Psychological Review
1992, Vol. 99, No. 3, 395–417

0033-295X/92/$3.00

Stagewise Cognitive Development: An Application of Catastrophe Theory

Han L. J. van der Maas and Peter C. M. Molenaar
Department of Developmental Psychology,
University of Amsterdam, Amsterdam, The Netherlands

In this article an overview is given of traditional methodological approaches to stagewise cognitive developmental research. These approaches are evaluated and integrated on the basis of catastrophe theory. In particular, catastrophe theory specifies a set of common criteria for testing the discontinuity hypothesis proposed by Piaget. Separate criteria correspond to distinct methods used in cognitive developmental research. Such criteria are, for instance, the detection of spurts in development, bimodality of test scores, and increased variability of responses during transitional periods. When a genuine stage transition is present, these criteria are expected to be satisfied. A revised catastrophe model accommodating these criteria is proposed for the stage transition in cognitive development from the preoperational to the concrete operational stage.

> Suppose that a team of developmental researchers stumbled upon a genuine stage in the development of some behavior. Would they be able to detect the stage that was right there in front of them? If they did notice it, how would they know what they had found? How would they be able to tell that it was a stage? What pattern of results would they look for? Despite the hundreds of studies of psychological development in recent decades, these questions still have no definitive answers. There is no uniform empirical criterion used by developmental researchers to determine when they have found a stage (K. W. Fischer, 1983, p. 5).

The concept of stages, as proposed by Piaget (Piaget & Inhelder, 1969), is a major subject of discussion in the field of cognitive development (Campbell & Bickhard, 1986; Emde & Harmon, 1984; Levin, 1986; Pinard, 1981). An important part of the debate concerns empirical criteria and methods for detecting stages and the interpretation of obtained results (K. W. Fischer & Silvern, 1985). In the relevant literature a distinction can be made between stage criteria and transition criteria. Piaget (1960) and the Genevean group used five explicit stage criteria for the acceptance of the stage hypothesis. These criteria—invariant sequences, cognitive structure, integration, consolidation, and equilibration—have met with considerable criticism (Brainerd, 1978; K. W. Fischer & Silvern, 1985). However, as for instance Flavell (1971) points out, stage-to-stage development entails qualitative changes, periods of major reorganization in thought between stages. This logical entailment leads to new criteria, transition criteria, which have been applied in cognitive development research with some success. Each of the transition criteria (bimodality, sudden spurts, response variability, and second-order transitions) have been satisfied in empirical research, yet have seldomly been investigated simultaneously (but see K. W. Fischer, Pipp, & Bullock, 1984). Also the rationale of each criterion has not been derived from an explicit formal transition theory.

Because empirical evidence on the basis of any one of these criteria may not be sufficient to accept a stage theory, the integration of transition criteria within a formal model is required (Connell & Furman, 1984). Such a theoretical model of transitions can be found in a branch of mathematics called *catastrophe theory* (Thom, 1975). This general theory of discontinuities (catastrophes) yields a set of mathematically derived criteria to detect discontinuities (Gilmore, 1981). An important part of this article is concerned with the correspondence between Gilmore's transition criteria, called *catastrophe flags,* and criteria used in cognitive development research. It is argued that catastrophe theory incorporates a basic formal model of transition criteria and thus enables the integration of empirical evidence for stage transitions into a powerful argument for stagewise development.

The connection between Piaget's stages and catastrophe theory has been noticed before (Freedle, 1977; Klahr & Wallace, 1976; Molenaar & Oppenheimer, 1985), and a few catastrophe models of stagewise development have been proposed (Preece, 1980; Saari, 1977). These models are based on the original goal of catastrophe theory, namely, the classification of all possible discontinuities in a series of elementary catastrophes. One of these elementary catastrophes, the cusp model (discussed in the next section), has been applied in many fields (Guastello, 1984; Zeeman, 1976). The construction of a cusp model (or any other elementary catastrophe model) starts with the identification of variables that control the transition. In the case of cognitive development consensus about these control variables (the underlying forces of development) is hard to find. For instance, the models of Saari and Preece differ importantly with respect to the nature of control variables.

The models of Saari (1977) and Preece (1980) are discussed later. Neither model has been tested empirically. Saari's model lacks specificity and conflicts with Piaget's theory on some points. Strictly speaking the model of Preece is not about stagewise development; it is a model of sudden jumps in the responses of so-called nonconservers. Therefore, we propose a

This research was supported by the Dutch Organization for Scientific Research and the Department of Psychology of the University of Amsterdam. We thank Juan Pascual-Leone, Kurt Fischer, Conor Dolan, and Maurits van der Molen for helpful comments on earlier drafts of this article.

Correspondence concerning this article should be addressed to Han L. J. van der Maas, Department of Developmental Psychology, University of Amsterdam, Roetersstraat 15, 1018 WB Amsterdam, The Netherlands.

new catastrophe model that is formulated on the basis of empirical findings of relevant neo-Piagetian research.

This article is organized as follows: First, we start with an introduction to catastrophe theory, in which the cusp catastrophe is explained. Second, the empirical evidence for transitions in cognitive development is summarized and related to Gilmore's catastrophe flags. Third, the models of Saari (1977) and Preece (1980) are discussed, and a new improved model for stage-to-stage transitions in cognitive development is proposed.

Aspects of Catastrophe Theory

Although the concepts of stage and transition are intrinsically vague in the context of very complex processes like cognitive development, they are well defined in simpler systems. The transition between the liquid phase and the gaseous phase of water, that is, boiling, is a well-defined discontinuity. This qualitative change in a physical system can be modeled using nonlinear dynamical system theory. Molenaar and Oppenheimer (1985) showed that such dynamic models of increasing complexity transcend the dichotomy between the organismic and mechanistic paradigms. Catastrophe theory, as part of nonlinear system theory, describes physical processes like boiling but has also been applied in the social sciences (Zeeman, 1976). Successful applications in psychology can be found in, for example, perception research (Stewart & Peregoy, 1983; Ta'eed, Ta'eed, & Wright, 1988) and clinical psychology (Callahan & Sashin, 1990).

As in regression models, catastrophe theory relates behavioral (dependent) to control (independent) variables. Catastrophe theory models concern discontinuities in behavioral variables as a function of continuous variation in the control variables. The restriction to continuous variation of the control variables is essential because many discontinuities in psychological behavior may be due to simple discontinuous changes in the independent variables and then do not constitute a genuine stage transition because of intrinsic reorganization.[1] The relation between behavior and the control variables is given in mathematical expressions that can generally (but far from exclusively) be conceived of as representing a dynamic system that, in varying situations, continually seeks to optimize some function (Poston & Stewart, 1978, p. 2). That is, most applications of catastrophe theory involve systems that minimize some quantity like energy, or (as a psychological example) cognitive dissonance or maximize one, like entropy, or (as another psychological example) degree of adaptation. This process of optimization implies the concept of equilibrium: For fixed values of the independent control variables the behavior will change until it reaches a stable state. The definition of *equilibrium* in catastrophe theory can be given in general mathematical terms (setting the first derivative of the dynamic system equations to zero) and covers continuous and discontinuous changes of the configuration of stable behavioral states as a function of the control variables. More specifically, this definition covers the Piagetian equilibration concept as applied in the field of cognitive development.

Another important concept of catastrophe theory is structural stability, which is also commonly used but less uniformly defined in the literature (cf. Poston & Stewart, 1978, p. 94). Intuitively, it means that scientists assume that empirical facts are repeatable under similar conditions and that small changes in these conditions will not change the results of their experiments dramatically. In catastrophe theory this means that various perturbations will not change the type of (continuous or discontinuous) relation between control variables and the configuration of stable behavioral states. A mathematical example may explain this. Suppose we perturb the function X^3 with a term ϵX, where ϵ is a small positive or negative constant. As we see in Figure 1, the resulting function $X^3 + \epsilon X$ has different configurations of equilibria for various ϵ (1 maximum and 1 minimum, 1 point of inflexion, and no equilibrium point at all, respectively). Therefore X^3 is not structurally stable (see for further review Poston & Stewart, 1978, p. 64).

Catastrophe theory concerns the study of equilibrium behavior of a large class of mathematical system functions that exhibit discontinuous jumps (more precisely, the points of functions where at least the first and second derivates are zero). Thom (1975), a major proponent of catastrophe theory, has proven that a large class of structurally stable system functions (involving up to four control variables) showing discontinuous behavior can be classified into seven archetypical forms (the elementary catastrophes) by means of a set of smooth transformations of the system variables. This result forms the basis of catastrophe theory. The maximum number of four control variables is not a real limitation because catastrophe theory concerns only variables that are actively involved in the transition (Poston & Stewart, 1978). Although hundreds of variables could be of importance in psychological systems, most transitions are locally controlled by changes in only a few variables.

The most popular elementary catastrophe is the cusp. All structurally stable discontinuities in three dimensions can be transformed to a cusp. The cusp function $V_{yx}(Z)$ (behavior variable Z and two control variables Y and X, called the splitting and the normal variable, respectively) is denoted by

$$V_{yx}(Z) = 1/4\, Z^4 - 1/2\, Y Z^2 - Z X.$$

It gives rise to the discontinuous equilibrium function after setting its first derivate with respect to Z to zero:

$$Z^3 - Y Z - X = 0.$$

For a continuum of values substituted for Y and X in this cubic equation, a range of values for Z is obtained. This relation between X, Y, and Z can be elucidated by a geometrical representation in three dimensions (the equilibrium surface of points for which the equation holds, see Figure 2).

In Figure 2, the folded smooth surface of the cusp is shown, and two paths are indicated (a and b), involving discontinuous and continuous change. In the bottom left-hand corner the axes are shown. The plane defined by X and Y is called the *control plane*. For a specific range of the control variables more than one value of the behavior variable is possible (the folded part of the surface). This particular area in the control plane is called the *bifurcation set*. As can be seen in Figure 2, in this set three

[1] In their discussion of the transition concept, Connell and Furman (1984) allowed discontinuous causes of behavior, for example, entering a new school.

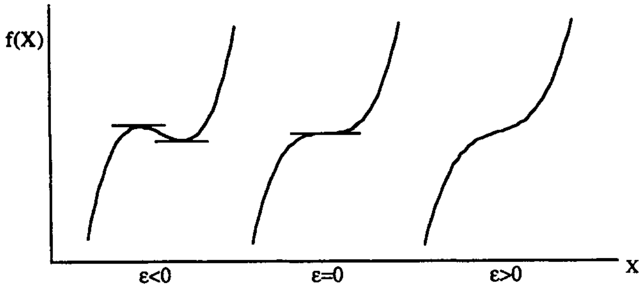

Figure 1. For three distinct values of the perturbation ϵ the configurations of equilibria of $f(X) = X^3 + \epsilon X$ are shown (1 maximum and 1 minimum, 1 point of inflexion, and no equilibrium point at all, respectively).

modes of behavior are possible; the middle sheet, however, represents unstable states and therefore is called the *inaccessible region* (see the Catastrophe Detection section). The presence of the two remaining modes of behavior implies a bimodal distribution of the behavioral variable.

Movement through the control plane leads to changes in the value of the behavior variable. A specific instance of an individual transitional process is obtained by description of the movement through the control plane, such as Paths a and b in Figure 2. To illustrate, the various characteristic paths are explained by means of the well-known model of aggression in dogs proposed by Zeeman (1976). Zeeman modeled sudden changes from attack to flight as a function of the emotions of a dog. The behavior dimension (or variable) ranges from attacking to fleeing. This behavior is assumed to be controlled by fear and rage. Zeeman argued that these controls can be measured independently of the observable behavior. A linear (noncatastrophic) model of this behavior is given in Figure 3.

In this linear model fear and rage are seen as the opposites of one dimension. Clearly, no discontinuity controlled by continuous change in fear–rage is possible in this model. In contrast, in Zeeman's (1976) model fear and rage are independent variables or conflicting factors. When a dog is both angry and frightened sudden changes from attack to flight can appear. A cusp model expressing this idea is shown in Figure 4a.

At the neutral point no fear or rage is present. Starting from this point, an increase in fear or rage only leads to a continuous increase of retreating or attacking. However, if rage is increased in an already fearful dog, an attack catastrophe appears (a sudden jump from the flight to the attack mode). The reverse process leads to a sudden jump from attack to flight. The magnitude of change depends on the distance from the neutral point.

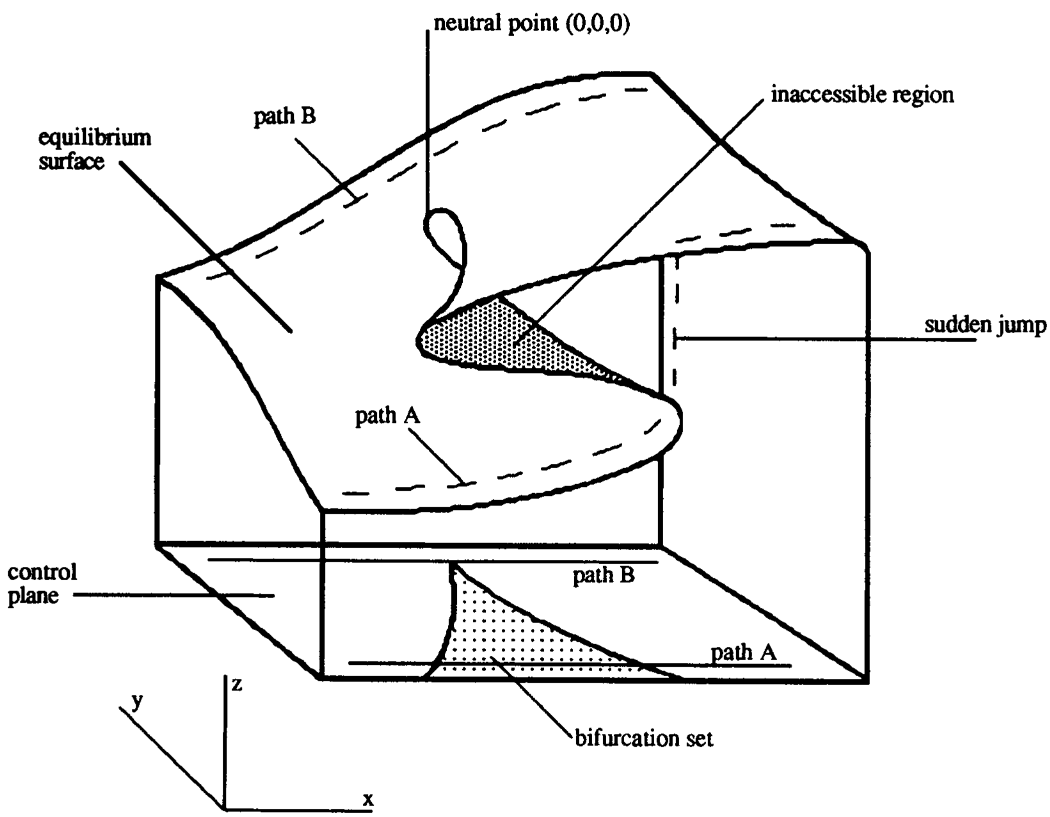

Figure 2. The smooth folded surface of the cusp is shown in three dimensions. (z represents the equilibria of the behavior or dependent variable, whereas x and y are control or independent variables. In the control plane, spanned by x and y, the bifurcation set is situated. In this area, for fixed values of the independent variables, there are two modes of behavior, and sudden jumps between these modes are possible.)

