k}$ is added to show the consistent pattern of the sub-matrices for factor loadings in $\mathbf{B}_o$. Thus, the correlation model for the manifest variables can be expressed as

$$\mathbf{R}_o = \mathbf{B}_o\mathbf{B}_o' + \mathbf{U}_o^2. \tag{9}$$

Using (1) through (3), Schmid and Leiman (1957) express the hierarchical factor loading matrix as

$$\mathbf{B}_o = \mathbf{P}_1(\mathbf{P}_2(\mathbf{P}_3 \cdots (\mathbf{P}_{k-1}(\mathbf{P}_k \vdots \mathbf{U}_{k-1}) \vdots \mathbf{U}_{k-2}) \cdots \vdots \mathbf{U}_2) \vdots \mathbf{U}_1), \tag{10}$$

which is identical to (8).

Compared with the original Schmid-Leiman derivation, there are many advantages for the present perspective. They are summarized as follows.

Result 1. The set of transformed factors are orthogonal. It is clear from (7) that after the Schmid-Leiman transformation the original set of (higher-order) factors $\mathbf{f} = \{\mathbf{f}_k', \mathbf{f}_{k-1}', \mathbf{f}_{k-2}', \cdots\cdots, \mathbf{f}_1'\}'$ are now replaced by the set of hierarchical factors $\mathbf{h} = \{\mathbf{f}_k', \mathbf{u}_{k-1}', \mathbf{u}_{k-2}', \cdots\cdots, \mathbf{u}_1'\}'$. Whereas Schmid and Leiman (1957) use the factor equation (9) to argue for the orthogonality of the hierarchical factors, the same fact is made apparent in (7). Except for $\mathbf{f}_k$, which remains unchanged after the Schmid-Leiman transformation, all the original factors are replaced by their corresponding unique components, which, by construction, are uncorrelated with any other transformed factors. In fact, we can even replace $\mathbf{f}_k$ with $\mathbf{u}_k$, a new symbol that represents the unique part of $\mathbf{f}_k$, to reflect the orthogonality of the entire set of hierarchical factors. That is,

$$\mathbf{h} = \{\mathbf{u}_k', \mathbf{u}_{k-1}', \mathbf{u}_{k-2}', \cdots\cdots, \mathbf{u}_1'\}'. \tag{11}$$

This clears up the question raised by Gorsuch (1983), who suspected the orthogonality of the transformed factors on the ground that the factor scores produced are dependent (Gorsuch, 1983, p. 252). It seems in the description by Gorsuch that the transformed factor matrix $\mathbf{B}_o$ in (8) has been (mis-) associated with the set of original higher-order factors $\mathbf{f} = \{\mathbf{f}_k', \mathbf{f}_{k-1}', \mathbf{f}_{k-2}', \cdots\cdots, \mathbf{f}_1'\}'$, which is a set of correlated factors. But the new vector $\mathbf{h}$ in (11) obviously contains a set of orthogonal factors.

Result 2. The hierarchical factor model based on the Schmid-Leiman transformation is highly structured. Let us denote the general form of hierarchical factor matrix as

$$\mathbf{B} = \{\mathbf{b}_k \vdots \mathbf{b}_{k-1} \vdots \mathbf{b}_{k-2} \vdots \cdots\cdots \vdots \mathbf{b}_2 \vdots \mathbf{b}_1\}, \tag{12}$$

where $\mathbf{b}_i$'s (of dimension $n_o \times n_i$ each) are layers of factor loading matrices. Comparing (12) with (8), we have

$$\mathbf{b}_i = \left(\prod_{j=1}^{i} \mathbf{P}_j\right)\mathbf{U}_i \qquad (1 \le i \le k) \tag{13}$$

as a form for the i-th layer using the Schmid-Leiman transformation. Thus, it is clear that the Schmid-Leiman hierarchical factor model is highly structured.

Result 3. Adjacent layers of the Schmid-Leiman hierarchical factor matrix are constrained. In the introduction, we point out the proportionality constraints in the hierarchical factor model obtained from the Schmid-Leiman transformation. A little manipulation of (13) explicates a more general fact:

$$\mathbf{b}_i = \left(\prod_{j=1}^{i-1} \mathbf{P}_j\right) \mathbf{U}_{i-1}\mathbf{U}_{i-1}^{-1}\mathbf{P}_i\mathbf{U}_i$$
$$= \mathbf{b}_{i-1}(\mathbf{U}_{i-1}^{-1}\mathbf{P}_i\mathbf{U}_i) \qquad (1 < i \leq k). \tag{14}$$

That is, each i-th layer ($i > 1$) factor submatrix $\mathbf{b}_i$ is a linear combination of the $i-1$-th layer factor submatrix $\mathbf{b}_{i-1}$. This explains the proportionality constraints for adjacent layers of the Schmid-Leiman hierarchical factor model.

