Raymond B. Cattell

Higher Order Factor Structures and Reticular-vs-Hierarchical Formulae for their Interpretation

THE NATURE OF FACTORS

With the clear and comprehensive formulation by Sir Cyril Burt[5, 6] of a hierarchical concept of ability structure in man, the stage was set for the exploration of factor structures, and as a result many psychological concepts have been extended and amplified: for example, the notion that there are not one, but two factors of general ability ('fluid' and 'crystallized' intelligence[13]), and the emergence of important theories concerning higher order structure in factors of personality. In this paper, illustrations will be mainly from the field of personality, but the chief purpose is to develop concepts and theoretical models for higher order factor structure in general. The treatment culminates in new formulae, notably the Cattell-White alternative to the well-known Schmid-Leiman transformation.

What is the relevance to purely psychological interests of this clarification of theoretical and mathematical models? Although the average student of personality and ability, particularly in the educational and clinical arts, is prone to theorize and practise without explicitly stating the formal model he uses, his theories cannot be taken very seriously until he specifies their properties. When criticized for not doing so, he is unfortunately apt to defend himself by saying that certain issues in factor analysis are 'esoteric'; yet the fact remains that in this field the fateful choice between different psychological theories turns on highly technical and statistical considerations. Social and educational psychologists, while recognizing that vast practical and political decisions on, say, nuclear fall-out depend on complex technical calculations in nuclear physics, nevertheless act as if their own psychological advice, affecting large numbers of children, patients and citizens, can be given without any effort to understand the true complexities of formulae such as we have to consider here. This is not to say that the average psychologist has to be an expert factor analyst, but it means that, to be considered qualified for his tasks, especially those dealing with the assessment of personality and ability, he *should* have clear concepts of the *logical* issues involved.

A deeply grateful student* of Professor Burt, the present writer finds himself in disagreement with him on a few issues, one of which concerns the relative advantages of orthogonal-vs-oblique factors. Sir Cyril holds that the purpose of factor analysis is classification and that, in the cognitive field particularly, the hierarchical arrangement of factors obtained by 'principal axes' and by 'simple summation' often gives not only a more economical but a truer picture of mental abilities than do oblique factor methods.

Elsewhere[8] I have put forward arguments (perhaps, one might say, as 'spokesman of the loyal opposition') for believing that simple structure is inherent in natural data (Cattell and Dickman[18]) and that, when it is discovered, and exactly adhered to, it normally yields oblique factors. These *are* free to 'go orthogonal' as a special case, but the chances of exactly zero correlations between them, even in populations, are infinitely small—if factors are meant to model nature. By the operational definition of simple structure (Cattell[8]), a definition somewhat different from Thurstone's,[39] a factor is given the property of something other than a mere classificatory principle. As part of a scientific model, with more properties than a purely mathematical model, it is given the status of an *influence or cause*, accounting for the covariation observed in the manifest variables affected by it. (Notably, of course, in the 'salient' or 'marker' variables, loaded most highly and significantly.) Such an influence is likely to leave untouched the majority of variables in any well-designed experiment, and to reveal, by leaving a galaxy of points forming a hyperplane of zero-loaded variables in hyperspace, the proper position to which it needs to be rotated.

However, the location of oblique factors (and therefore the proof that they are oblique) does not rest merely on the criterion of simple structure. For the independent resolution of results of a correlational research by the main alternative principle—confactor rotation (Cattell[14])—can also lead to the same result. Indeed, the notion that oblique factors will be the common outcome in scientific investigation does not

* It was a series of lectures by Professor Burt, in 1925, which turned the writer from post-graduate work in the physical sciences to a career in psychology.

rest only on experimental evidence in this narrow area itself. It rests on the general scientific proposition that in an interacting, unsegregated universe most influences will tend to show some correlation. The weight and volume of the planets, or the temperature and pressure taken at a hundred meteorological stations, will normally show significant correlations. It is the task of factor analysis to reveal and define these distinct, but correlated, concepts. If we insist on entities which are statistically uncorrelated they may well be conceptually contaminated.

THE NATURE OF HIGHER ORDER FACTORS

If factors can be correlated, then obviously one can find factors among factors. Those derived from the primary matrix of correlations between factors we call second order or secondary factors. There is no mathematical or logical reason why this process should not be repeated, leading to tertiary and quaternary factors, etc. Indeed, it has already been shown that one *can* get simple structure at these higher orders (Cattell,[10] Humphreys[31]), and that such higher order factors, like primaries, are consistent in pattern from one experiment to another, and correlated. It is an historical curiosity that pursuit of these higher orders in the personality field has developed almost simultaneously with similar work in the much older realm of research into abilities.