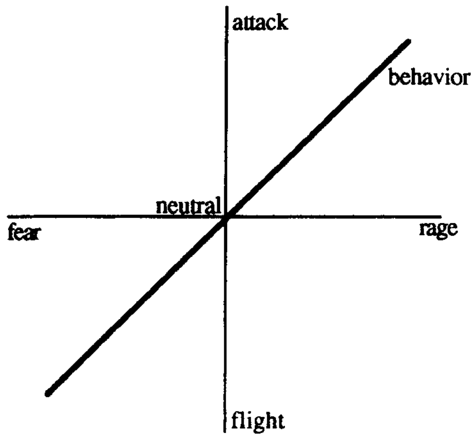

Figure 3. Linear model of dog aggression: The behavior variable attack–flight is a continuous function of the control variable fear–rage. (In this model small variations in fear–rage do not give rise to sudden changes between attack and flight.)

A continuous change from flight to attack is still possible, if first fear is decreased before rage is increased, thus transversing the neutral point. Both catastrophes (attack or flight) take place at different positions on the control plane. This is called *hysteresis* and is shown in Figure 4b as a cross section of the cusp along Path a. Movement along this path from left to right will lead to a jump upwards. Going back from right to left will lead to a jump downwards but on a different position more to the left in the control plane. In the dog aggression model this has the following conceptual meaning: You have to anger a frightened dog very much before it will attack, but once it has attacked, it will continue doing so though its anger may have abated.

Besides attack and flight catastrophes, the cusp model implies divergence of behavior, often called a *bifurcation* (see Figure 4c, which depicts a cross section of the cusp perpendicular to that of Figure 4b). That is, a very small initial difference between two paths in the control plane could become very large when entering the bifurcation set. In Zeeman's (1976) model this is expressed as follows: If rage and fear are simultaneously increased from the neutral starting point, the upper or the lower sheet is followed (because the middle sheet is inaccessible). The dog will attack or flee, but it will not remain indifferent. These phenomena—bimodality, inaccessibility, hysteresis, and divergence—can be used as indicators of discontinuities and are therefore called *catastrophe flags.*

Notice that the formulation of the model is independent of time; time is not a control variable. Although catastrophe theory belongs to dynamical system theory (systems that evolve in time), in the ultimate formulation of elementary catastrophes time is implicit. Time is expressed in the model in the changes in the values of control variables, resulting in various trajectories through the control planes. Consequently, the time taken for changes in the behavioral variable is given by the velocity of change in values of control variables. However, this time-dependent relation does not apply to the abrupt jumps between the two equilibria in the bifurcation set. Although the catastrophic jump is depicted in the cusp surface as occurring instantaneously, an actual abrupt change in a behavioral variable will take a small but definite amount of time. For example, the boiling of water, that is, the transition between the liquid phase and the gaseous phase of water, takes a finite amount of time because of purely physical constraints and hence is not completely instantaneous.

It may seem strange that models without explicit time variables are applied to developmental phenomena, as is the case with this article. Yet the specification of paths in the control plane does imply time. Moreover this implicit treatment of time in catastrophe models is in agreement with an important notion in developmental research, namely, that time is not a genuine causal factor of development (Lewis, 1990; Wohlwill, 1973).

A last remark concerns the control axes in Zeeman's (1976) model. In Figure 2 the control axes (X and Y) are called *normal* and *splitting* variable, respectively. In Zeeman's model the axes are rotated 45°. In this rotated configuration of axes, the controls are called *conflict variables* (rage and fear are clearly conflicting). It is this latter configuration that we use in our model of a stage transition proposed in the Catastrophe Modeling section.

After a rapid popularization of catastrophe theory, mainly because of Zeeman (1976), some critical reactions followed (Zahler & Sussmann, 1977). These criticisms concern the alleged deductive power of catastrophe theory. The weak point of Zeeman's models is the lack of experimental evidence. Zeeman constructed many appealing catastrophe models in several disciplines but never tested them rigourously. Yet empirical investigation is possible, as is discussed in the next section.

Detection, Modeling, and Analysis of Catastrophes

In addition to catastrophe modeling, two other methods can be used in the application of catastrophe theory. One of these is catastrophe detection, which is based on the work of Gilmore (1981). It involves the application of so-called catastrophe flags, that is, typical properties of behavior that indicate the presence of a catastrophe. These flags are especially useful in social sciences applications, as is explained in the next section.

A second strong variant is called *catastrophe analysis* and consists of mathematical analysis of the dynamic equations of a transition process. Catastrophe analysis requires knowledge of the mathematical equations of the transition process. By repeated application of transformation techniques these mathematical equations are reduced to one of the seven archetypical topological forms specified by Thom (1975), the so-called elementary catastrophes. In Poston and Stewart (1978) the usefulness of catastrophe analysis is demonstrated in, for example, laser physics, the geometry of fluids, and optics. In the social sciences this variant of catastrophe theory is not yet feasible, as we do not know the precise dynamical equations of the processes governing, for instance, cognitive stage transitions. Hence, we do not discuss catastrophe analysis any further.

Catastrophe modeling is intermediate in strength between detection and analysis. Zeeman's (1976) cusp model of dog aggression is a typical example of catastrophe modeling, involving the direct specification of behavioral and control variables in one of the elementary catastrophes. An intricate problem with catastrophe modeling concerns the actual fitting of the

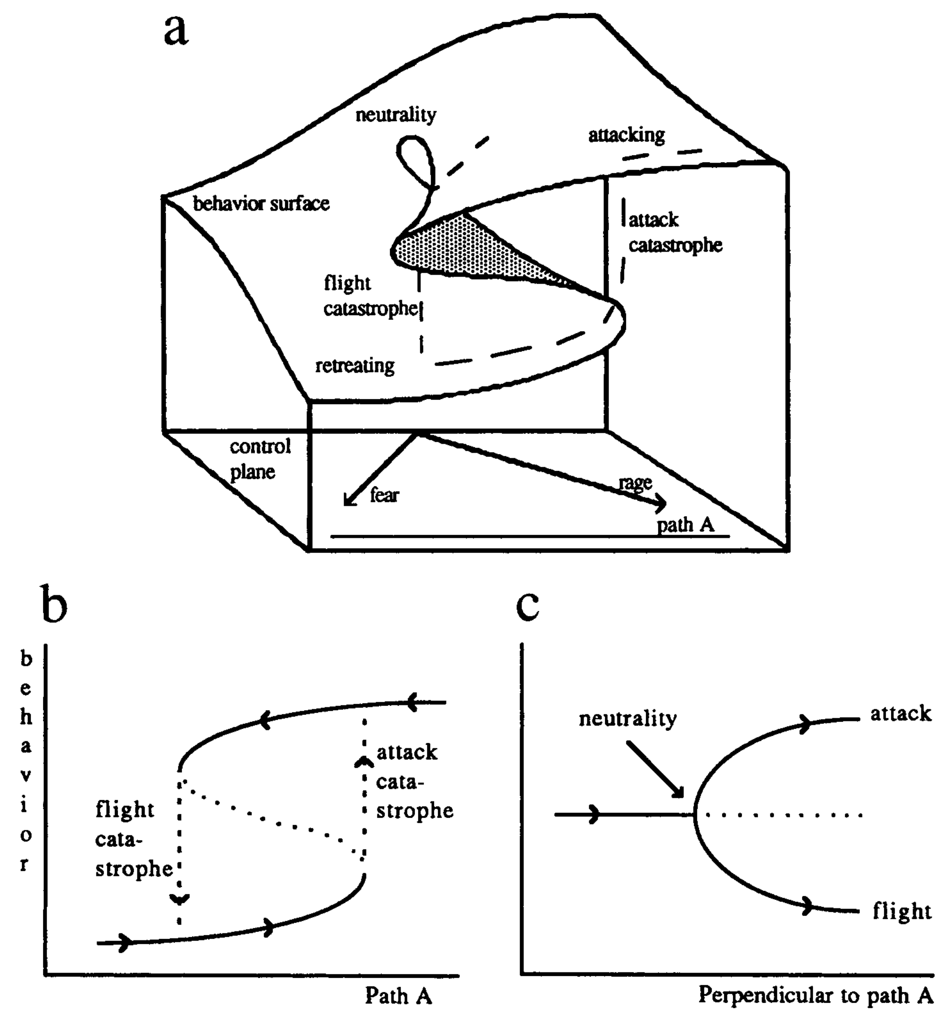

Figure 4. a: Dog aggression model of Zeeman (1976): In this nonlinear model attack–flight is still the behavior of interest as in the linear model shown in Figure 3, but fear and rage are now independent control variables. (If rage is smoothly increased in an already fearful dog, this may lead to a sudden change from flight to attack. From "Catastrophe Theory" by E. C. Zeeman, 1976, *Scientific American, 234,* p. 66. Copyright © 1976 by Scientific American, Inc. All rights reserved. b: A cross section of the cusp is shown along Path a in Panel a. That is, if movement through the control plane is limited to Path a, the cusp surface reduces to a two-dimensional projection (comparable with the linear model in Figure 3), depicting hysteresis. *Hysteresis* means that both catastrophes (attack or flight) take place at different positions on the control plane. This occurs if Path a is followed from left to right and backwards. c: The cross section perpendicular to Path a is shown, displaying divergence. An initial very small difference between two paths in the control plane could become very large when entering the bifurcation set. The dotted lines in Panels 4b and c correspond to the inaccessible region of the cusp.)

specified model to data. This problem is mainly a statistical one. Specifically, it is necessary to reformulate the deterministic model in probabilistic terms.

Statistical Catastrophe Theory

In a series of articles, Cobb (Cobb, 1978, 1980, 1981; Cobb, Koppstein, & Chen, 1983; Cobb & Watson, 1980; Cobb & Zacks, 1985) started to solve this problem by developing statistical techniques for the application of probabilistic catastrophe models. Parameter estimation techniques on the basis of the method of moments and maximum likelihood have been constructed. The latter method permits hypothesis testing according to the likelihood ratio test. This is not possible with the method of moments, but this method has the advantage of computational simplicity (Cobb, 1980). Both methods have been combined in a single computer program (Cobb, 1980) and have been used in several applications of catastrophe theory (Stewart & Peregoy, 1983; Ta'eed et al., 1988). In response to Cobb's (1980) approach, alternative estimation techniques have been developed. Guastello (1988) used a polynomial regression method that permits hypothesis testing, whereas Oliva, Desarbo, Day, and Jedidi (1987) presented a multivariate method to accommodate multiple behavioral and control indices. All these methods have problems, for instance regarding optimization of the likelihood function and the required transformations of the raw data. The transformations concerned are called *local diffeomorphisms* and include all kinds of smooth transformations. At present the available statistical techniques only consider a small subset of these transformations.

Catastrophe Detection: Empirical Evidence for Transitions in Cognitive Development

In this section we restrict the discussion to catastrophe detection. The presence of a catastrophe is invariantly associated with so-called catastrophe flags. Each flag is a behavioral property that has been mathematically derived from catastrophe theory by Gilmore (1981). Gilmore distinguished eight flags that are, as is shown, applicable in cognitive developmental research. Each one is discussed (in a nonmathematical fashion) in relation to experiments used to detect stage-to-stage transitions.

Five flags were briefly introduced in the previous section, namely, hysteresis, divergence, sudden jumps, inaccessibility, and bimodality. Additional flags are divergence of linear response, critical slowing down/mode softening, and anomalous variance. The first five flags will occur when the values of the control variables are inside the bifurcation set (see Figure 2). According to catastrophe theory these five flags occur simultaneously when the system is in transition. The three remaining flags can be manifest when the control variables are outside the bifurcation set and hence may occur before the bifurcation set is entered. An important point has to be made here: We actually want to distinguish between the sudden rapid spurt and the more general concept of transition. That is, we want to relate spurts to sudden catastrophic jumps, on the one hand, and transitions to the entire bifurcation set, on the other (see Figure 2). A system is in transition if it resides in the bifurcation set and has the potential to jump. However, a jump is only one of the five phenomena that characterize the transitional system state. Only the last three flags can be used as predictors of transitions because they may occur outside the bifurcation set.

The detection of catastrophe flags can be carried out straightforwardly and only requires the availability of behavioral measures. This may be especially useful in the initial analysis of stagewise cognitive development. In the following sections (see Table 1) we first discuss the general characteristics of Gilmore's (1981) flags and then proceed with a consideration of each flag in relation to traditional methods used in cognitive developmental research.

Gilmore's (1981) Catastrophe Flags

Gilmore (1981) gave a mathematical explanation of each catastrophe flag. The mathematical arguments involved are rather complex, so we restrict ourselves to a heuristic presentation.

In Figure 5 the occurrence of the 1st five flags is illustrated in a cusp model. In the Aspects of Catastrophe Theory section the equilibrium concept was introduced and the cusp surface, called the surface of equilibrium points. This means that each point of this surface represents a minimum or maximum of the function $V_{yx}(Z)$. In Figures 5 and 6 this relation between the cusp surface and $V_{yx}(Z)$ is elaborated. For given points in the control plane along Paths a and b in Figure 5, the corresponding projections of the function $V_{yx}(Z)$ are shown in Figure 6. These projections can be interpreted as areas in which a little ball tends to roll to the lowest point (minimum).

Table 1
Gilmore's (1981) Catastrophe Flags Related to Psychological Transition Research

Gilmore's catastrophe flags	Relations to cognitive developmental research
Modality and inaccessibility	Bimodal score distributions on developmental tests
Sudden jumps	Spurts in longitudinal data
Hysteresis	Regression in development
Divergence	Optimal test conditions
Divergence of linear response	Validity of training studies
Critical slowing down	Psychometric tests of conservation
Anomalous variance	Second-order transitions
	Oscillations in responses of transitional subjects

In these graphs the pattern of minima and maxima is shown for Points 1 to 7 along Path a and Points 1′ to 3′ for Path b. Moving from Points 1 to 7 leads to the following typical pattern of change. At Point 2 a point of inflexion emerges, which transforms to a local maximum and additional minimum when the bifurcation set is entered at Point 3. The two minima cause bimodality, and the local maximum gives rise to an inaccessible mode (at Point 4). At Point 5 the original minimum reduces, and a new unique minimum is established after leaving the bifurcation set (sudden jump at Point 6). Hysteresis occurs when Path a is followed in both possible directions, as is shown in Figure 5. When moving from Points 7 to 1 the ball remains in its minimum until Point 2 (instead of Point 6) where the sudden jump back will take place. Path b is added to Figure 6 to explain divergence. Starting at 1′ moving to the center of the bifurcation set leads to a splitting of the original unique minimum into two new minima and a maximum (at 3′). Which one of the two minima is chosen by the little ball depends on very small variations in its original position.