Result 4. The Schmid-Leiman hierarchical factor model could be characterized as a general hierarchical factor model with ($n_o - n_{i-1}$) constraints in the i-th layer. Because of the linear dependence of factor layers shown in (14), the number of parameters in the Schmid-Leiman hierarchical factor model should be counted with caution. In (8), we observe that all parameters in the Schmid-Leiman hierarchical factor model are contained in the k-th factor layer $\mathbf{b}_k$ and none of the remaining layers contains additional parameters. Therefore, the total number of parameters in the Schmid-Leiman hierarchical factor model is

$$\text{number of parameters} = \sum_{i=0}^{k-1} n_i, \tag{15}$$

which is the total number of nonzero factor loadings in all $\mathbf{P}_i$. Again, it is noted that all the parameters in $\mathbf{U}_i$ are redundant with the parameters in $\mathbf{P}_i$ and should not be counted. This number is the same as the number of parameters for the original higher-order factor model.

Were the solution of a hierarchical factor model obtained directly, there would be n_o parameters in each factor layer. Therefore, the total number of parameters for a general hierarchical factor model is

$$\text{number of parameters} = \sum_{i=1}^{k} n_o = k\, n_o. \tag{16}$$

By comparing (15) and (16), the number of constraints placed on the Schmid-Leiman hierarchical factor model is

$$\text{number of constraints} = \sum_{i=1}^{k} n_o - \sum_{i=0}^{k-1} n_i = \sum_{i=1}^{k}(n_o - n_{i-1}). \tag{17}$$

In (17), we express the total number of constraints as a sum of the number of constraints placed on *each* layer of the Schmid-Leiman hierarchical factors. The characterization of the constraints is not unique. The particular characterization in (17) is chosen because it motivates the derivation of higher-order factor models with direct effects, which will be elaborated later.

A short example. Using (17), there are 15 constraints in the hierarchical factor model shown in Table 1, $9 = 12 - 3$ for the third layer and $6 = 12 - 6$ for the second layer. To get this number, one can also count the proportionality constraints directly. When comparing the third and the second layers, we observe that the first four pairs of nonzero loadings from the two layers are of the same proportion, so this yields 3 constraints. Similarly, there are 6 more proportionality constraints for the next eight pairs, counting four pairs at a time. These nine constraints plus the six proportionality constraints between the second and the first layers sum to 15, as expected. Therefore, even though there are 36 nonzero entries in Table 1, the number of free parameters is just 21 (= 36 − 15).

4. Deriving Higher-Order Factor Models from Hierarchical Factor Models: A Generalized Inverse Schmid-Leiman Transformation

To show the equivalence for either the upper or the lower row models illustrated in Figure 1, one needs to define a one-to-one transformation between the two classes of models. Schmid and Leiman (1957) provide the left-to-right transformation for the models in the upper row. We generalize the Schmid-Leiman transformation by adding (constrained) direct higher-order effects matrices $\mathbf{E}_i$ $(1 < i \leq k)$ to the higher-order factor model. This provides the left-to-right transformation for the models in the lower row. Such a generalized Schmid-Leiman transformation is detailed in the Appendix. To complete the proof of equivalence, we now need to define the right-to-left transformation: a generalized inverse Schmid-Leiman transformation.

To motivate the derivation of the inverse transformation, we start with the restrictive case for the models in the upper row. Suppose that a hierarchical factor matrix $\mathbf{B}_o$, which satisfies the proportionality constraints, is given; the inverse Schmid-Leiman transformation is the same problem as finding unique $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \cdots \mathbf{P}_{k-1}, \mathbf{P}_k$ given $\mathbf{B}_o$. A point critical to the understanding of the inverse transformation involves the interpretation of the layers of loadings in $\mathbf{B}_o$. For example, in (13), $\mathbf{b}_i = (\prod_{j=1}^{i} \mathbf{P}_j)\mathbf{U}_i$ is the factor loading matrix for the hierarchical factors $\mathbf{u}_i$, which are the unique components, respectively, of the corresponding higher-order factors $\mathbf{f}_i$. Therefore, $\mathbf{b}_i$ can be interpreted as the unique effects of $\mathbf{f}_i$ on the manifest variables, with the effects of all other factors removed. The same interpretation is also offered in Humphreys, Tucker, and Dachler (1970), and Mulaik and Quartetti (1997). A little manipulation of (13) reveals the idea behind the inverse Schmid-Leiman transformation in the appendix. That is,