There are now no fewer than fourteen researches, recently surveyed and critically compared by Gorsuch,[25] on higher order factors among the primary factors fixed by the Sixteen Personality Factor Questionnaire. They agree extremely well in defining five second order factors, two of which, anxiety and exvia-invia (the precise extraversion-introversion factor defined by Warburton, 1962) are very easily recognized in terms of the classical Freudian and Jungian concepts. Moreover, they have been confirmed by clinical evidence and by their good alignment with first order factors obtained from objective tests (O–A Battery Cattell[15]) and factors U.I. 24 and U.I. 32 (Universal Index Numbers, Cattell[10]).

In this quick glance at the substantive illustration of these structural concepts we may note that Knapp, Cattell and Scheier[33] have explored second-order structure in the 21 objective test factors extending from U.I. 16 to U.I. 36 and have reached agreement on seven second order factors. Some of these make good sense in terms of psychoanalytic concepts, while others present new constructs around which post-psychoanalytic theories of personality can be developed. Recently, Pawlik and Cattell[35] have carried the O–A Battery studies to the third order analysis and have found, at what may be a final level, unambiguous structure in *three* major factors, which bear a distinct resemblance to the Freudian trio of id, ego and super ego.

Unfortunately, psychologists working in learning theory, clinical psychology, perception, etc., who are unfamiliar with factor analysis, have failed to avail themselves of the theoretical and experimental possibilities which the measurement of these definite factors would bring to their work. It would appear that they are confused by the technical controversies among factor analysts; first, over the relations between three common personality questionnaires—the 16 P.F., the MMPI and the Guilford-Zimmerman; secondly, in the sphere of behaviour ratings and objective tests, by the methodological disputes on the nature of hierarchies and the definition of higher order factors. Such disputes continue, despite published researches containing well substantiated primary and higher order factors, for example the work of Burt,[6] Cattell,[8] Digman,[21] Eysenck,[22] Harman,[28] Humphreys,[31] Peterson,[36] Vernon[41] and others.

The first question—that of alignment of factors from different questionnaires—is not particularly relevant here. The Guilford-Zimmerman and the 16 P.F. can *never* be aligned, for the one has been aimed at orthogonal, the other at oblique factors. The MMPI, on the other hand, deals with *surface* traits rather than with factors as here defined and these are resolvable into five *source* traits (rotated factors—Cattell[10]) which lie in a sub-space among the 16 source traits (factors) subtended by the 16 P.F.

The second area of debate—on higher order structure—involves more complex and extensive issues, which it is the object of this paper to clarify.

THREE POSSIBLE CONCEPTS OF FACTOR HIERARCHY

Formulation of the concepts of second order factors had a poor start because of the almost accidental circumstance that in the pioneer studies, including Thurstone's on primary abilities, simple structure was applied at the level of the primaries but not at the level of higher order factors. Yet surely simple structure (or confactor) unique rotation should be used at *all* levels if the term 'factor' is consistently to retain its meaning. A primary factor is here defined as an influence among variables, and we propose to define the next order of factors as influences upon factors. It follows that they must have simple structure on the primaries.

Failure to appreciate this may lead to an additional confusion—the assertion that a very 'broad' primary factor is the same thing as a second order factor. The latter view is found in Adcock's,[1] Eysenck's[22] and Peterson's[36] writings. These writers either state or imply that, for example, by taking out only one or two massive and general factors at the first order level—such as is often achieved by not rotating the first big centroid or principal axis—they are reaching the same result as they would obtain by taking out many primaries and then finding broad second order factors which cover the former. It is true that there is a very broad resemblance in the patterns of such alternative factor loadings, but these can scarcely serve as a basis for refined scientific concepts. Indeed, the alternative methods lead to vital differences in the concepts of neurosis, anxiety and ego strength between the personality theories of Cattell and Scheier[20] on the one hand, and Becker,[2] Eysenck[22] and Peterson[36] on the other.

The rest of this chapter deals with the problems which reside in the analysis itself and attempts to answer the question—what structures *could* exist and what are the methodological and statistical conditions of their existence within the factor analytic model?

If we accept the basic model in which correlations (including communality estimates) are resolved into common, specific and error factors, there exists already a considerable 'taxonomy' of stereotyped possibilities, beginning with Burt's designation of the 'bipolar factor' type of solution, Spearman's positive general factor and Holzinger's 'bifactor' solution, and extending to many others, for example, Guttman's 'simplex', and Cattell's 'co-operative factor structure'. These are essentially sets of standard 'mosaics' of factor patterns and, though never ideally attained in practice, provide useful terms by which specialists in the field can refer to a particular factor resolution.