The remaining three flags, divergence of linear response, critical slowing down/mode softening, and anomalous variance, are of special importance because they are not restricted to the bifurcation set. Divergence of linear response and critical slowing down/mode softening concern the observable instabilities because of perturbations of the system. In the vicinity of a catastrophe such perturbations cause significant changes in the magnitude (divergence of linear response) and the relaxation time (critical slowing down/mode softening) of the behavioral variable or variables. The relaxation time is the time it takes to restore a perturbed equilibrium. The ball in its minimum position again provides a good example: the magnitude of its reaction to a perturbation and the time it will subsequently roll

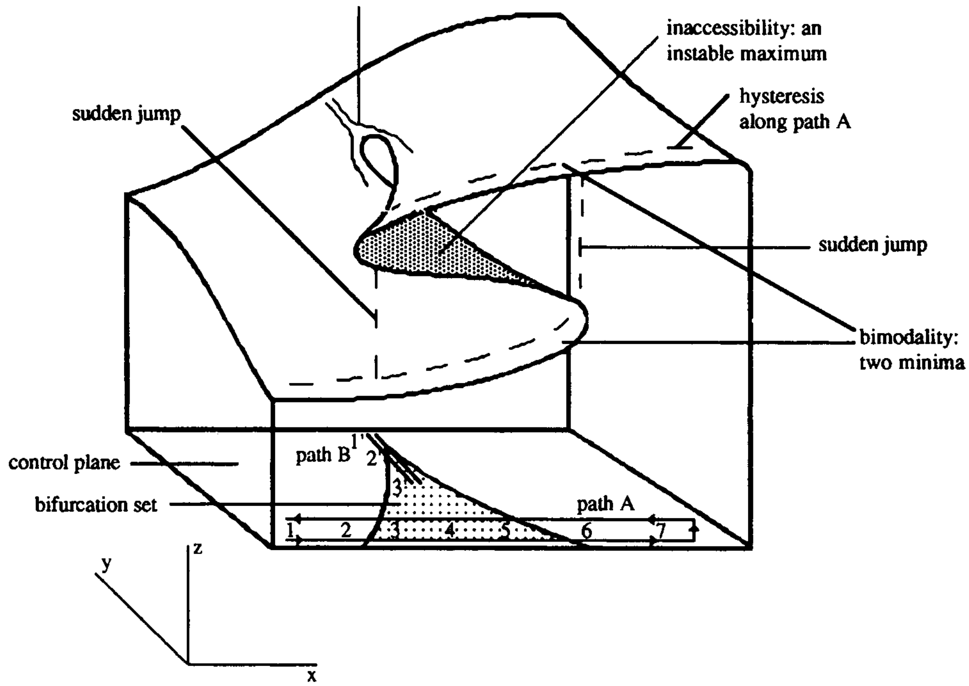

Figure 5. Five catastrophe flags—bimodality, inaccessibility, sudden jumps, divergence, and hysteresis —that are associated with the bifurcation set are here elucidated in a cusp model. (Two paths are indicated, which give rise to hysteresis and divergence. Each point on the cusp surface reflects an equilibrium state.)

around until it retakes its equilibrium position are larger near a catastrophe. Finally, anomalous variance relates to large increases in the variance of behavioral variables near a catastrophe.

If one flag is observed, one expects the others (except hysteresis) also to occur in case a genuine transition is taking place. Consequently, together they form a strong argument for the presence of a catastrophe. The proper ways in which the detection of each flag in psychological data has to be carried out may involve intricate methodological problems. For each separate flag we give plausible solutions and present examples of its application.

The presence of hysteresis is closely associated with a special condition called the *delay convention.* Both the delay convention and its alternative, the *maxwell convention,* concern the behavior in the bifurcation set. The delay convention implies that the system state remains in a local minimum of the energy function until that minimum disappears altogether. In contrast, the maxwell convention implies that the system always tries to move to the global minimum of this energy function. According to the maxwell convention, a sudden jump takes place on Position 4 or 5 (Figure 6), whereas according to the delay convention this jump takes place at Position 6. Which convention is obeyed by the system is an empirical issue that has to be settled by experiments. Moreover, this is not a matter of either–or: The delay and maxwell conventions are two extremes of a continuous scale of possible intermediate conventions (see Gilmore, 1981, chap. 8).

Gilmore's (1981) flags are particularly important for the definition of a transition. Earlier, we distinguished between a transition and an actual sudden jump; in that the latter (discontinuity) is only one property (flag) of transitions. Another important distinction is that between a sudden jump and a developmental acceleration. In psychological developmental research the difference between a genuine transition and an acceleration in a continuous process may not be immediately clear (see Figure 7).

It may be hard to differentiate between these two kinds of developmental changes on the basis of noisy data. However, by means of the catastrophe flags the discontinuity hypothesis is testable in that the remaining flags will not occur in the case of an acceleration.

Many authors either do not make this distinction or accept both as transitions (Kenny, 1983). In a recent article by van Geert (1991) stage transitions have been explicitly interpreted in terms of accelerations. van Geert's approach and the present one are closely related, as both start from nonlinear system theory. van Geert studied various phenomena in the development of cognitive and language skills by invoking modifications of the so-called Verhulst equation. In these modified equations rapid increases in independent variables (growth rate and limited resources) lead to even more rapid changes in growers (cognitive strategies and grammatical rules). According to catastrophe theory, however, such a rapid increase only constitutes a genuine sudden jump if it is caused by continuous small variations in the control variables. A mathematical analysis of van Geert's equations shows that these do not yield catastrophes. However, it is possible to extend the basic Verhulst

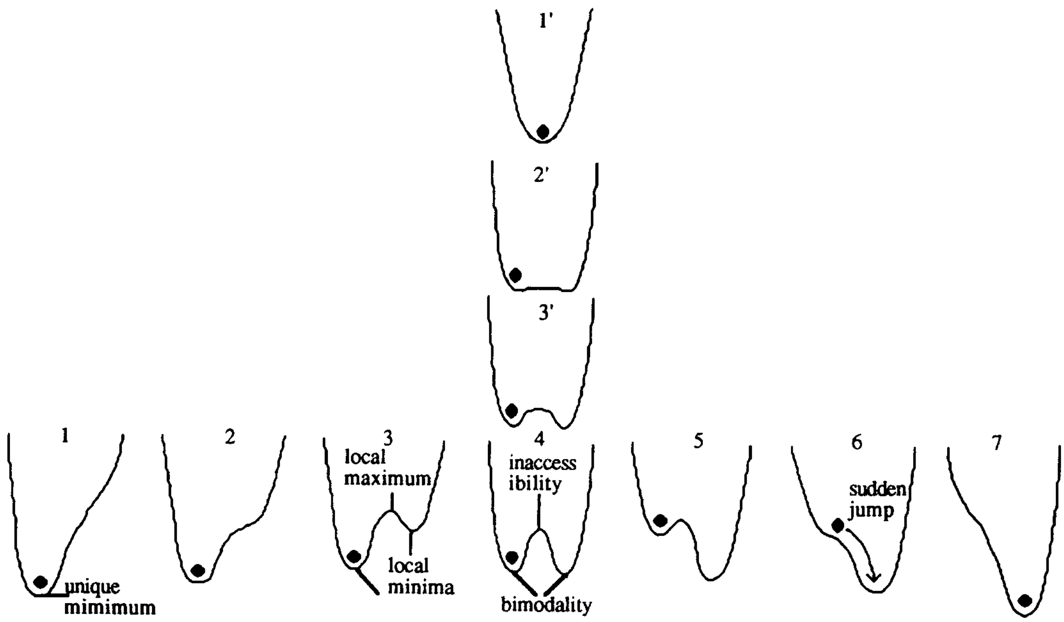

Figure 6. For given points in the control plane along Paths a and b in Figure 5, the corresponding projections of the function $V_{yx}(Z)$ are shown. (These projections can be interpreted as areas in which a little ball tends to roll to the lowest point (minimum). The two modes of behavior in the bifurcation set are represented as two distinct minima; inaccessibility as a local maximum that is not a stable state for the little ball. Moving from 1' to 3' leads to divergence, that is, a splitting of the original unique minimum into two new minima and a maximum. Movement from Points 1 to 7 and back from Points 7 to 1 gives rise to hysteresis in the form of sudden jumps when leaving the bifurcation set at Points 6 and 2, respectively.)

equation in such a way that it may give rise to genuine catastrophes. In this way, the present catastrophe approach could also be applied to van Geert's models.[2]

Modality and Inaccessibility: Bimodal Score Distributions

Modality and inaccessibility are discussed together, as both relate to the properties of bimodal score distributions. The relation between surfaces of equilibrium (like the cusp) and probability densities or score distributions is explained by Gilmore (1981, chap. 9) and Cobb et al. (1983).

Several authors in the field of cognitive development use bimodality as an important transition criterion. Wohlwill (1973) explained the occurrence of bimodality by referring to classical test theory. That is, the assumption that responses to the various items of a test are independent is not justified in the case of discontinuous behavior. According to Wohlwill, the high degree of interdependence among responses to different items may result in a sharply bimodal distribution of responses. In contrast, a continuous change will lead to a uniform or a normal distribution. K. W. Fischer and Bullock (1981) and K. W. Fischer et al. (1984) extended Wohlwill's argumentation. In a nutshell, they translate a spurt in cognitive level as a function of age directly into a distribution of scores. It then follows that only a low proportion of children will have intermediate test scores. This is shown in Figure 8.

Kenny (1983) and Tabor and Kendler (1981) gave similar reasoning as K. W. Fischer et al. (1984). Thus, the presumed relation between discontinuous cognitive behavior and bimodality is certainly not new. The occurrence of bimodality has been shown in several experiments for various age groups and in the context of various stage theories (cf. Kenny, 1983). Here we concentrate on cognitive stage theories and especially on the transition from preoperational to concrete operational thought.

Tabor and Kendler (1981) used two tasks, Piaget's class inclusion task and the optional shift task. The latter task is expected to produce a unimodal, bell-shaped distribution; this task serves as the control task. It was found that the Piagetian task produced clearly bimodal score distributions, whereas the optional shift task did not. Field (1987) also presented bimodal distributions for the number, length, mass, and liquid items of the conservation test.

Wohlwill (1973) referred to three conservation experiments, which all showed strong bimodal distributions. One of these

[2] The necessary modifications are the addition of an extra variable N and the introduction of a higher order constraint on the exponential growth term. That is, instead of the normal Verhulst equation, $X_{n+1} = r X_n (1 - X_n)$, the equation $X_{n+1} = r X_n (1 - X_n^2) + N$ must be used. van Geert (1991) already gave an interpretation for the additional variable N. In his article he assumed that the increase of growth must be autocatalytic: "Any increase that amounts to mere additions from an outside source is not genuine growth" (p. 4). If this constraint is skipped the variable N may represent these mere additions from an outside source.

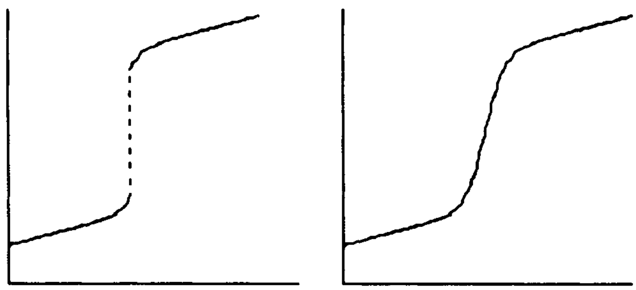

Figure 7. Schematic diagrams of a genuine sudden jump associated with a catastrophe (Panel a) and a rapid acceleration in a continuous developmental model (Panel b).

experiments was performed by Bentler (1970). The evidence for bimodality given by Bentler is very convincing. He used the complete Goldschmid and Bentler version of the conservation test (the standard conservation test at present) and assessed 560 children (aged 4 to 8 years) in a cross-sectional design. Amazingly, Bentler argued that the score distributions he obtained do not confirm the stage hypothesis. On the basis of the Piagetian classification in nonconservers, partial conservers (transition group), and conservers, he expected to find a trimodal distribution. However, the scores of partial conservers, children in the transition period, do not relate to a distinct mode of functioning. According to Wohlwill's argument as well as catastrophe theory, the partial conservers will be subsumed in the two other groups. Bentler displayed the results of Form A and B of the Goldschmid and Bentler test in a frequency table. Here, the same results are depicted as distributions. In Figure 9a the overall scores are presented (N = 560), whereas in Figure 9b the scores are displayed for each age group. The development of conservation scores shown in this figure suggests a discontinuity in the acquisition of conservation.

Although the degree of bimodality of the distributions shown in Figure 9 can be inspected visually, this has to be objectively ascertained by means of a statistical test. Everitt and Hand (1981) provided several statistical tools for the analysis of bimodal (mixture) distributions (see also Cobb & Watson, 1980, for an alternative approach). A recent application of their methods to psychological data can be found in Thomas (1989).

The use of bimodality as a transition criterion can be criticized in two ways. First, according to Zahler and Sussmann (1977), most applications of catastrophe theory in the social sciences model behavior that cannot be measured on a continuous scale. For instance, they argued that flight and attack (which are two modes of a continuous behavior variable in Zeeman's, 1976, dog aggression model) in reality are dichotomous. However, with respect to the conservation test this criticism does not apply. Even conservation tests concerning only single domains of conservation (for instance, conservation of volume) consist of slightly different items that cover different difficulty degrees. Together these items will form a one-dimensional continuous scale satisfying certain statistical criteria (cf. G. H. Fischer, 1974). (Incidentally, notice that this criticism, which presumably affects most social scientific applications, is a bit strange in the present context in that opponents of Piaget's stage theory claim that the acquisition of conservation is essentially continuous).

Wohlwill (1973) already pointed out that a similar criticism, that is, that many psychological tests consist of dichotomous items, is not appropriate. According to classical test theory, the responses to a set of independent dichotomous items add up to an approximately normal distribution. Consequently, according to Wohlwill, the occurrence of bimodality can only be due to interdependence of responses.

The second criticism of bimodality as a transition criterion is a more serious one. In a nutshell, this criticism implies that a discontinuity in behavior gives rise to a bimodal (or multimodal) distribution, but the reverse may not be true. That is, a bimodal distribution may be due to interdependence of responses that can be induced by various sources unrelated to discontinuous behavior (for examples, see Everitt & Hand, 1981). In Figure 7 we discriminate between a discontinuity and an acceleration in continuous development. Both a discontinuous and a continuous accelerated development may lead to a bimodal distribution (see Figure 8). Consequently, bimodality in itself is not sensitive to this important distinction. The answer to this problem was already indicated before: Gilmore's (1981) flags have to occur in concert so that the remaining flags can discriminate between the two sources of bimodality.

Problems related to bimodality of (conservation) score distributions may arise in training experiments. These problems concern (a) the assumption of a normal score distribution underlying most statistical methods and (b) the identification of transitional children (often called partial conservers). The first problem can be solved by the use of nonparametric or robust

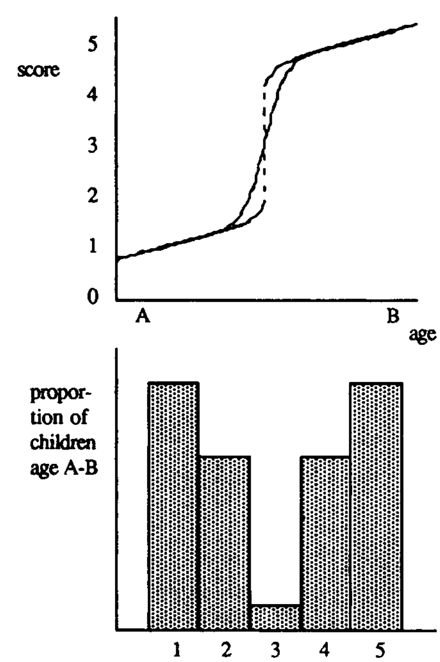

Figure 8. The upper frame shows developmental paths that may give rise to the bimodal distribution of scores depicted in the lower frame.