$$\mathbf{b}_i \mathbf{U}_i^{-1} = \prod_{j=1}^{i} \mathbf{P}_j, \tag{18}$$

where $\mathbf{b}_i \mathbf{U}_i^{-1}$ summarizes the total effects of $\mathbf{f}_i$ on the manifest variables, *without* controlling for the effects of factors at higher levels. The product form of the factor loading matrices on the right side of the equation represents the usual calculation of *indirect* effects of factors $\mathbf{f}_i$, except for $i = 1$. Taken together, (18) states that the total effects of $\mathbf{f}_i$ $(i > 1)$ on the manifest variables are all *indirect*. A further manipulation of (18) gives

$$\begin{aligned} \mathbf{b}_i \mathbf{U}_i^{-1} &= \left(\prod_{j=1}^{i-1} \mathbf{P}_j\right) \mathbf{U}_{i-1} \mathbf{U}_{i-1}^{-1} \mathbf{P}_i, \\ &= \mathbf{b}_{i-1} \mathbf{U}_{i-1}^{-1} \mathbf{P}_i. \end{aligned} \tag{19}$$

If $\mathbf{U}_i$ is known in (19), then the only unknowns in the equation are the elements in $\mathbf{P}_i$ and $\mathbf{U}_{i-1}$. Furthermore, because $\mathbf{U}_{i-1}$ is a function of $\mathbf{P}_i$ (see (4)), the n_{i-1} parameters in $\mathbf{P}_i$ are the only unknowns. This suggests an algorithm (or a transformation) for solving for $\mathbf{P}_i$ given $\mathbf{B}_o$. The inverse Schmid-Leiman transformation starts with the highest factor level (layer) k. Because $\mathbf{U}_k$ is an identity matrix, n_{k-1} unknowns in $\mathbf{P}_k$ and $\mathbf{U}_{k-1}$ can be solved using the n_o linear equations in (19). After $\mathbf{U}_{k-1}$ is solved, (19) is reapplied to the next highest factor level (layer) for solving for $\mathbf{P}_{k-1}$, and hence for $\mathbf{P}_{k-2}, \ldots$ successively.

At the first glance, it seems that there is an over-identification problem of $\mathbf{P}_i$ in each step of the algorithm. However, as Result 4 suggests, because there are exactly $(n_o - n_{i-1})$ constraints in the i-th layer of the Schmid-Leiman hierarchical factor matrix, there are exactly $n_{i-1} = n_o - (n_o - n_{i-1})$ nonredundant equations for solving for n_{i-1} unknowns in $\mathbf{P}_i$ in each step of the inverse transformation. In other words, the solution for $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \cdots \mathbf{P}_{k-1}, \mathbf{P}_k$ given the Schmid-Leiman hierarchical factor matrix $\mathbf{B}_o$ is unique. Therefore, the equivalence of the models in the upper row of Figure 1 is established.

We now generalize the equivalence relationship between the models in the lower row. Here we show the right-to-left transformation (i.e., a generalized inverse Schmid-Leiman transformation) for illustrating the basic idea. Detailed derivations are available in the Appendix. Basing on the fact that $\mathbf{b}_i\mathbf{U}_i^{-1}$ summarizes the total effects of $\mathbf{f}_i$ on the manifest variables, (19) is now extended to include the *direct* effects, in addition to the indirect effects, of $\mathbf{f}_i$. That is, for any *unconstrained* hierarchical factor model with factors h, we relate it to a higher-order factor model via the equation

$$\mathbf{b}_i\mathbf{U}_i^{-1} = \mathbf{b}_{i-1}\mathbf{U}_{i-1}^{-1}\mathbf{P}_i + \mathbf{E}_i, \tag{20}$$