Broadly conceived, these abstract mosaics differ in pattern in four main respects:

1. The number of common factors operating;
2. The number of variables they influence;
3. Their mutual overlap in influence on the variables;
4. Their algebraic sign patterns.

Naturally, the number of possible combinations constituting such Stereotyped Influence Patterns (or SIPs) is very great. The three most discussed and, indeed, most important at the primary factor level are set out in Fig. 1; others, sampled also from the still

FIG. 1

Relation between Dimension and Variable

Three Basic Resolution Patterns in the Structure of Personality and Abilities

Orthogonal

(1) *Unrotated ('Genealogic') Resolution*

(a) *With Reflection, (Bipolar Pattern)*

	1	2	3
1	+	+	+
2	+	+	+
3	+	+	−
4	+	+	−
5	+	−	+
6	+	−	+
7	+	−	−
8	+	−	−

or

(b) *Without Reflection*

	1	2	3
1	−	−	−
2	+	+	+
3	−	−	+
4	+	+	−
5	−	+	−
6	+	−	+
7	−	+	+
8	+	−	−

(2) *Bifactor ('Staircase') Resolution*

	1	2	3
1	+	+	
2	+	+	
3	+	+	
4	+	+	
5	+		+
6	+		+
7	+		+
8	+		+

Oblique

(3) *Multiplex Simple Structure Resolution, with Higher Order Outcome*

	1	2	3
1	+		
2	+		
3	+		+
4	+		+
5	+	+	
6	+	+	+
7		+	
8		+	+

I	1	2	3
+	+		
+		+	
+			+

larger number which appear when we combine the many possibilities at higher order levels, are shown in Fig. 2.

The first of these stereotyped patterns is nothing more than the primitive unrotated centroid or principle axis itself. Burt has made much use of it for logical, classificatory purposes, for, as he points out, the orderly arrangement of bipolar factors has the logical classificatory scheme of a 'tree of Porphyry'. (The present writer has suggested the term 'genealogic' because the successive divisions are like ancestors in a family tree.) Few, however, would consider this likely to correspond to *functional* psychological influences, and, of course, if variables are reflected back to original meanings, as is usual after a centroid analysis (and as illustrated in (1*b*) by reflecting every other variable), the main logical relations themselves become obscured. The second, or Holzinger bifactor resolution, is obtained by a special rotation from (1*a*), and, as Burt points out,[5] it preserves essentially the same relations, but eliminates the bipolarity by dropping negative loadings, though requiring more factors to represent the same complexity.

The third pattern, or SIP, has been called a *multiplex* by the present writer, because it gives equal importance to all factors and a random but multiple determination of variables by all factors. It is a multiple factor, simple structure influence pattern, i.e. with zeros in every column, and its essence is that it does *not* precisely stipulate degrees of overlap and signs, but accepts random overlap and a random pattern of signs. The structure of the multiplex also accepts any angle among the factors (oblique or orthogonal) and can thus support a second or higher order factor structure. The general multiplex is probably the most widely used, and useful resolution in psychological research.

It will readily be seen that resolutions (1) and (2) lend themselves to a 'hierarchical' view of personality or ability structure—in that one positive general factor dominates. When some psychologists speak of 'a hierarchy' they refer only to such a structure in this first order realm and to the special orthogonal case. Others, however, use the term to refer to the oblique case and the arrangement of the additional higher *order* factors in a sort of pyramid, which may appear in such circumstances. It would clarify the position if writers would refer to the first as a 'dominant general factor' solution and reserve hierarchy for superimposed higher orders.

Even in the latter, however, there are *two* senses of hierarchical, as shown in Fig. 2. In system (1), which follows the typical solution from a true use of simple structure on successive orders, the hierarchy

FIG. 2

Two types of hierarchy among factor orders

(1) *A pyramid of successive orders*

First order analysis

	1′	2′	3′	4′	5′	6′	7′
1	+			−			
2	−					+	
3	+	−					
4		+					
5		−	−			−	
6			+		−	+	
7			+				+
8	−	+					−
9			−	+			+
10				−	+		−
11		−		+	+		
12					−	−	
13	+			+		+	
14	−		−			−	
15						+	+
16				+		+	

2nd order analysis

	1″	2″	3″	4″
1′	−			+
2′		+	−	
3′	+	+	+	
4′		−	−	
5′			+	+
6′	−	+		−
7′				+

3rd order analysis

	1‴	2‴
1″	+	
2″	−	+
3″		−
4″	+	

(2) *Division under a general factor*

	1′	2′	3′	4′	5′	6′	7′
1	+						
2	−						
3	−						
4		+					
5		−					
6		+					
7			+				
8			−				
9				−			
10				+			
11					+		
12					+		
13						−	
14						+	
15							−
16							−