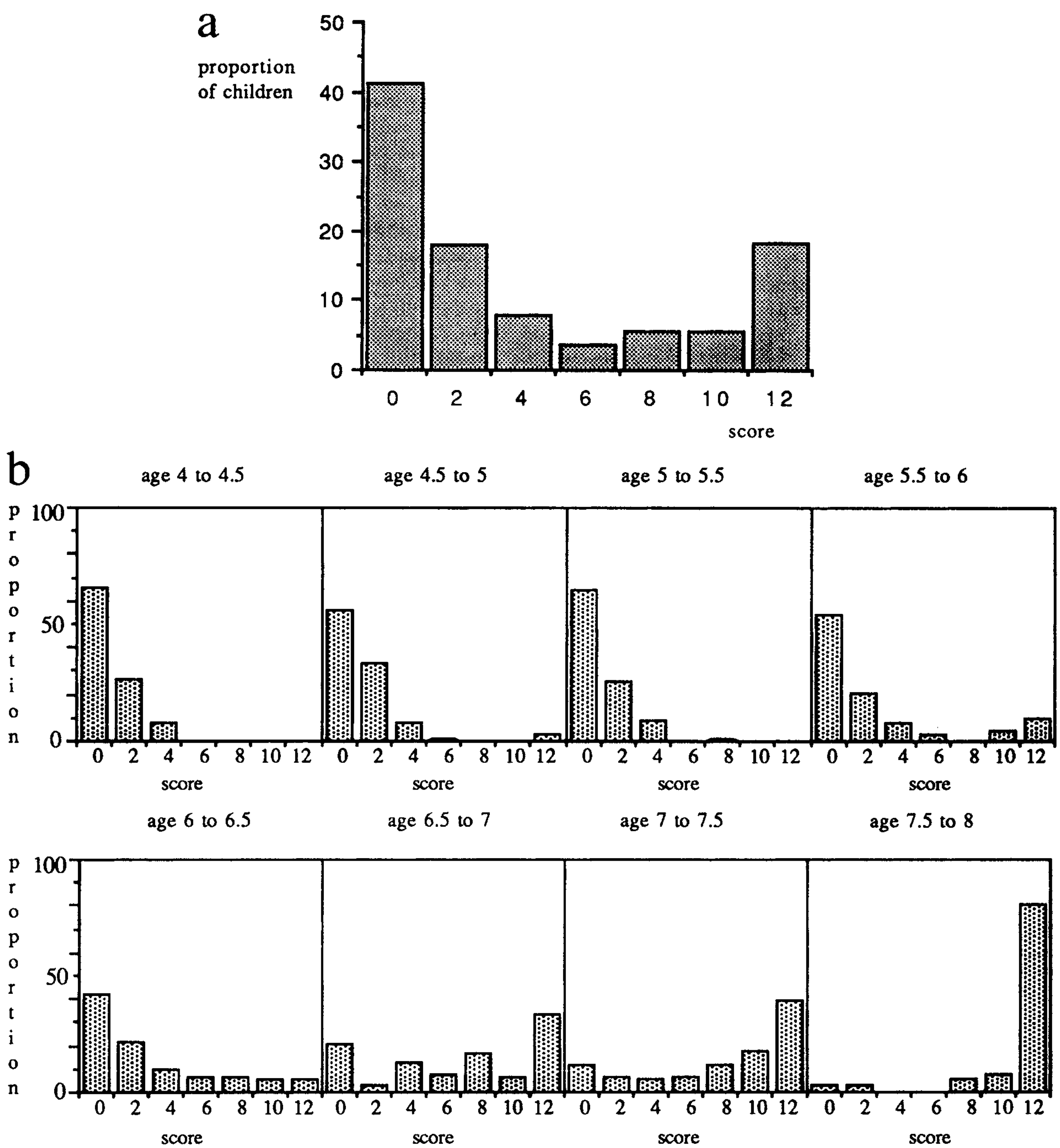

Figure 9. a: Bimodal distribution of conservation scores of 560 children. (Figure 9a is based on data from Bentler, P. M. "Evidence Regarding Stages in the Development of Conservation." *Perceptual and Motor Skills,* 1970, *31,* 855–859. © *Perceptual and Motor Skills,* 1970. Used by permission of author and publisher.) b: Distributions for eight age groups.

parametric statistical methods. The second problem, however is harder to solve. Essentially, this problem concerns the impossibility of discriminating between transitional subjects and nontransitional subjects if only scores at a single (pretest) measurement occasion are available, because these yield insufficient information to identify the catastrophe flags.

Sudden Jumps: Spurts in Longitudinal Data

The most obvious catastrophe flag, sudden jumps, is directly coupled to bimodality and inaccessibility. It has been often proposed as a transition criterion or seen as equivalent to a transition. In catastrophe theory the sudden jump is just one property of transitions. It is defined as a large change in the value of the behavioral variable because of a small change in the value of a control variable. In cognitive developmental research, K. W. Fischer and his colleagues (K. W. Fischer, 1983; K. W. Fischer & Bullock, 1981; K. W. Fischer et al., 1984; K. W. Fischer & Silvern, 1985) repeatedly recommended the use of the sudden jump or spurt criterion as an important transition criterion.

The method to detect spurts in developmental functions is based on a simple procedure. It consists of a yardstick (a developmental test) and a clock (age) for cognitive change. Each developmental test that consists of items constituting a Guttman-type scale can be used in a longitudinal design to test for spurts. Wilson (1989) presented a latent trait model for analyzing spurts in development. K. W. Fischer and his group discussed a number of applications of this method in different fields, for instance speech development, brain development, object permanence, and classification. In psychophysiological research strong evidence for spurts in electrocortical indices of brain development has been found by several authors (Hudspeth & Pribram, 1990; Stauder, Molenaar, & van der Molen, in press; Thatcher, 1991; Thatcher, Walker, & Giudice, 1987).

In our opinion, K. W. Fischer's scalogram method in longitudinal designs has two advantages. First, it allows for the study of individual developmental functions, and second, in case reliable frequently repeated measurements are available, one could try to discriminate between the presence of a discontinuity and an acceleration in development. Another method recommended by K. W. Fischer, second-order spurts, is discussed in the Anomalous Variance section.

Hysteresis and Divergence

Hysteresis, jumps at distinct values of the control variables, when the latter follow either an increasing or a decreasing path (see Figure 4b), may be difficult to detect. First, the magnitude of the hysteresis effect will decrease insofar as the pure delay convention is not obeyed. Second, for detection of hysteresis the actual control variables have to be known. For example, in perception research it is hypothesized that variation in shape of the Necker cube (perceived as either hollow or solid) controls the jumps between the two distinct modes of apperception. It then follows that systematic variation in stimulus shape must give rise to hysteresis.

If time or age would have been considered as a control variable of cognitive development, this would have hampered the detection of hysteresis because this involves effective manipulation of control variables. Specifically, to show jumps at distinct values of the control variable it is necessary to enter the bifurcation set from both directions. Obviously, this cannot be done with age as the control variable.

Instead of taking age as the control variable (cf. Lewis, 1990), several alternative explanatory variables that may feature as controls are proposed in the field of cognitive development. For instance, short-term memory and cognitive capacity (Case, 1985; Pascual-Leone, 1970). These explanatory variables could be used as controls in the detection of sudden jumps and hysteresis. That is, if we assume that short-term memory is a control variable for discontinuities in the acquisition of conservation, then a small increase in short-term memory may lead to a sudden change in conservation. However, these alternative explanatory variables may in themselves be difficult to manipulate as control variables (in particular, the experimentally induced continuous increases and decreases of a control variable required in detecting hysteresis). Possibly, in this respect, the threshold models proposed by Freedle (1977) may be of help in suggesting feasible experimental designs. In addition, hysteresis has been detected in simple neural networks that may be relevant to simulation of stagewise development (van der Maas, Verschure, & Molenaar, 1990a, 1990b).

The remaining flag associated with the bifurcation set is divergence (see Figures 4c, 5, and 6). It occurs when a small change in the initial value of a path through the control plane ultimately leads to large changes in the behavioral variable. The occurrence of divergence is not dependent on the actual (maxwell or delay) convention. It appears that divergence has not been used as a criterion for stage transitions in cognitive developmental psychology, hence no direct experimental evidence can be given. However, divergence may be related to the role K. W. Fischer et al. (1984) assigned to environmental test conditions. They argued that the detection of developmental spurts has to take place under optimal test conditions. That is, spurts will only become manifest if the performance on the developmental test is optimized. This suggests that the optimality of environmental conditions controls the discontinuity in a way that is represented by the splitting control variable in catastrophe theory. This variable is represented in Figure 2 as the splitting axis.

Change along this splitting variable axis leads to divergence of two modes of behavior: For low values jumps are smaller than for higher values. This can be seen in Figure 6 where Points 1' to 3' depict movement along the splitting axis of the cusp. For higher values of the splitting variable the minima are seen to be more distinct. Jumps along paths that are perpendicular to the splitting axis will be more pronounced for increasing values of the splitting variable (see Figure 10).

K. W. Fischer et al. (1984) presented empirical evidence for the role of optimal test conditions for the detection of spurts in longitudinal data. Incidentally, divergence could also be tested in cross-sectional designs in which suboptimal environmental

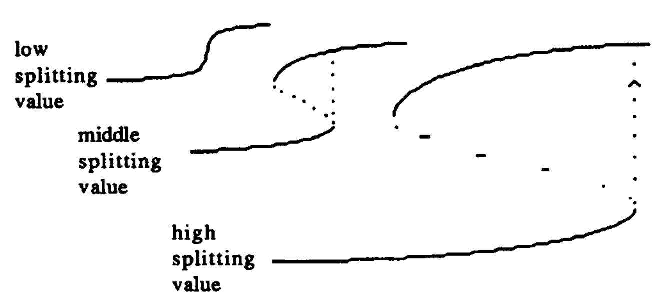

Figure 10. Increase in jumps as a function of increased splitting value. (This effect corresponds to what K. W. Fischer, Pipp, and Bullock, 1984, call the role of optimal performance. Only in optimal conditions can significant jumps appear.)

conditions are expected to generate unimodal distributions, whereas optimal conditions yield bimodal distributions of test scores. In the Catastrophe Modeling section we return to this hypothesis and to other interpretations of the splitting variable in stage-to-stage transitions. Both hysteresis and divergence can be statistically tested with empirical data by means of Cobb's algorithm (Cobb & Zacks, 1985).

Divergence of Linear Response: Effect of Training

The divergence-of-linear-response flag implies that perturbations of the control variables near a catastrophe point will lead to a large loss of stability and large oscillations of the behavioral variable. Divergence of linear response tells us also which behavioral variables are directly involved in the discontinuity. This flag is not easily operationalized in psychological developmental research because this requires the use of densely sampled time series in which the consequences of perturbations can be studied.

The divergence-of-linear-response flag may be of interest in relation to training experiments. A popular issue in developmental research concerns the effects of training (feedback, verbal rule explanation, measurement training, etc.) on the acquisition of stage-related cognitive capacities. Recently, Field (1987) reviewed training studies of conservation in relation to the theoretical stance of the researchers concerned. It appears that training studies differ in almost every respect: selection of nonconservers, use of delayed posttest, outcome criteria, and interpretation of obtained results. According to Field, some of these differences are due to distinct theoretical outlooks of researchers. Consequently, no consensus has been reached about the effects of training and the implications for Piaget's theory.

According to the catastrophe theoretical interpretation of Piaget's stage hypothesis, a positive effect of training may occur for transitional children (see also Murray, 1983). If the distance to a catastrophe point is small, training could cause variation along the control axes and thus induce enough divergence and loss of stability for a stage transition to occur. Kuhn (1974), in reaction to the review of Brainerd and Allen (1971), examined several problems with such training experiments, including the lack of agreement on methodological criteria for inferring change and the ambiguity in application of even the most stringent (Genevan) criteria. On the basis of catastrophe theory another problem arises. That is, one cannot be sure, on the basis of a low test score on a pretest or on the basis of a young age, that a child really is a nonconserver. It may also be a transitional child. As only transitional children will profit from training, this sample heterogeneity will seriously affect the outcomes of training studies. A related argument has been put forward by Flavell and Wohlwill (1969).

Critical Slowing Down: Delayed Recovery of Equilibrium

Another dynamic consequence of perturbations in the neighborhood of a catastrophe is the lengthening of the relaxation time. This flag does not concern the magnitude of change but the speed of reequilibration of the behavioral variable. Normally, it will take a relatively short time before a system reaches its equilibrium mode. This time will become longer and longer if the bifurcation set is reached.

This flag does not appear to have been used in cognitive development research, although it may be possible to do so. First, one needs an effective perturbation, and second, a proper measure for relaxation time is required. In general, training might be viewed as a perturbation. The operationalization of relaxation time is more difficult; perhaps a reaction time measure could be used for this purpose. However, reaction time measures may not be suitable for classical Piagetian tasks because of the complex interaction occurring between child and experimenter and the required verbal explanation. This problem can be solved by using so-called psychometric versions of these Piagetian tasks in which reaction time can be measured more effectively. Computer-regulated conservation tests have been found to correlate high with traditional conservation tests, like the Goldschmid and Bentler test, and could be used for this objective (van der Maas et al., 1992).

Anomalous Variance

Gilmore (1981) distinguished two consequences of catastrophes on the variance of behavioral variables. Generally, the variance may become large in the neighborhood of a catastrophe. Gilmore proved this by invoking a probabilistic formulation of catastrophe theory.

The first consequence for the behavioral variance arises from changes in the correlation structure if the system approaches a catastrophe point. The changes concerned imply that common factors will disappear. Accordingly, this increased behavioral variance will be associated with the second-order transitions or drops in correlations investigated in transition research (K. W. Fischer et al. 1984; see also Bornstein & Sigman, 1986; Rutter, 1984; Sternberg & Okagaki, 1989). A particularly successful application of methods for the detection of second-order transitions is given by McCall, Eichorn, and Hogarty (1977) and McCall (1983). They show sudden drops in stability of individual differences (changes in factorial structure) at various ages in which stage transitions are expected to occur.

The second consequence refers to the influence of a newly emerging equilibrium. The emergence of this second equilibrium is a source of anomalous variance. The increased behavioral variance is closely related to response variability or oscillations in the probabilistic transition model of Flavell and Wohlwill (1969). Their so-called disequilibrium–stabilization model possesses four phases. In the initial phase the children fail at all test items, whereas in the transitional phase children have low scores and many oscillations in their response patterns. Subsequently, in the stabilization phase the scores increase, whereas the number of oscillations decreases. In the final or terminal phase scores are very high, and oscillations are almost absent. This model corresponds to Piaget's model of transitions, whereas the expected sequence of behavior can be characterized by the use of a cross section of the cusp as shown in Figure 11.