where $\mathbf{E}_i$ is an $n_o \times n_i$ matrix representing the *direct* effects of $\mathbf{f}_i$ on the manifest variables ($i > 1$). Certainly, not all elements in $\mathbf{E}_i$ could be free parameters. For an unconstrained hierarchical factor model, there are n_o linear equations in (20) and n_{i-1} non-redundant parameters in $\mathbf{U}_{i-1}$ and $\mathbf{P}_i$. As a result, assuming consistency of the equations, only $(n_o - n_{i-1})$ nonredundant parameters in E_i can be solved uniquely. This would also be the maximum number of non-redundant parameters for the identifiability of $\mathbf{E}_i$. For example, when obtaining an equivalent higher-order factor model (lower left corner) from the general hierarchical factor model (lower right corner) in Figure 1, exactly $(n_o - n_1) = 10 - 3 = 7$ nonredundant direct effects from $\mathbf{f}_2$ (i.e., the g factor in Figure 1) must be defined. There are many ways to do this. For example, for the bottom left model we may set the direct paths from g to X1, X4 and X8, respectively, to zero. This results in just seven parameters (non-zero direct paths from g) in $\mathbf{E}_2$. For identification purposes, we note that exactly one path from the higher-order factor g to the variables in each factor cluster at the next order should be set to zero. It remains arbitrary to select the zero path within each factor cluster. One may perhaps suggest the use of substantive theory in guiding our choice. Unfortunately, while substantive theories for zero direct effects may exist, they do not seem to provide a general methodology. Here we consider two other possibilities.

The first method considered is the so-called "minimum correlations method". This method selects a set of zero paths so that the correlations among the higher-order factors at the next level are minimized (see a numerical example in the Appendix). To achieve this, the method essentially minimizes the nonzero values of higher-order factor loadings $\mathbf{P}_i$ and at the same time maximizes the nonzero direct effects in matrix $\mathbf{E}_i$. The advantage of this method is the clarity of the derived higher-order factors. Unfortunately, the large direct effects produced as a byproduct are incompatible with the structural simplicity of the higher-order factor model. We thus consider an alternative in which the direct effects are expressed as "residual" effects in the generalized inverse Schmid-Leiman transformation. In other words, we set the sum of the direct effects in each factor cluster to zero, instead of setting one of them to zero. This method is called the "residual direct effects method", an example of which could be found in the Appendix.

Whichever method one selects to define the direct effects, (20) itself is sufficient to define a generalized inverse Schmid-Leiman transformation. Once (20) is established, as illustrated in the Appendix, it is algorithmically straightforward to transform the models uniquely. Some useful results will then follow.

Result 5. Any hierarchical factor model is equivalent to a higher-order factor model with or without direct effects. When a hierarchical factor model satisfies the Schmid-Leiman proportionality constraints, a traditional higher-order factor model without direct effects can be derived. In general, however, a hierarchical factor model is not equivalent to a higher-order factor model, unless a set of (constrained) direct effects are added to the latter model.

Result 6. The class of higher-order factor models (without direct effects from higher-order factors) is nested within the class of general hierarchical factor models. Our conclusion follows immediately from the derivation of the higher-order factor model with direct effects, which is a simple extension of the traditional higher-order factor model and is equivalent to the general hierarchical factor model.

Result 7. For empirical data, a chi-square difference test can be applied for testing the higher-order factor model (without direct effects) against the corresponding general hierarchical factor model. The degrees of freedom for the chi-square difference test are equal to the number of constraints imposed by the Schmid-Leiman transformation, that is, $df = \sum_{i=1}^{k}(n_o - n_{i-1})$. This follows immediately from Result 6 and the difference in the number of parameters between the two models.

5. Discussion and Implications

Contrary to Wherry's claim (1959), we show that the hierarchical factor model is not mathematically equivalent to the higher-order factor model, unless appropriate direct effects (loadings) are added. In general, the latter is a subclass of the former. An immediate implication of this is the relative fit of these two types of models for empirical data. Because a chi-square difference test can now be applied to distinguish these two models, the power of the test is greatly improved over the usual chi-square test for the absolute fit of the higher-order factor model, assuming that the hierarchical model is true. For example, Mulaik and Quartetti (1997) found that these two types of models are nearly equivalent and concluded that the power of discriminating these two models is very small for empirical data; their conclusion does not correspond well with results reported here. Suppose we fit the second-order factor model (the upper left model) in Figure 1 to some data. If one tests the second-order factor model using the usual χ^2 test at $\alpha = .05$, the degrees of freedom for the test are 32. Assuming that the two-layer hierarchical factor model (the lower right model) is the true model and the noncentrality parameter is 6, the approximate power of the test is just .18. Here, the procedure of finding the approximate power of the chi-square test is based on the theory described in chapter 10 of Ferguson (1996). More conveniently, as was done in this paper, we may use the following SAS/IML (1990) statements:

```
alpha = .05; df = 32; nc = 6;
crit_val = cinv(alpha, df, 0);
power = 1 - probchi(crit_val, df, nc);
```

to find the approximate power.

However, if a chi-square difference test is applied for testing the second-order factor model against the two-layer hierarchical factor model, which is assumed to be the true model in this case, the degrees of freedom for the test would be 7 ($= n_o - n_1 = 10 - 3$). For the same noncentrality parameter, the power of the test is now .37 at $\alpha = .05$, an approximate increment of 20% power over the test based on the absolute fit of the second-order factor model.

Despite the fact that our results have been derived for correlation structures, they apply to the analysis of covariance structures as well. When covariance structures are analyzed, the formulas regarding the factor loadings and direct effects derived here should refer to the standardized solutions, where factors and variables are all standardized to have variance 1. A formal demonstration about this is possible; but because it will not add to our basic understanding it may as well be omitted here. With slight modifications, the current results also extend to the "incomplete" hierarchical factor models, or the "incomplete" higher-order factor models, as described in Mulaik and Quartetti (1997). For example, suppose in Figure 1 that all the loadings from F1* to X1 through X3 vanish in the lower right model. In this case, we have an incomplete hierarchical factor model. Similarly, for the higher-order factor model in the upper left corner, we may have F1 vanished and the effects of g on X1 through X3 all become direct. This results in an incomplete higher-order factor model. These two incomplete models still have a nested relationship, as explained by a simple argument here. When taking X1 through X3 away from the two incomplete models temporarily, two "complete" models for X4 through X10 remain. All the derived results are now applied to these complete models. For example, the higher-order factor model is nested within the hierarchical factor model for X4 through X10 and the degrees of freedom for the chi-square difference test are 5 ($n_0 - n_1 = 7 - 2$). When adding back the paths for X1 through X3, one merely adds the same new set of parameters to both models. Thus the established nested relationship between the models does not change.

This result is now applied to the example of power assessment by Mulaik and Quartetti (1997), who quoted the models studied by Gustafsson and Balke (1993). Two incomplete models, one with higher-order factors and the other with hierarchical factors, are compared. There are 16 variables and two orders (layers) of factors in the higher-order (hierarchical) factor model. In the higher-order (hierarchical) factor model, the first level (layer) consists of three primary factors for the last twelve variables and the second level (layer) consists of a single general factor. The first four variables directly load on the general factor in the two models. When testing the incomplete higher-order factor model using the chi-square test of absolute fit, the degrees of freedom are $df = 101$. Assuming the incomplete hierarchical model is true and the noncentrality parameter is 3.59, the power of the test is .08 at $\alpha = .05$ (apparently using a normal approximation, Mulaik & Quartetti obtained a different power value). But when a chi-square difference test for relative fit is applied, the degrees of freedom are 9 ($= n_o - n_1 = 12 - 3$, which is based on the comparison of the "complete" sub-models for the last twelve variables). With the same non-centrality parameter, the power of the test is now .20 at $\alpha = .05$.

We now discuss the implications of our results for the bi-factor model, which is a special case of the hierarchical factor model. In a bi-factor factor, the general factor (the only second layer factor; see Figure 1) has direct effects on all manifest variables but not on the primary factors (the first layer factors). This is the "breadth" conception suggested by Humphreys (1981). In contrast, in a second-order factor model the general factor only has indirect effects on the manifest variables, which is the "superordination" conception of the general factor. Our result seems to suggest that the qualitative distinction between the breadth and the superordination concepts of the general factor can also be considered quantitatively. That is, a chi-square difference test can be applied to test the tenability of the superordination conception (the second-order factor model) against the breadth conception (the two-layer hierarchical factor model). Rejecting the second-order factor model implies that the superordination conception is too restrictive. The same test can also be interpreted in a slightly different way. That is, it is also a test of the necessity of adding direct effects from the general factor. This interpretation is especially clear if we adopt the residual direct effects method for defining the equivalence of the models in the bottom row of Figure 1. Therefore, it seems that we can now offer a reasonable answer, at least in empirical sense, to the question raised by Tucker (1940), who wrote,

"Is the general factor one of the factors in domain α [i.e., the same order as the group factors], or is it one of the factors in domain β [i.e., at the second order which explains the correlations among the first-order factors]?"

As a byproduct of showing the equivalence of the models in the bottom row of Figure 1, our results provide a computationally efficient method of fitting higher-order factor models with direct effects. For example, if direct effects are expressed as residual effects in the higher-order factor model (i.e., sum to zero direct effects for clusters of variables from the higher-order factors), one may first fit the equivalent general hierarchical factor model, which, in our experience, is computationally efficient because of the orthogonality of the factors. Then the inverse transformation method described in the Appendix is applied to obtain the corresponding higher-order factor solution with residual direct effects. To the best of our knowledge, there is currently no statistical software that can fit this kind of models in a handy way. But the generalized inverse Schmid-Leiman transformation provides a good solution for this.