The graph in this figure shows the deterministic equilibrium surface, whereas the variance in the probabilistic analogue is indicated along the abscissa. Notice that the two criteria, anomalous variance and oscillations, have the same expected pat-

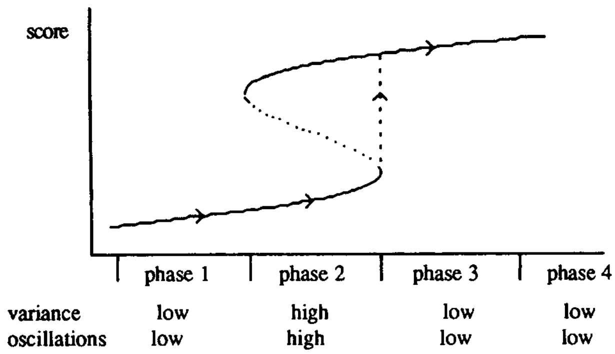

Figure 11. The relation between oscillations and anomalous variance. (The flag anomalous variance incorporates the slightly different conceptions of instability or disequilibrium as recommended in literature and places them within a formal theory of discontinuity. Variance and oscillations predicted by catastrophe theory and by Flavell and Wohlwill, 1969, respectively.)

tern. Flavell and Wohlwill (1969) presented empirical evidence for their model by reanalysis of a cross-sectional conservation data set of Uzgiris (1964).[3]

Instead of oscillations, other researchers used inconsistencies in verbal reports (Genevean method) or discrepancies between gestures and verbal reasoning (Breckinridge Church & Goldin-Meadow, 1986; Perry, Breckinridge Church, & Goldin-Meadow, 1988) as measures of disequilibrium. In fact a large body of developmental research is concerned with the concept of disequilibrium, and various definitions and operationalizations have been introduced. The flag anomalous variance incorporates these slightly different conceptions of instability and places them within a formal theory of discontinuity.

Catastrophe Modeling: Transition Models

Our review of applied criteria for stage transitions in cognitive development within the context of Gilmore's (1981) catastrophe flags constitutes a convenient starting point for the construction of a more encompassing transition model. Although the combined empirical evidence presented in the previous section consistently supports the reality of stage transitions, only the availability of a formal transition model on the basis of catastrophe theory will allow a direct integral test of this hypothesis. Unfortunately, the construction of a suitable catastrophe model meets with several difficulties. The most important one has been mentioned earlier: the choice of control variables constituting the driving processes of cognitive development. This is not a problem that we pretend to solve definitively; we only indicate some possible solutions and an approach for testing their validity.

We start with a brief discussion of two relevant catastrophe models, which have appeared in the literature: a qualitative model for the dynamics of cognitive processes (Saari, 1977) and a geometrical model of Piagetian conservation (Preece, 1980). Both models received relatively little attention, yet they are of interest for the ensuing discussion of stagewise development. Our alternative catastrophe model, a conflict model of stage transitions, is explained in the Conflict Cusp Model of Stage Transitions section.

The Model of Saari: Assimilation and Accommodation

Saari's (1977) model of stages in cognitive development reflects mainly his mathematical point of view and thus leads to an important discussion of several problems in the construction of catastrophe models. Saari, however, neglected almost all results of neo-Piagetian research. That is, his model only pertains to Piaget's equilibration process. As to this, Saari summarized Piaget's argumentation about the role of adaptation, assimilation, and accommodation as follows: "Any improvement in the level of organization must be discussed in terms of concomitant changes in the level of assimilation and accommodation, plus the strength of stabilizing efforts of the adaptation process" (p. 149).

This Piagetian equilibration process forms the basis of the model, which relates assimilation and accommodation as control variables to developmental level as a behavioral variable. Each variable is seen as a one-dimensional construct, where the behavioral variable includes all manifestations of organizing levels. In Piaget's theory the interaction between assimilation and accommodation is rather complex, involving two distinct but concurrent ways of processing information. Yet, a successful co-operation of these control variables is necessary for the qualitative increase of developmental level. Saari (1977) modeled this co-operation as a double cusp surface of equilibria, which is shown in Figure 12.

In Figure 12a two kinds of jumps are possible. One is due to an increase of assimilation, the other to an increase of accommodation. The upper side of the cusp surface represents a new stage in cognitive development. Continuous and discontinuous changes are allowed, and the model clearly shows the necessity of balancing accommodation and assimilation controls. Notice that Saari's (1977) model consists of two coupled cusps. In Figure 12b (in which the uninterrupted lines represent the lines of jumps or fold lines) a typical path through control space is shown.

Figure 12 represents one stage transition. According to Saari (1977), it would be possible to model a series of stage transitions in this way, although no precise details are given. An additional important component of Saari's model is the role of adaptation. In Saari's view adaptation is the strength of attraction of the equilibrium surface and is called the *adaptation vector field.*

This theoretical model relates some basic concepts of Piaget's equilibration theory in a way that explains major properties of cognitive development. It is difficult to ascertain whether the mathematical operationalizations of these basic concepts correspond to Piaget's ideas in a valid way. As to this, Saari (1977) discussed several issues, for instance, the presumed distinctiveness of assimilation and accommodation, the possibility of regression, and the reduction of complex processes like

[3] The only intricacy was the number of oscillations in Phase 3 (and between Phases 2 and 3); this was higher than expected, possibly because the Maxwell convention is being obeyed.

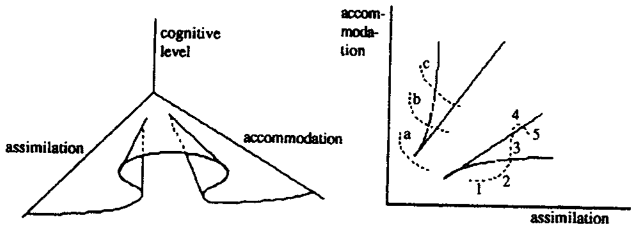

Figure 12. a: In Saari's double cusp model the balancing of assimilation and accommodation leads to sudden jumps to a new stage in development. b: A path in control plane. (Change from Points 1 to 5 leads to a jump between Points 3 and 4. Points 1, 2, and 3 are situated at the lower sheet; Points 4 and 5, at the upper sheet. Points 3 and 5 are located in the bimodal area. Paths a, b, and c illustrate the possibility of individual differences in paths, locations, and magnitudes of the transitions. Path a does not lead to a sudden jump, and Path c will show a larger jump than Path b. Figures 12a and 12b are from "A Qualitative Model for the Dynamics of Cognitive Processes" by D. G. Saari, 1977, *Journal of Mathematical Psychology, 15,* pp. 160 and 162, respectively. Copyright 1977 by Academic Press. Adapted by permission.)

assimilation and accommodation to one-dimensional constructs.

There are two issues that Saari (1977) did not discuss: the empirical test of the model and the intrinsic local character of catastrophe models. To start with the latter issue, Saari's model predicts stage transitions in all cognitive developmental processes in which assimilation and accommodation are involved. This is in contradiction to Piaget's theory. As Saari already pointed out, Piaget's theory is highly task specific. Hence, a catastrophe model must be restricted to the local behavior in the vicinity of a single transition. In the Conflict Cusp Model of Stage Transitions section we return to this point. The first problem, concerning the empirical test of the model, has been alluded to earlier in our discussion of Zeeman's (1976) model. For Saari's model, empirical tests will present problems with respect to the operationalization of the assimilation and accommodation controls. This problem may be hard to solve. Assimilation and accommodation are highly abstract theoretical concepts that underlie a multitude of manifest cognitive processes. There appear to be no cognitive tests available that, for example, distinguish conservation from assimilation and accommodation. Some explicit definitions of assimilation and accommodation provided by Sommerhoff (1969) may lead to a solution of this problem.

Preece's (1980) Model: Acquisition of Conservation

Preece (1980) rejected the global character of Saari's (1977) model and therefore constructed a local model for the transition from preoperational to concrete operational thought. The model gives an interesting view of conservation acquisition, is testable, yet involves a discontinuity in the responses of children within the preoperational stage instead of between the preoperational and the concrete operational stages. This subtle change in the actual focus of catastrophe modeling is an important one. Preece's model is, strictly speaking, not a model of stage transitions. It is, however, a catastrophe model related to Piaget's theory and therefore is discussed next.

Preece (1980) used the conservation of weight as an example to illustrate his model. Conservation of weight is tested with two clay balls of the same form and the same weight. One of these balls is rolled in the form of a sausage by the experimenter. Then three responses of the child are possible: The weight is considered to be still the same, it's more, or it's less. Preece expected a switch in responses of nonconservers when one clay ball is rolled in a sausage that becomes progressively longer and thinner. First a nonconserver would say something like "There is more because it is longer," and after a while "Now there is less because it is too thin." This switch from a wrong answer to another wrong answer is due to a switch in focusing on one dimension (length) to another dimension (thickness) and the inability to integrate information about both dimensions. According to Preece, this inability to integrate is caused by a limited cognitive capacity. For the preoperational child the relation between thickness of the clay ball and the response is presented in Figure 13.

The sequence a, b, c, and d represents the effect of rolling the clay ball in a progressively longer and thinner sausage. First, at Position a, the two clay balls are equal in form and judged as equal. From Positions a to c the distorted clay ball is seen as longer and hence as heavier. Then at c the increased salience of thinness causes a jump to the opposite judgment. Preece (1980) expected that a decrease of thinness from Point d would lead to a reverse jump at another Position e. This is a typical example of hysteresis. Preece's model constitutes a cusp model, as shown in Figure 14, with thickness of clay balls and cognitive capacity as control variables.

The front of Figure 14 is identical to the cross section shown in Figure 13. An increase in cognitive capacity (i.e. moving toward the back of Figure 14) yields a flat curve representing the correct answers of conservers (consistent equality responses irrespective of thickness). The pocket of conservation shown in Figure 14 is a property of a higher order catastrophe model, the butterfly catastrophe, and is added to the model to explain the acquisition of conservation. In the pocket of conservation the inaccessible region (the interrupted line between c and d in Figure 13) now becomes a structurally stable equilibrium. This ad hoc extension of the model suggests that it could explain the

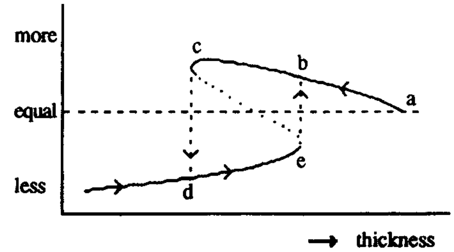

Figure 13. Switch from one wrong answer to another wrong answer. (Variation in thickness of clay sausage may lead to the hysteresis effect. Reproduced with permission of authors and publisher from: Preece, P. F. W. "A geometrical model of Piagetian conservation." *Psychological Reports,* 1980, *46,* 143–148. © *Psychological Reports,* 1980.)

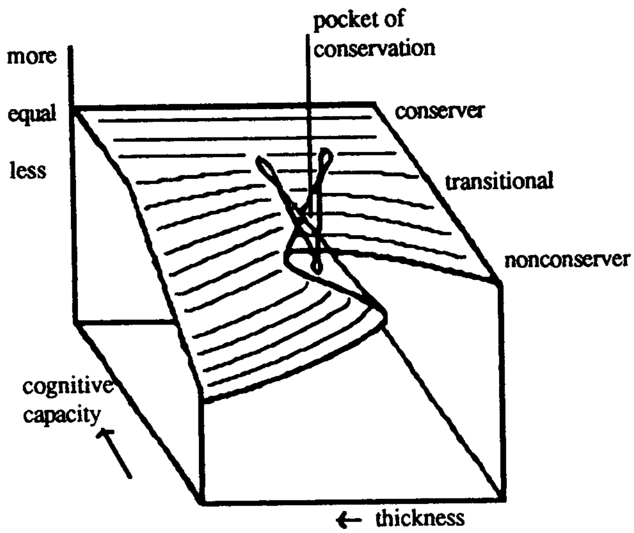

Figure 14. The complete Preece (1980) model, with thickness of clay sausage and cognitive capacity as control variables. (An increase in cognitive capacity, that is, moving toward the back of the figure, yields a flat curve representing the correct answers of conservers. The pocket of conservation is a property of the butterfly catastrophe and should explain the acquisition of conservation. Reproduced with permission of author and publisher from: Preece, P. F. W. "A Geometrical Model of Piagetian Conservation." *Psychological Reports,* 1980, *46,* 143–148. © *Psychological Reports,* 1980.)

acquisition of conservation. However, Preece (1980) did not present further details concerning this extension to a butterfly catastrophe. Yet, two problems immediately arise. First, the butterfly catastrophe includes four control variables, whereas only two control variables have been specified in Preece's model. Second, it is not clear whether every child has to follow a path through the pocket of conservation.

Preece's (1980) model is not a model of the discontinuity between preoperational and concrete operational thought. It pertains to a discontinuity in nonconservation responses only. As far as we know there is no firm empirical evidence for a hysteresis effect along the thickness of clay dimension, as presumed by Preece.

Conflict Cusp Model of Stage Transitions

A problem of Saari's (1977) model is the lack of specificity. It is a limitation of catastrophe theory that it provides local descriptions of transitions instead of global descriptions of series of stages. It is necessary to apply catastrophe theory to just one transition at a time. The advantage of Preece's model is that it concerns only one transitional period. With Preece we share the interest in the same transitional period (between preoperational and concrete operational thought), but we choose alternative behavioral and control variables. We present a preliminary cusp model that relates discontinuities in conservation acquisition to the set of different strategies for solving conservation items by an appeal to the concept of cognitive conflict. In this model the general concept of cognitive level, as used by Saari, has been restricted to conservation only. Although the model has not yet been applied to real data, it is based on empirical results obtained in experiments discussed in the foregoing sections.

The model is presented in three steps. First, the behavioral variable is introduced; second, the control variables are specified and explained. In the third step we introduce the cusp model, starting with the deterministic variant and then proceeding to the final, probabilistic model. In the closing section some connections with theories of cognitive development are discussed.

Behavioral variable: Conservation. The choice of conservation instead of cognitive level would seem appropriate for at least three reasons. First, according to Piaget's theory conservation will specifically discriminate between the actual stages in which we are interested. Second, conservation has been intensively studied, yielding several alternative (continuous) models with which our cusp model can be compared. Third, conservation is, in contrast with cognitive level in general, a behavioral variable that is, in principle, measurable so that empirical verification is possible.

Regarding this last point, there have been issues with the way conservation should be measured. This discussion concerns the procedure of scoring conservation tests and is directly relevant to our present purposes. *Conservation* has been defined as the invariance of a characteristic despite transformation of the object or collection of objects possessing this characteristic (Field, 1987). Conservation ability is measured by conservation tasks that combine a particular characteristic with a particular transformation. In Figure 15 a typical conservation item is schematically shown: it depicts the volume of liquid that is poured from one glass into another.