Even for fitting higher-order factor models with a priori zero direct effects, applying the generalized inverse Schmid-Leiman transformation to the equivalent hierarchical factor models may circumvent some computational problems involved in the direct fitting. For example, suppose we want to fit a higher-order factor model with direct effects to the data that produces the hierarchical factor solution in Table 2. The higher-order factor model is illustrated in the lower left corner of Figure 1 (McLeod & Thissen, 1997). For identification purposes, one may set the paths from the g-factor to X1, X4, and X8 to zero; or one may set the paths from the g-factor to X2, X5, and X10 to zero; and so on. There are totally $36 = 3 \times 4 \times 3$ possible ways to fix these paths for identification. All these 36 equivalent models with a priori zero direct effects can be fitted by existing structural equation modeling software and should yield the same model fit for a given data set. Without setting a particular set of starting values, we fit all these 36 models using LISREL program. Among these 36 models, two of them did not yield convergent solutions. We suspect the reason for the difficulty in the fitting is the highly correlated factors in the higher-order factor models. For those converged solutions, the average number of iterations for convergence is 409, with a standard deviation of 132. However, when fitting the corresponding hierarchical factor model to the same data, the solution (i.e., the solution in Table 2) was obtained in just 33 iterations. The generalized inverse Schmid-Leiman transformation was then applied to the hierarchical factor solution. All 36 higher-order factor models with different patterns of direct effects were then obtained almost effortlessly. Whether this kind of computational advantage for fitting higher-order factor models has further implications is subject to further investigations.

Appendix

A generalized Schmid-Leiman transformation and its inverse. To include the direct effects from the higher-order factors in (1a), we write:

$$\mathbf{z} = \mathbf{P}_1\mathbf{f}_1 + \sum_{i=2}^{k} \mathbf{E}_i\mathbf{f}_i + \mathbf{U}_i\mathbf{u}_i, \tag{A1}$$

where $\mathbf{E}_i\,(1 < i \leq k)$ is an $n_o \times n_i$ matrix of factor loadings or direct effects of $\mathbf{f}_i$ on $\mathbf{z}$. In this general form, the factor model may not be identified unless further constraints on the direct effects are imposed; the constraints have been discussed in the main text and will be illustrated later. At this stage we assume these constraints are satisfied. Substituting (2a) into (A1) successively we get

$$\mathbf{z} = \mathbf{B}\mathbf{h} + \mathbf{U}_o\mathbf{u}_o, \tag{A2}$$

where

$$\mathbf{h} = \{\mathbf{f}'_k, \mathbf{u}'_{k-1}, \mathbf{u}'_{k-2}, \cdots\cdots \mathbf{u}'_1\}' \tag{A3}$$

is a vector of $\sum_{i=1}^{k} n_i$ hierarchical factors and

$$\begin{aligned}\mathbf{B} = \{&(\mathbf{P}_1 \cdots \mathbf{P}_k + \mathbf{E}_2\mathbf{P}_3 \cdots \mathbf{P}_k + \mathbf{E}_3\mathbf{P}_4 \cdots \mathbf{P}_k + \cdots + \mathbf{E}_{k-1}\mathbf{P}_k + \mathbf{E}_k)\mathbf{U}_k \vdots \\ &(\mathbf{P}_1 \cdots \mathbf{P}_{k-1} + \mathbf{E}_2\mathbf{P}_3 \cdots \mathbf{P}_{k-1} + \cdots + \mathbf{E}_{k-2}\mathbf{P}_{k-1} + \mathbf{E}_{k-1})\mathbf{U}_{k-1} \vdots \\ &\vdots \\ &(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3 + \mathbf{E}_2\mathbf{P}_3 + \mathbf{E}_3)\mathbf{U}_3 \vdots \\ &(\mathbf{P}_1\mathbf{P}_2 + \mathbf{E}_2)\mathbf{U}_2 \vdots \\ &(\mathbf{P}_1)\mathbf{U}_1\}\end{aligned} \tag{A4}$$

is the corresponding factor loading matrix for the hierarchical factors. Clearly, the set of hierarchical factors h obtained here is the same as that in the original Schmid-Leiman transformation (refer to (11)). But we now have a generalized form in (A4) for (8). A more compact representation for each layer of factor loadings is

$$\mathbf{b}_i = \left(\left(\prod_{j=1}^{i} \mathbf{P}_j \right) + \sum_{j=2}^{i} \mathbf{E}_j \left(\prod_{q=j+1}^{i} \mathbf{P}_q \right) \right) \mathbf{U}_i \qquad (1 \leq i \leq k), \tag{A5}$$

where the summation and the second product in (A5) will not be carried out, respectively, if $i < j$ and if $i < q$. Equation (A5) thus provides a transformation for obtaining hierarchical factor layers from a given higher-order factor model with (constrained) direct effects. It is a generalized version of the Schmid-Leiman transformation characterized by (13), which could be derived by setting all $\mathbf{E}_j$ in (A5) to null matrices.

To show the existence of the corresponding generalized inverse transformation, (A5) is re-expressed as

$$\begin{aligned} \mathbf{b}_i \mathbf{U}_i^{-1} &= \left(\left(\prod_{j=1}^{i} \mathbf{P}_j \right) + \sum_{j=2}^{i-1} \mathbf{E}_j \left(\prod_{q=j+1}^{i} \mathbf{P}_q \right) \right) + \mathbf{E}_i \\ &= \left(\left(\prod_{j=1}^{i-1} \mathbf{P}_j \right) + \sum_{j=2}^{i-1} \mathbf{E}_j \left(\prod_{q=j+1}^{i-1} \mathbf{P}_q \right) \right) \mathbf{U}_{i-1} \mathbf{U}_{i-1}^{-1} \mathbf{P}_i + \mathbf{E}_i \\ &= \mathbf{b}_{i-1} \mathbf{U}_{i-1}^{-1} \mathbf{P}_i + \mathbf{E}_i, \qquad (i > 1) \end{aligned} \tag{A6}$$

which is Equation 20, as well as a generalization of (19). Given any matrix $\mathbf{B}$ that exhibits a hierarchical factor pattern, the inverse transformation amounts to finding solutions for $\mathbf{P}_1, \mathbf{P}_2, \ldots$, and $\mathbf{P}_k$, and $\mathbf{E}_2, \mathbf{E}_3, \ldots$, and $\mathbf{E}_k$. When the hierarchical factor loading matrix $\mathbf{B}$ satisfies the Schmid-Leiman proportionality constraints, all $\mathbf{E}_i$ are null matrices. In general, an inverse transformation algorithm starts with $i = k$ (the highest layer), where $\mathbf{U}_k$ is an identity matrix. With appropriate parameterization in $\mathbf{E}_i$ (such as the minimum correlation method, or the residual direct effect method discussed in the text) and $\mathbf{b}_i$ given, (A6) can be solved uniquely. Therefore we get solutions for $\mathbf{E}_k$, $\mathbf{P}_k$, and hence $\mathbf{U}_{k-1}$. The latter is replaced in the left side of (A6) for $i = k - 1$ so that all $\mathbf{P}_i$ and $\mathbf{E}_i$ will be solved successively. When $i = 1$, the finally step reduces to:

$$\mathbf{b}_1 \mathbf{U}_1^{-1} = \mathbf{P}_1. \tag{A7}$$

Hence, $\mathbf{P}_1$ is solved directly since U_1 has been obtained in the previous step.

TABLE 2.
An example of general hierarchical factor solutions (Source: McLeod and Thissen, 1997)

	Layer 2	Layer 1		
	g	F1*	F2*	F3*
1	.610	.434		
2	.628	.569		
3	.658	.377		
4	.527		.017	
5	.486		.116	
6	.474		.286	
7	.574		.373	
8	.656			.205
9	.583			.367
10	.649			.490

Illustrating the inverse transformation. With all $\mathbf{E}_i$ set to zero, we successfully recover the original higher-order factor solution reported in Schmid and Leiman (1957) by applying the inverse transformation to the Schmid-Leiman hierarchical factor solution in Table 1. To illustrate the case where no proportionality constraints are imposed on the hierarchical factor model, an example reported in McLeod and Thissen (1997) is used.