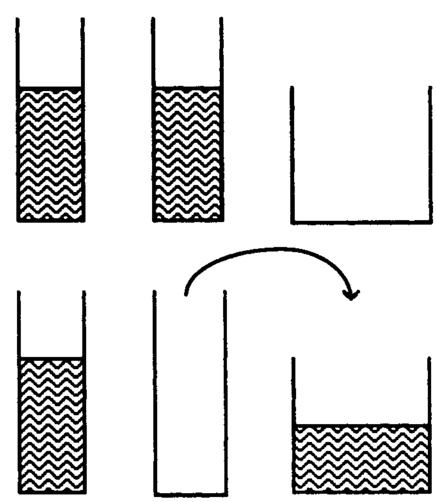

Figure 15. A schematic view of a typical conservation item of volume. In the initial situation two equal glasses are filled with the same amount of liquid. After the transformation the child has to compare the amounts of liquid in Glasses A and B. A nonconserver will focus on the difference in the level of liquid and ignore the information of the initial situation.

The question that is asked after the transformation (the arrow) is "Is the amount of liquid in A equal to B, or has one of them more liquid in it?" Normally, a child will respond to this question by answering "equal" and "unequal." This first response is called the *judgment.* The simplest way to score items of conservation tests is based on judgments only, that is, the response is correct (equal) or not correct (other judgment). However, apart from this judgment, Piaget asked for explanations of judgments and involved these explanations in the scoring. In the scoring procedure he used, a correct judgment and explanation are required. Brainerd (1973), on the other hand, limited the scoring procedure to the judgments. In the Goldschmid and Bentler conservation test a correct judgment without an appropriate argumentation is scored as being intermediate between correct and incorrect.

Which procedure should be used is still unclear (Kingma, 1984; McShane, 1991; van der Maas et al., 1992). According to McShane, this is a fundamental problem that has been extensively discussed in the literature but, perhaps unexpectedly, still has not been resolved. Brainerd's (1973) procedure has been criticized because it may overestimate conservation ability, whereas success of unusual nonconservation strategies cannot be ruled out. On the other hand, the validity of verbal explanations obtained by means of the Piagetian method can be questioned because they are sufficient, but not necessary, conditions for the presence of logical abilities and hence may lead to underestimation of true ability.

A possible solution to this problem involves the statistical analyses of the judgments only, using a set of many items that are specifically constructed to disentangle conservation, nonconservation, and, for example, gambling strategies. Presently, we restrict attention to the simple scoring criterion, as it has been recommended by Brainerd (1973), that is, a test score that is based on judgments only.

Now the behavioral variable of the model can be defined more precisely: it is the number of correct judgments of conservation items. The behavioral variable thus defined is denoted as a p value, that is, the quotient of the number of correct judgments and the total number of conservation items.

Control variables: Predominances of strategies. The basic problem of constructing catastrophe models of complex discontinuous processes is the choice of control variables. For simple processes this choice is normally simple, but, in the case of complex processes like cognitive development, there is a great number of possible control variables. Preece (1980) used cognitive capacity and a particular characteristic of the stimulus as control variables, whereas Saari (1977) referred to the Piagetian assimilation and accommodation concepts. In the literature on cognitive development additional variables have been proposed as important influences on cognitive level and conservation ability. Each of these variables could play the role of control variable; examples are Piagetian concepts like maturation, equilibration, along with several alternative concepts like cognitive capacity, training, short-term memory, language, and field dependency. In our opinion it should be possible to construct catastrophe models on the basis of a judicious selection of these variables (see the Relationship With Theories of Cognitive Development section for further discussion). We first formulate the model at a psychometric level, while invoking the concept of strategy (rule, operation, argumentation, scheme, or algorithm). This psychometric level pertains to the strategies that children apply to conservation items and the cognitive conflicts associated with transitional periods.

1. Classification of strategies. Our basic assumption is that a child uses a strategy to solve test items of a cognitive developmental test. In the case of conservation this assumption is frequently made, see for example Brainerd (1979). Brainerd (1979) constructed a Markovian model for conservation learning that was based on the concept of strategies (see also Molenaar, 1986a). Moreover, the strategy concept includes the mental operations that are, according to Piaget, necessary for conservation. Piaget distinguished three genuine conservation strategies: compensation, addition–subtraction, and inversion. At present another strategy is often added, namely, qualitative identity (Bruner, 1967; Goldschmid & Bentler, 1986). Brainerd (1979) distinguished as many as six correct conservation strategies. Typical of this category of strategies is that it leads to systematically correct judgments. In contrast, a second well-known category consists of the nonconservation strategies. A child that applies a nonconservation strategy will focus on only one relevant dimension (e.g., the height of the liquid or the thickness of the clay ball). Conservation items are constructed in such a manner that these strategies lead to systematically wrong judgments.

Each classification of conservation strategies should minimally distinguish between these two sets of strategies. Brainerd (1979) introduced a third set, the partial conservation strategies. In Brainerd's (1979) model it is assumed that these are used by partial conservers, that is, the transitional subjects. However, Brainerd (1979) did not specify what these strategies really are. It is argued that partial conservation strategies do not form a distinct set, but transitional subjects apply a mixture of conservation and nonconservation strategies.

In our classification the conservation and nonconservation sets are included, whereas the set of nonconservation strategies is explicitly limited to those leading to systematically wrong judgments. We do propose a third set, but this set includes strategies that are not test relevant. That is, strategies in the latter set do not lead to systematically correct or wrong judgments. Examples are application of the same judgment to all items, reaction to cues of the experimenter, and responses to irrelevant stimulus cues (color of clay ball).

These strategies are considered to be a threat to the validity of the scoring criterion on the basis of judgment only. However, it is only on tests comprising a small number of items that a spuriously high conservation score will be obtained using these strategies. These strategies may be used for several reasons, such as a lack of understanding of verbal concepts like *same* and *more,* lack of concentration, or anxiety. In suboptimal test conditions every child is expected at times to use these strategies. One could argue that irrelevant strategies are not strategies at all. We think this is a matter of definition; as is shown later, our implicit "weak" definition of strategy has some important advantages. In Brainerd's (1979) model these irrelevant strategies are not explicitly defined, they are not permitted, or they are classified as nonconservation strategies. The previously mentioned conservation, nonconservation, and irrelevant sets of strategies are henceforth denoted, respectively, as CS, NS, and IS.

The justification of our classification is based on the consequences of each set of strategies for the observed test score. With the usual psychological developmental tests, the lowest test score is given by chance level (for dichotomous items, this is $p = .5$). It is crucial that this is not the case for conservation tests. The NS lead to test scores below chance level. Conservation items are often called perceptually misleading (see Brainerd, 1979; Globerson, 1985; Odom, 1972) because of this systematic negative effect of NS on test score. The actual occurrence of children with scores below chance level justifies the distinction between NS and IS in that the IS do lead to scores at chance level.

2. *Cognitive conflict.* By means of the classification introduced earlier we can define four groups of children. A conserver will sample CS and IS; nonconservers have at their disposal NS and IS. Transitional children have at their disposal all three sets, whereas the remaining children (called here provisionally the *residual group*) use predominantly IS. Although all groups have at their disposal IS, conservers, nonconservers, and transitional children are supposed to use them minimally.

In our model these groups are modeled by means of two latent control variables. These controls are based on the strategy concept and are called predominance of NS and predominance of CS, denoted by m and n. The latter variables are supposed to vary continuously and independently. What these predominances really are (what their relation is to variables proposed in the pertinent literature on cognitive development) will not be specified in this section. In the Relationship With Theories of Cognitive Development section we discuss two possible interpretations of the predominances in the current model. For now they are used as two opposite tendencies or influences, one implying a tendency to use NS and the other CS. These predominances, as given by their values m and n, are related to the four conservation groups (see Table 2).

These definitions are easily explained for Groups 1, 3, and 4. The transitional group is a special case, however. A transitional child is faced with two opposite tendencies, a situation often described as a cognitive conflict (Cantor, 1983; Pinard, 1981). Although cognitive conflict may seem to be a rather vague concept, it is elaborated in our description of transitional subjects. That is, we use the conflict interpretation of the cusp along with the presumed predominances of NS and CS to arrive at a formal representation of cognitive conflict.

Cusp model of cognitive conflict. The cognitive conflict between predominances is modeled by means of the predominances as controls and the conservation p-valued score as a behavioral variable. The preliminary deterministic model is shown in Figure 16.

Table 2
Four Conservation Groups Associated With the Predominances of Nonconservation (m) and Conservation (n) Strategies

Group	m	n
Conservers	Low	High
Nonconservers	High	Low
Transitional	High	High
Residual group	Low	Low

Discontinuities can occur when both control variables are active (m and n are high). Path a illustrates a typical stage transition. First, the test score of a nonconserver depends on m only; the child consistently uses NS. The score of the nonconserver will be below chance level. Then, as the value of n, the predominance of CS, increases, the child reaches the bifurcation set and becomes a transitional child. CS and NS are now both available, and the child is faced with a cognitive conflict. Although both predominances could be equally strong, the actual application of strategies (and therefore the test score) is biased toward one set of strategies. The model implies that a transitional child must choose; it cannot remain indifferent. The middle sheet, the inaccessible region, represents a situation in which both predominances have equal influence. However, this region consists of inaccessible unstable states so that a transitional child will score below or above chance level and not at the chance level. If the delay convention is obeyed (as in Figure 16), the jump to the higher level takes place at the moment of leaving the bifurcation set. After leaving the bifurcation set the child becomes a stable conserver.

During the transition a number of typical behavioral aspects are present, which have been described in Catastrophe Detection section as catastrophe flags. Inaccessibility, sudden jumps, and bimodality are clearly shown in this model, consequently the evidence summarized in the Modality and Inaccessibility and Sudden Jumps sections applies to this model. Especially, the large body of evidence for bimodal conservation score distributions is remarkable; the results of Bentler (see 1970; Figures 9a and 9b in the present article) corroborate the discontinuity hypothesis.

Hysteresis and divergence are also shown in Figure 16. Divergence will occur when both predominances are equally increased in the residual group (m as well as n increases). That is, some children will follow the upper sheet; some will take the lower. This interpretation of divergence is closely connected to K. W. Fischer's et al. (1984) argument about optimal conditions (see the Hysteresis and Divergence section). A simple example of the divergence effect is formulated in the following experimental hypothesis: Suppose a group of children is tested in suboptimal conditions. According to the strategy model this will lead to low values along the splitting axis, where this splitting axis can be interpreted as optimality of test conditions (see also discussion of Figure 10). This low value along the splitting axis will give rise to a dominance of irrelevant strategies, implying that most children will reside in the residual group. When this group is subsequently tested in optimal conditions, this results in higher values along the splitting axis and hence the dominance of relevant strategies (CS as well as NS) will increase. Some children will now apply CS instead of IS, but for others NS will become dominant. Consequently, the scores of some of the children of the residual group in the first condition (with scores at chance level) will significantly increase, whereas scores of others will decrease below chance level.

Hysteresis will show up when the predominances are manipulated near the bifurcation set. Of course it is not immediately clear how the predominances should be manipulated. Their possibler operationalizations are discussed later. The interpre-

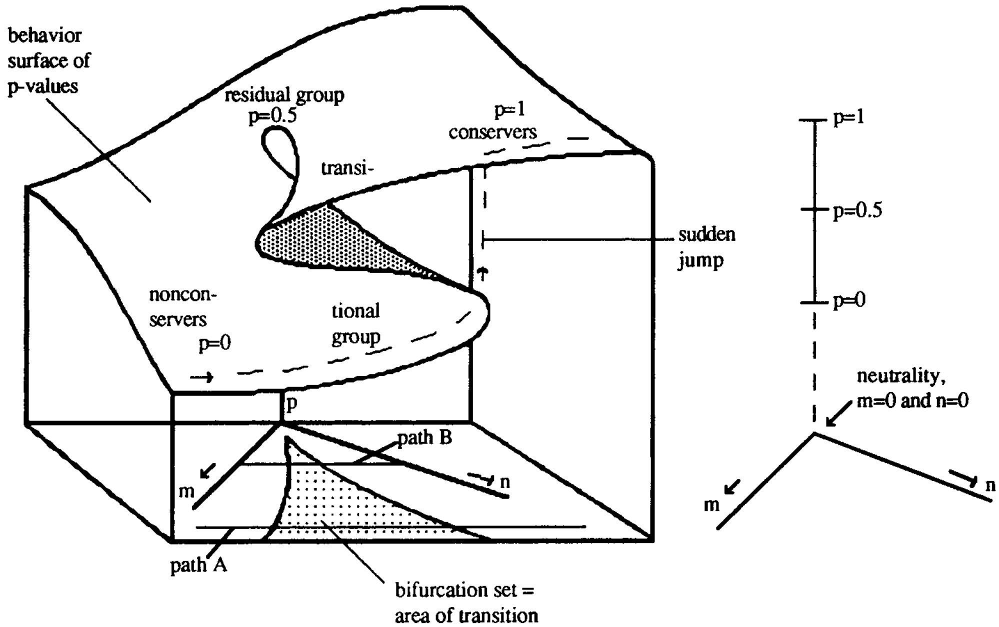

Figure 16. A conflict cusp model of conservation acquisition. (On the right side of the figure the axes are depicted. The two opposite forces define the control plane. Variation along one axis only [$m = 0$ or $n = 0$], for example, variation in predominance of nonconserver strategies (NS) in nonconservers [$n = 0$], yields continuous changes in the behavioral variable. The same argument applies to conservers, but for them $m = 0$. For both conserver and nonconserver groups the conservation score is continuously related to variation in predominances. The residual group is placed around the neutral point where both predominances are zero. The use of irrelevant strategies by the residual group will lead to scores at chance level, $p = .5$. Sudden jumps occur in the bifurcation set for high values of both predominances, typical of the transitional subjects. p = conservation test score, m = predominance of NS, n = predominance of conservation strategies.)

tations of divergence of linear response, critical slowing down, and anomalous variance, discussed, respectively, in the Divergence of Linear Response, Critical Slowing Down, and Anomalous Variance sections are also in agreement with this cusp interpretation of conservation acquisition.

In our view the evidence reported in literature for the occurrence of anomalous variance constitutes another empirical corroboration of our cusp model. This occurrence of increased variance is also predicted by the model of Flavell and Wohlwill (1969). However, the weakness of Flavell and Wohlwill's disequilibrium–stabilization model, as argued by Brainerd (1979), is that in contrast with their objective, it is essentially linear and continuous. Brainerd's (1979) criticism concerns the model's linear mathematical representation. The cusp model includes the behavioral properties of the disequilibrium–stabilization model while, at the same time, it is based on a discontinuity theory of nonlinear equations.

Path a in Figure 16 represents only one path through the control plane, other paths at different distances from the neutral point are feasible. Consequently, several kinds of individual differences are allowed. Path b represents a child that will regularly apply irrelevant strategies. Moreover, paths that are not parallel to a and b are also possible. Notice that a child following Path a could jump much later than a child following Path b *but immediately after the jump may have a better score on the* test. Even a continuous change from nonconserver to conserver is possible. For this to happen, m should decrease to zero before n increases. This path along neutrality represents the possibility to follow the sequence consisting of nonconserver, residual group, and conserver instead of the normally expected sequence of nonconserver, transitional, and conserver.