In Table 2, a two-layer hierarchical factor solution with a general factor is reported. The corresponding path diagram representation is shown in the lower right corner of Figure 1. We start with $i = 2$ for solving for $\mathbf{P}_2$, and $\mathbf{E}_2$. We attempt to find an equivalent second-order factor model with constrained direct effects. Because $\mathbf{U}_2$ is an identity matrix, (A6) becomes

$$\begin{pmatrix} .610 \\ .628 \\ .658 \\ .527 \\ .486 \\ .474 \\ .574 \\ .656 \\ .583 \\ .649 \end{pmatrix} = \left(\begin{pmatrix} .434 & & \\ .569 & & \\ .377 & & \\ & .017 & \\ & .116 & \\ & .286 & \\ & .373 & \\ & & .205 \\ & & .367 \\ & & .490 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\lambda_1^2}} & & \\ & \frac{1}{\sqrt{1-\lambda_2^2}} & \\ & & \frac{1}{\sqrt{1-\lambda_3^2}} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \end{pmatrix} \right), \tag{A8}$$

where λ_1, λ_2, and λ_3 are factor loadings of the first order factors on the general factor, and e_1 to e_{10} are direct effects from the general factor. In (A8) there are 10 equations for 13 unknowns. For identification purposes, the direct effects from the general factor to X3, X7, and X10, respectively, are set to zero (i.e., $e_3 = 0$, $e_7 = 0$, and $e_{10} = 0$). The factor loadings can then be solved uniquely using the equations at the third, the seventh, and the tenth rows, respectively. For example, we have

$$\lambda_1 = \operatorname{sign}(.377)\sqrt{\frac{.658^2}{(.377^2 + .658^2)}} = .868.$$

Replacing the solution for λ_1 into (A8), the direct effects e_1 and e_2 are solved easily. For example,

$$e_1 = .610 - \frac{.434\lambda_1}{\sqrt{1-\lambda_1^2}} = -.148.$$

Continuing the same procedure, we finally have

$$\mathbf{P}_2 = [868, .839, .798]'$$

and

$$\mathbf{E}_2 = [-.148, -.365, 0, .501, .308, .034, 0, .385, .097, 0]',$$

as the solutions for the higher-order factor loadings and the direct effects from the general factor, respectively. Now, for $i = 1$, we use (A7) to obtain the following matrix equation

$$\begin{pmatrix} .434 & & \\ .569 & & \\ .377 & & \\ & .017 & \\ & .116 & \\ & .286 & \\ & .373 & \\ & & .205 \\ & & .367 \\ & & .490 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\lambda_1^2}} & & \\ & \frac{1}{\sqrt{1-\lambda_2^2}} & \\ & & \frac{1}{\sqrt{1-\lambda_3^2}} \end{pmatrix} = \mathbf{P}_1, \tag{A9}$$

where λ_1, λ_2, and λ_3 have been solved previously. Therefore,

$$\mathbf{P}_1 = \begin{pmatrix} .873 & 1.145 & .758 & & & & & & & \\ & & & .031 & .213 & .523 & .685 & & & \\ & & & & & & & .340 & .609 & .813 \end{pmatrix}'$$

is a solution for the first-order factor loading matrix

The minimum correlations method. When solving for $\mathbf{P}_2$ above, we could have set some other direct effects to zero for obtaining unique solutions for (A8). A specific set of such direct effects can be chosen so that the correlations among the first-order factors will have minimum correlations. This is equivalent to finding the minimum possible values for the loadings in (A6). For example, setting the direct effects from the general factor to X2, X7, and X10, respectively, to zero, accomplishes this goal. As a result,

$$\mathbf{P}_2 = [.741, 839, 798]',$$

$$\mathbf{E}_2 = [.131, 0, .242, .501, .308, .034, 0, .385, .097, 0]', \quad \text{and}$$

$$\mathbf{P}_1 = \begin{pmatrix} .646 & .847 & .562 & & & & & & & \\ & & & .031 & .213 & .523 & .685 & & & \\ & & & & & & & .340 & .609 & .813 \end{pmatrix}'.$$

The residual direct effects method. To eliminate the arbitrariness of the selection of zero direct effects, one may instead constrain the sum of direct effects within each first-order factor cluster at zero. For example, in (A8), we could set:

$$\begin{aligned} e_1 + e_2 + e_3 &= 0, \\ e_4 + e_5 + e_6 + e_7 &= 0, \quad \text{and} \\ e_8 + e_9 + e_{10} &= 0, \end{aligned} \tag{A10}$$

instead of setting $e_3 = 0$, $e_7 = 0$, and $e_{10} = 0$. Then the following solutions are obtained:

$$\mathbf{P}_2 = [.809, .934, .872]',$$

$$\mathbf{E}_2 = [.014, -.154, .140, .483, .184, -.270, -.400, .292, -.069, -.222]', \quad \text{and}$$

$$\mathbf{P}_1 = \begin{pmatrix} .738 & .967 & .641 & & & & & & & \\ & & & .047 & .323 & .797 & 1.040 & & & \\ & & & & & & & .418 & .749 & 1.000 \end{pmatrix}'.$$

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