Although this model includes irrelevant strategies, actually a probabilistic element, it is called the *deterministic variant* of the

Table 3
Strategies and Control Values of Distinct Conservation Groups

Group	Strategies	Control values
Conservers	Conservation and test-irrelevant strategies	$m = 0$ $n > c$
Transition	Conservation, nonconservation, and test-irrelevant strategies	$m > c$ $n > c$
Nonconservers	Nonconservation and test-irrelevant strategies	$m > c$ $n = 0$
Residual	Irrelevant strategies	$m < c$ $n < c$

Note. c is a constant. m = nonconservation strategy; n = conservation strategy.

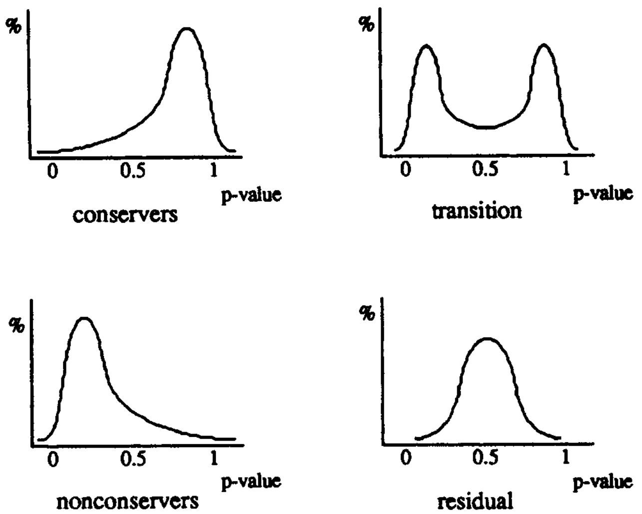

Figure 17. Distribution of scores of distinct conservation groups (compare with Table 3).

model. Genuine probabilistic catastrophe models include representations of the distributions of scores of the behavioral variable. The probabilistic interpretation is shown in Table 3 and Figure 17.

In Table 3 four groups are characterized by strategies, control values, and score distributions. The distributions of the conserver, nonconserver, and residual groups do not differ from those normally occurring in psychological research. The distribution belonging to the transitional group is a special case, however, as it is a bimodal one. Incidentally, each distribution shown in this table can be obtained with a cross-sectional sample of subjects and with time series data of one subject.

In our discussion of conservation training (see the Divergence of Linear Response section) we mentioned the problem of identification of distinct conservation groups. In particular, by what procedure can we discriminate between the four conservation groups? As is evident from Table 3, the *p* value is not sufficient. That is, when a low test score criterion is used to identify nonconservers, some transitional subjects will be included. Manipulation of control variables (like training) for the latter, transitional, subjects may lead to sudden increases in their scores. Consequently, a statistically significant effect of training does not in itself allow rejection of Piaget's stage hypothesis. Evidence concerning the special sensitivity of transitional subjects for training has been found by Breckinridge Church and Goldin-Meadow (1986).

The validity of our model depends in particular on the choice of control variables, that is, the predominances of NS and CS. Before we specify possible substantive interpretations (sources) of these predominances, an important point has to be made. Namely, the predominances are not simply related to the proportions of use of NS and CS. If they were, the model would be a continuous model because the test score then is linearly related to the proportions of use of NS and CS. For conservers and nonconservers the predominances may be linearly related to the proportions of use, but not in the case of transitional children. Hence, our control variables are latent constructs that are nonlinearly related to conservation scores and, by implication, to the proportions of use of strategies. This assumption of nonlinearity is based on the concept of cognitive conflict.

Relationship With Theories of Cognitive Development

The concept of strategies is so general that it relates to all major theories of cognitive development. An important question to answer is what are the predominances of strategies; that is, is it possible to interpret the strategy model in terms of more encompassing cognitive variables?

We distinguish between two interpretations: one in terms of Piaget's theory, the other in terms of the theory of information processing. These interpretations, preliminary extensions of the strategy model, are examined concisely. The literature on theories of cognitive development is so extensive that an exhaustive overview cannot be given here.

As suggested before, according to the Piagetian view the conflict of predominances can be seen as conflict between the preoperational and the concrete operational structures. Piaget described the role of conflict in development relative to the equilibration concept. Although Piaget changed his definition of *equilibration* several times (Murray, 1983), it appears that disequilibrium defined by catastrophe theory (i.e., related to the behavior in the bifurcation set) is in agreement with Piaget's ideas (see Saari, 1977, for a discussion of this problem). Moreover, Piaget's description of transitional behavior corresponds both to predictions of the model of Flavell and Wohlwill (1969) and to the predictions of our strategy model. The notions *strategy* and *operation* are closely related; the classification of strategies and conservation groups is comparable with similar classifications by the Genevean group.

The disadvantage of an interpretation in terms of Piagetian theory is that it does not yield an operationalization of the predominances of the conservation and nonconservation strategies. This problem is comparable with the problem encountered in Saari's (1977) model where theoretical concepts like assimilation, accommodation, and preoperational or concrete operational structures do not have mutually independent empirical definitions. That is, they cannot be measured independently of behavioral variables like conservation. Hence, an interpretation in terms of Piagetian theory does not lead to a convincing empirical verification of catastrophe models.

The second interpretation is associated with another large body of literature on conservation and transitions on the basis of the information processing view. In the theories of Pascual-Leone (1970) and Case (1985), the growth of cognitive capacity (related concepts are M power, short-term memory, and working memory) causes the emergence of conservation strategies. Measures of cognitive capacity and related variables may be operationalizations of the predominance of CS.

Another concept in this context is field dependency (Case & Globerson, 1974; Globerson, 1985; Pascual-Leone, 1989). This concept is introduced to discriminate between cognitive styles; that is, to explain differences in test scores of children with the same cognitive capacity. According to Globerson and Zelniker (1989), it is independent on cognitive capacity; consequently, it could be seen as an alternative control variable. Field dependency–independency is related to the explanation, recommended by Odom (1978), of differences in conservation scores in terms of the salience of perceptual cues. Operationalizations of field dependency and perceptual salience might be used as indicators of the predominance of NS. In this way, using statistical methods, a direct empirical verification should be possible. It even could be possible to manipulate cognitive capacity and perceptual salience to study directly the occurrence of hysteresis and divergence. This possibility may have an interesting bearing on Pascual-Leone's approach in which cognitive capacity and field dependency (among others) are integrated within a single process model of cognitive development. In particular, an application along these lines of catastrophe theory to scores obtained with the horizontality of water level task (Pascual-Leone & Morra, in press) might shed more light on the precise relations between the process model concerned and catastrophe theory.

In Figure 18 both interpretations are displayed, the first one is based on the cognitive conflict between structures; the second one, on field dependency (or perceptual salience) and cognitive capacity (mental power, short-term memory, and working memory).

Discussion

In this article we applied two distinct methods, on the basis of catastrophe theory, to the problem of stage-to-stage transitions. In this discussion we evaluate these methods. First, we discuss catastrophe modeling, in particular the advantages of the proposed strategy model and its extensions to theories of cognitive development. Second, catastrophe detection by means of Gilmore's (1981) catastrophe flags is examined.

Our evaluation of catastrophe modeling consists of a short list of the advantages and the disadvantages that in our opinion are important. It is a major advantage of the strategy model that it is directly relevant to most studies of conservation acquisition and stage-to-stage transitions. It is in agreement with results of conservation experiments and thus integrates different fields of research. Examples are the implications for training studies and for the procedure of test scoring. The behavioral phenomena predicted by the strategy model are generally in agreement with Piaget's description of conservation acquisition as well as with the behavioral properties of Flavell and Wohlwill's (1969) disequilibrium–stabilization model. Brainerd's (1979) criticism of the latter model, implying that the model equations would depict a continuous change, does not apply to the strategy model. The equations of the cusp model are nonlinear and are derived from an adequate formal theory of discontinuities.

The interpretations of the predominances of strategies in terms of two alternative theoretical views on cognitive development illustrate the relative independence of nonlinear models with respect to theoretical dichotomies (Molenaar & Oppenheimer, 1985). Although these interpretations are preliminary extensions, they indicate several possible implications on a theoretical level.

We distinguish four major limitations of the model and, in general, catastrophe modeling of cognitive stage transitions. The first limitation of the strategy model concerns the *décalages horizontales.* The sequence of conservation domains in conservation acquisition is not covered by our model. The model cannot explain this sequence and therefore would seem to apply to subtests (within a domain) only. The second problem

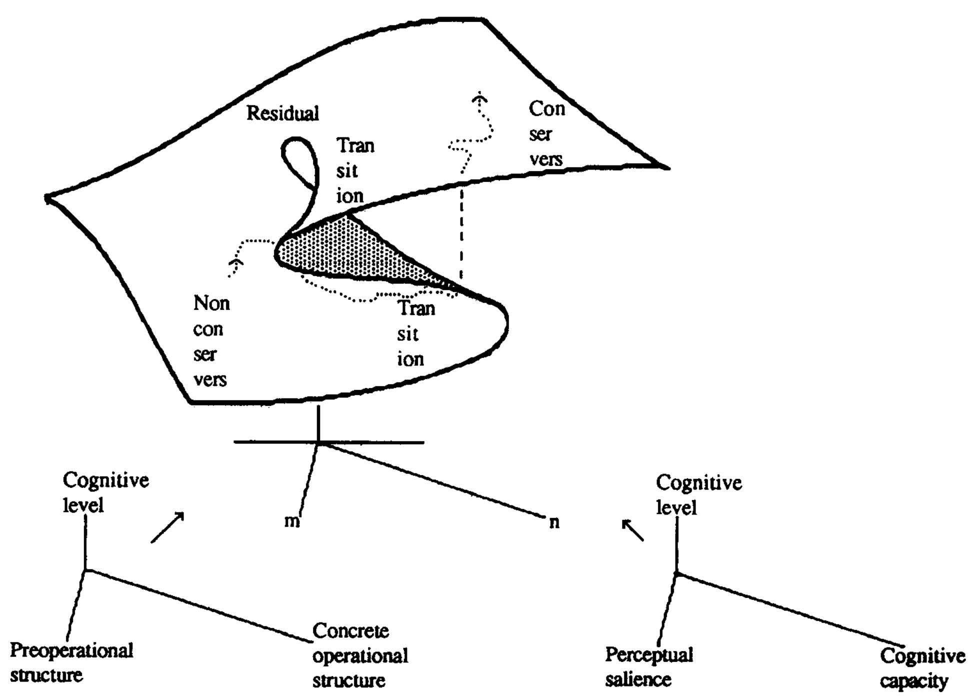

Figure 18. Two possible theoretical models. (In the first, a Piagetian interpretation of the strategy model is given. The relevant cognitive structures serve as control variables. In the second, neo-Piagetian interpretation, perceptual salience, and cognitive capacity are used as controls.)

involves the explanatory status of the model. Although it may explain typical observable behavioral properties of conservation acquisition, it does not explain the emergence of new strategies itself. That is, it does not specify a mechanism according to which conservation strategies evolve. Hence, the model could be considered to be descriptive, at least in this respect. A third problem concerns the limitation of transition research. Our approach is essentially focused on models of a single transition (which is a consequence of the local nature of catastrophe theory). Empirical results indicating the occurrence of a transition do not confirm a complete stage theory but is only preliminary to its confirmation. For a more definite confirmation not only do other transitions have to be identified, but also the investigation of nontransitional periods is important (see, for example, Tabor & Kendler, 1981).

The final problem concerns the verification of the proposed model. Although that has been one of our objectives, the complete model is not easily verified with real data. For an overall verification, the statistical methods of Cobb (1980; and related methods) might be applied to measurements of conservation ability and the control variables (cognitive capacity and perceptual salience). On the other hand, it is possible to partly investigate the model by means of specified hypotheses in combination with catastrophe-detecting methods (the catastrophe flags).

Our review of applied criteria (corresponding to flags) for stage transitions in cognitive development has to be considered as being independent from our particular strategy cusp model. In the field of applied catastrophe theory, catastrophe detection appears to be rather unknown. Its advantage is that it does not presume knowledge of the control variables (except for hysteresis). Flags were derived from catastrophe theory by Gilmore (1981) to detect catastrophes in physical systems. Some of our psychological interpretations of the flags might be weak, especially the interpretations that concern behavioral indices (flags) after perturbations. We are uncertain which of the possible environmental influences on conservation ability are adequate operationalizations of a perturbation. Yet, the general correspondence between Gilmore's flags and the applied transition criteria is remarkable. In our opinion catastrophe detection is useful to integrate traditionally applied transition criteria and to summarize their results.

Conclusions

At present we are witnessing a revolution in the pure and applied mathematical analysis of nonlinear systems. Under the headings of bifurcation theory, catastrophe theory, nonequilibrium thermodynamics, synergetics, soliton, and chaos theory, considerable progress has been made in the analysis of various aspects of nonlinear systems. One distinguishing feature of these systems is that sudden qualitative changes may occur in the dynamic structure of their behavior. These so-called catastrophic changes mark transitions to newly emerging equilibria that arise through endogenous reorganization of the system dynamics. Accordingly, catastrophes are associated with a kind of self-organization that can only take place in nonlinear systems and as such are unrelated to the fast smooth changes that may occur in the behavior of linear systems because of large fluctuations in their input.

The basic tenet of this article is that catastrophes constitute formal analogues of stage transitions in Piaget's theory of cognitive development. Undoubtedly, the cognitive system is one of the most complex information-processing systems in nature, and at least its neural substrata is highly nonlinear. Hence, we can characterize cognitive development in a formal sense as the evolution of a nonlinear system for which the general mathematical results of catastrophe theory will hold. Specifically, the evolution of the cognitive system may undergo catastrophes or genuine stage transitions as put forward in Piaget's theory and thus may be characterized as epigenetic development (cf. Molenaar, 1986b). In our view, this is in itself a decisive contribution of catastrophe theory to the debate of whether stage transitions can have genuinely explanatory status (Brainerd, 1978). Contrary to Brainerd's (1978) conclusion, it follows from catastrophe theory that stage transitions constitute the landmark of epigenetic development. Moreover, ironically, it is the Markovian state–space model put forward by Brainerd (1978) that lacks explanatory status as a model of stage transitions, because it is a linear model.

We considered three increasingly strong versions of catastrophe theory for the applied analysis of stage transitions in cognitive development. The weakest version involves the detection of a number of catastrophe flags that have been mathematically derived from the general catastrophe model by Gilmore (1981). It is expected that all these catastrophe flags are present when a stage transition takes place. This can be established by means of standard statistical techniques applied to behavioral measures only. Some of these catastrophe flags, like bimodality, sudden jumps, and anomalous variance were already put forward in the developmental psychological literature as indicators of stage transitions. Catastrophe theory gives a formal underpinning of the validity of these indicators, introduces new flags, like divergence of linear response and critical slowing down, and makes explicit the close interrelationships between all flags.

A stronger version of catastrophe theory, called *modeling*, involves the fit of elementary catastrophe models to a dataset that not only consists of behavioral measures but also includes measures of the control variables. Fitting catastrophe models requires special techniques. If the goodness of fit of a catastrophe model is acceptable, then this constitutes strong evidence that a stage transition is present. We proposed a viable catastrophe model for the transition from the preoperational to the concrete operational stage in cognitive development that might serve this purpose.

Our catastrophe theoretical approach to stagewise development thus leads to a methodology for detecting and modeling of stage transitions that can be applied straightforwardly in empirical research. Given the appropriateness of the underlying model of catastrophe theory as a model of cognitive development, this methodology has established validity. As to this, it is important to note that the underlying model of catastrophe theory is a restricted model. Stated succinctly, catastrophe theory only applies to dynamic nonlinear systems that can be characterized by a so-called potential function and that conserve energy. Hence, if the cognitive system would be a dissipative system, then it would seem that our catastrophe theoretical approach does not apply. However, it turns out that catastrophe theory is applicable to any dynamic nonlinear system insofar as this system can be locally characterized by an arbitrary potential function at each bifurcation point or transition. Although this result has important consequences for inductive (so-called inverse) modeling, its obtainment in any given application can only be ascertained if the mathematical expressions characterizing the nonlinear system under scrutiny are known (cf. Jackson, 1989, p. 117). Consequently, our approach can be expected to be quite robust if attention is restricted to single stage transitions.

In closing, the catastrophe theoretical approach is in the first place a formal approach to cognitive development. This implies that not all aspects and issues associated with the neo-Piagetian world view are directly subsumed under it. We only expect that catastrophe detection and modeling will lead to more definite tests of one essential aspect of epigenetic development, namely, stage transitions.

References

Bentler, P. M. (1970). Evidence regarding stages in the development of conservation. *Perceptual and Motor Skills, 31,* 855–859.

Bornstein, M. H., & Sigman, M. D. (1986). Continuity in mental development from infancy. *Child Development, 57,* 251–274.

Brainerd, C. J. (1973). Judgments and explanations as criteria for the presence of cognitive structures. *Psychological Bulletin, 79,* 172–179.

Brainerd, C. J. (1978). The stage question in cognitive-developmental theory. *The Behavioral and Brain Sciences, 2,* 173–213.

Brainerd, C. J. (1979). Markovian interpretations of conservation learning. *Psychological Review, 86,* 181–213.

Brainerd, C. J., & Allen, T. (1971). Experimental inductions of the conservation of "first-order" quantitative invariants. *Psychological Bulletin, 75,* 128–144.

Breckinridge Church, R., & Goldin-Meadow, S. (1986). The mismatch between gesture and speech as an index of transitional knowledge. *Cognition, 23,* 43–71.

Bruner, J. S. (1967). On the conservation of liquids. In J. S. Bruner, R. R. Olver, & P. M. Greenfield (Eds.), *Studies in cognitive growth* (pp. 183–207). New York: Wiley.

Callahan, J., & Sashin, J. I. (1990). Predictive models in psychoanalysis. *Behavioral Science, 35,* 60–76.

Campbell, R. L., & Bickhard, M. H. (1986). *Contributions to human development: Volume 16. Knowing levels and developmental stages* (J. A. Meacham, Ed.). Basel, Switzerland: Karger.

Cantor, G. N. (1983). Conflict, learning, and Piaget: Comments on Zimmerman and Blom's "toward an empirical test of the role of cognitive conflict in learning." *Developmental Review, 3,* 39–53.

Case, R. (1985). *Intellectual development: Birth to adulthood.* San Diego, CA: Academic Press.

Case, R., & Globerson, T. (1974). Field independence and central computing space. *Child Development, 45,* 772–778.

Cobb, L. (1978). Stochastic catastrophe models and multimodal distributions. *Behavioral Science, 23,* 360–373.

Cobb, L. (1980). Estimation theory for the cusp catastrophe model. *Proceedings of the Section on Survey Research Methods* (pp. 772–776). Washington, DC: American Statistical Association.

Cobb, L. (1981). Parameter estimation for the cusp catastrophe model. *Behavioral Science, 26,* 75–78.

Cobb, L., Koppstein, P., & Chen, N. H. (1983). Estimation and moment recursion relations for multimodal distributions of the exponential family. *Journal of the American Statistical Association, 78,* 124–130.

Cobb, L., & Watson, B. (1980). Statistical catastrophe theory: An overview. *Mathematical Modelling, 1,* 311–317.

Cobb, L., & Zacks, S. (1985). Applications of catastrophe theory for statistical modeling in the biosciences. *Journal of the American Statistical Association, 80,* 793–802.

Connell, J. P., & Furman, W. (1984). The study of transitions: Conceptual and methodological issues. In R. N. Emde & R. J. Harmon (Eds.), *Continuities and discontinuities in development* (pp. 153–174). New York: Plenum Press.

Emde, N. R., & Harmon, R. J. (Eds.). (1984). *Continuities and discontinuities in development.* New York: Plenum Press.

Everitt, B. S., & Hand, D. J. (1981). *Finite mixture distributions.* London: Chapman & Hall.

Field, D. (1987). A review of preschool conservation training: An analysis of analyses. *Developmental Review, 7,* 210–251.

Fischer, G. H. (1974). *Enführung in die Theorie psychologischer Tests: Grundlagen und Anwendungen* [Introduction to the theory of psychological tests: Fundamentals and applications]. Bern, Switzerland: Verlag Hans Huber.

Fischer, K. W. (1983). Developmental levels as periods of discontinuity. In K. W. Fischer (Ed.), *Levels and transitions in children's development: New directions for child development* (Vol. 21, pp. 5–20). San Francisco: Jossey-Bass.

Fischer, K. W., & Bullock, D. (1981). Patterns of data: Sequence, synchrony, and constraint in cognitive development. In K. W. Fischer (Ed.), *Cognitive development: New directions for child development* (Vol. 12, pp. 1–20). San Francisco: Jossey-Bass.

Fischer, K. W., Pipp, S. L., & Bullock, D. (1984). Detecting developmental discontinuities: Methods and measurement. In R. N. Emde & R. J. Harmon (Eds.), *Continuities and discontinuities in development* (pp. 95–122). New York: Plenum Press.

Fischer, K. W., & Silvern, L. (1985). Stages and individual differences in cognitive development. *Annual Review of Psychology, 36,* 613–648.

Flavell, J. H. (1971). Stage-related properties of cognitive development. *Cognitive Psychology, 2,* 421–453.

Flavell, J. H., & Wohlwill, J. F. (1969). Formal and functional aspects of cognitive development. In D. Elkind & J. H. Flavell (Eds.), *Studies in cognitive development: Essays in honor of Jean Piaget* (pp. 67–120). New York: Oxford University Press.

Freedle, R. (1977). Psychology, thomian topologies, deviant logics, and human development. In N. Datan & H. W. Reese (Eds.), *Life-span development psychology: Dialectical perspectives on experimental research* (pp. 317–342). San Diego, CA: Academic Press.

Gilmore, R. (1981). *Catastrophe theory for scientists and engineers.* New York: Wiley.

Globerson, T. (1985). Field dependence/independence and mental capacity: A developmental approach. *Developmental Review, 5,* 261–273.

Globerson, T., & Zelniker, T. (1989). Introduction. In T. Globerson & T. Zelniker (Eds.), *Human development: Cognitive style and cognitive development* (Vol. 3, pp. 1–9). Norwood, NJ: Ablex.

Goldschmid, M. L., & Bentler, P. M. (1986). *Manual: Concept assessment kit—conservation.* San Diego, CA: Educational & Industrial Testing Service.

Guastello, S. J. (1984). Cusp and butterfly catastrophe modeling of two opponent process models: Drug addiction and work performance. *Behavioral Science, 29,* 258–269.

Guastello, S. J. (1988). Catastrophe modeling of the accident process: Organizational subunit size. *Psychological Bulletin, 103,* 246–255.

Hudspeth, W. J., & Pribram, K. H. (1990). Stages of brain and cognitive maturation. *Journal of Educational Psychology, 82,* 881–884.

Jackson, E. A. (1989). *Perspectives on nonlinear dynamics* (Vol. 1). Cambridge, England: Cambridge University Press.

Kenny, S. L. (1983). Developmental discontinuities in childhood and adolescence. In K. W. Fischer (Ed.), *Levels and transitions in children's development: New directions for child development* (Vol. 21, pp. 81-96). San Francisco: Jossey-Bass.

Kingma, J. (1984). Criterion problem in conservation research reconsidered from a psychometric point of view. *Journal of General Psychology, 111,* 109-129.

Klahr, D., & Wallace, J. G. (1976). *Cognitive development: An information processing view.* Hillsdale, NJ: Erlbaum.

Kuhn, D. (1974). Inducing development experimentally: Comments on a research paradigm. *Developmental Psychology, 10,* 590-600.

Levin, I. L. (Ed.). (1986). *Stage and structure: Reopening the debate.* Norwood, New Jersey: Ablex.

Lewis, M. (1990). Development, time, and catastrophe: An alternative view of discontinuity. In P. Baltes, D. L. Featherman, & R. Lerner (Eds.), *Life span development and behavior series* (Vol. 10, pp. 325-350). Hillsdale, NJ: Erlbaum.

McCall, R. B. (1983). Exploring developmental transitions in mental performance. In K. W. Fischer (Ed.), *Levels and transitions in children's development: New directions for child development* (Vol. 21, pp. 65-80). San Francisco: Jossey-Bass.

McCall, R. B., Eichorn, D. H., & Hogarty, P. S. (1977). Transitions in Early Mental Development. *Monographs of the Society for Research in Child Development, 42*(3), Serial No. 171.

McShane, J. (1991). *Cognitive development: An information processing approach* (1st ed.). Oxford, England: Basil Blackwell.

Molenaar, P. C. M. (1986a). Issues with a rule-sampling theory of conservation learning from a structuralist point of view. *Human Development, 29,* 137-144.

Molenaar, P. C. M. (1986b). On the impossibility of acquiring more powerful structures: A neglected alternative. *Human Development, 29,* 245-251.

Molenaar, P. C. M., & Oppenheimer, L. (1985). Dynamic models of development and the mechanistic-organismic controversy. *New Ideas in Psychology, 3,* 233-242.

Murray, F. B. (1983). Equilibration as cognitive conflict. *Developmental Review, 3,* 54-61.

Odom, R. D. (1972). Effects on perceptual salience on the recall of relevance and incidental dimensional values: A developmental study. *Journal of Experimental Psychology, 92,* 185-291.

Odom, R. D. (1978). A perceptual salience account of décalage relations and developmental change. In L. S. Siegel, & C. J. Brainerd (Eds.), *Alternatives to Piaget: Critical essays on the theory* (pp. 11-130). San Diego, CA: Academic Press.

Oliva, T. A., Desarbo, W. S., Day, D. L., & Jedidi, K. (1987). GEMCAT: A general multivariate methodology for estimating catastrophe models. *Behavioral Science, 32,* 121-137.

Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's development stages. *Acta Psychologica, 32,* 301-345.

Pascual-Leone, J. (1989). An organismic process model of Witkin's field-Dependency-Independency. In T. Globerson, & T. Zelniker (Eds.), *Cognitive style and cognitive development* (Vol. 3, pp. 36-70). Norwood, New Jersey: Ablex.

Pascual-Leone, J., & Morra, S. (in press). Horizontality of water level: A neopiagetian developmental review. *Advances in Child Development and Behavior.*

Perry, M., Breckinridge Church, R., & Goldin-Meadow, S. (1988). Transitional knowledge in the acquisition of concepts. *Cognitive Development, 3,* 359-400.

Piaget, J. (1960). The general problems of the psychobiological development of the child. In J. M. Tanner, & B. Inhelder (Eds.), *Discussions on child development* (Vol. 4, pp. 3-28). London: Tavistock.

Piaget, J., & Inhelder, B. (1969). *The psychology of the child.* New York: Basic Books.

Pinard, A. (1981). *The conservation of conservation: The child's acquisition of a fundamental concept.* Chicago: University of Chicago Press.

Poston, T., & Stewart, I. (1978). *Catastrophe theory and its applications.* London: Pitman.

Preece, P. F. W. (1980). A geometrical model of Piagetian conservation. *Psychological Reports, 46,* 143-148.

Rutter, M. (1984). Continuities and discontinuities in socioemotional development: Empirical and conceptual perspectives. In R. N. Emde & R. J. Harmon (Eds.), *Continuities and discontinuities in development* (pp. 41-68). New York: Plenum Press.

Saari, D. G. (1977). A qualitative model for the dynamics of cognitive processes. *Journal of Mathematical Psychology, 15,* 145-168.

Sommerhoff, G. (1969). The abstract characteristics of living systems. In F. E. Emery (Ed.), *System thinking* (4th ed., pp. 147-202). Middlesex, England: Penguin Books.

Stauder, J. E. A., Molenaar, P. C. M., & van der Molen, M. W. (in press). *Event related brain potentials elicited in an Piagetian conservation task analogue. Child Development.* Submitted

Sternberg, R. J., & Okagaki, L. (1989). Continuity and discontinuity in intellectual development are not a matter of 'either-or.' *Human Development, 32,* 158-166.

Stewart, I. N., & Peregoy, P. L. (1983). Catastrophe theory modeling in psychology. *Psychological Bulletin, 94,* 336-362.

Tabor, L. E., & Kendler, T. S. (1981). Testing for developmental continuity or discontinuity: Class inclusion and reveral shifts. *Developmental Review, 1,* 330-343.

Ta'eed, L. K., Ta'eed, O., & Wright, J. E. (1988). Determinants involved in the perception of the necker cube: An application of catastrophe theory. *Behavioral Science, 33,* 97-115.

Thatcher, R. W. (1991). Maturation of the human frontal lobes: Physiological evidence for staging. *Developmental Neuropsychologica, 7,* 397-419.

Thatcher, R. W., Walker, R. A., & Giudice, S. (1987). Human cerebral hemispheres develop at different rates and ages. *Science, 236,* 1110-1113.

Thom, R. (1975). *Structural stability and morphogenesis.* Reading, MA: Benjamin.

Thomas, H. (1989). A binomial mixture model for classification performance: A commentary on Waxman, Chambers, Yntema, and Gelman (1989). *Journal of Experimental Child Psychology, 48,* 423-430.

Uzgiris, I. C. (1964). Situational generality of conservation. *Child Development, 35,* 831-841.

van der Maas, H. L. J., Pope, F., de Geus, W., Bast, J. W., van Gils, N., & Molenaar, P. C. M. (1992). *The nonverbal assessment of conservation strategies.* Manuscript in preparation.

van der Maas, H. L. J., Verschure, P. F. M. J., & Molenaar, P. C. M. (1990a). *Catastrophe fingerprints of simple neural networks on route to chaos.* Manuscript submitted for publication.

van der Maas, H. L. J., Verschure, P. F. M. J., & Molenaar, P. C. M. (1990b). A note on chaotic behavior in simple neural networks. *Neural Networks, 3,* 119-122.

van Geert, P. (1991). A dynamic systems model of cognitive and language growth. *Psychological Review, 98,* 3-53.

Wilson, M. (1989). Saltus: A psychometric model of discontinuity in cognitive development. *Psychological Bulletin, 105,* 276-289.

Wohlwill, J. F. (1973). *The study of behavioral development.* San Diego, CA: Academic Press.

Zahler, R. S., & Sussmann, H. J. (1977). Claims and accomplishments of applied catastrophe theory. *Nature, 269,* 759-763.

Zeeman, E. C. (1976). Catastrophe theory. *Scientific American, 234*(4), 65-83.

Received November 14, 1990
Revision received September 24, 1991
Accepted October 23, 1991 ■