	1″	1_1′	2_1′	3_1′	4_1′	5_1′	6_1′	7_1′
1′	+	+						
2′	+		+					
3′	+			+				
4′	+				+			
5′	+					+		
6′	+						+	
7′	+							+

means nothing more than a series of factor orders, plus the pyramidal structure formed by the diminishing number of factors that are recorded as higher orders are reached. This pyramid may be incomplete however, in the sense of not finishing in a single factor to make a true pyramid, but in two or three, as in Fig. 2 (1). This most common result of the common multiplex base we may call a *tapering oblique hierarchy*. As we shall argue later, its tapering is accidental and artificial.

The psychologist who wants to produce the second type of SIP in Fig. 2 will do his best to rotate the primaries to be non-overlapping. He is thus obtaining a 'dominant general factor' resolution at the higher order. To get this special arrangement he may choose, perhaps unconsciously, variables which happen to give a single, general, second order factor plus specifics (extreme right of Fig. 2 (2)). The result in this case will be a very neat resolution of each and every variable into a loading on one of a number of orthogonal primaries (not the original primaries, $1'$, $2'$, etc., but the specific factor remnant, $1'_1$, $2'_5$, etc., corresponding to each) and on a general factor.

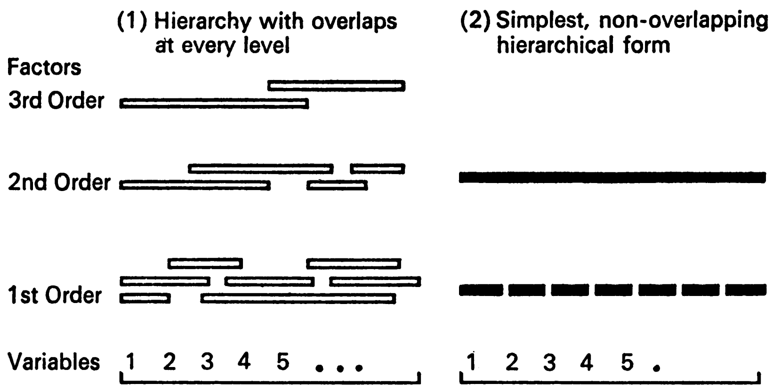

FIG. 3

Two plans of hierarchy among factor orders

A second manner of showing these same structures, in Fig. 3, shows that in what we may call the 'pure pyramid or monarchical hierarchy', the primaries will neither overlap nor leave gaps between them, and thus will yield a more orderly hierarchy than any other arrangement. This has the same utility, as a logical classification scheme, as Burt's bipolar system, except that signs are not specified. Not all centroids can be rotated into this form, but many which would otherwise be naturally represented by a tapering oblique hierarchy can be transformed by what is known as the Schmid-Leiman formula into a monarchical hierarchy.

Unfortunately, as I have indicated in the introduction to this paper, to get such a hierarchy one has to be selective or lucky in one's data—notably in being able to end with a grand monarchic general factor. A more important criticism of this SIP form is that although the pattern undoubtedly appeals to neat and tidy minds, the factors may not correspond to those realities in nature which will be constant from matrix to matrix. For the second order general factor here, like that in the first order centroid (the bipolar or bifactor general factor), is specific to the matrix; it is dependent on the particular choice of variables and is not to be fixed by simple structure because there is no hyperplane to rotate it by. The 'specifics' from the primaries are therefore equally arbitrarily truncated entities.

As we shall see, the position and nature of the lower order factors, when transformed into orthogonality by the Schmid-Leiman formula, are not arbitrary, neither are they non-overlapping, though those of the factors at the top of the pyramid are. But, with this exception, neither the 'hierarchy' at the first order (which we have defined as a 'dominant general factor solution') nor the true hierarchy across orders, in the monarchical sense, are normally anything but artificial creations, whose factors lack the constancy from matrix to matrix which we require of scientific concepts. Whether the one remaining form—the tapering oblique hierarchy in Figs. 2 (1) and 3 (1)—deserves the designation of a hierarchy we shall now discuss.

HIERARCHY OR NETWORK?

No matter how starkly we define a hierarchy operationally in the initial stages, it will tend to carry connotations of a broader and even of a philosophical nature to most who use the term. Among these connotations are the implications that i. the factor *higher* in the hierarchy must *always* be broader in its influence; ii. it is more *important* for prediction; iii. it is more *constant* in its form and appearance; iv. it is more *fundamental* for psychological theory, and v. it is more 'real'. We propose to show that these implications vary from insufficiently defined or inaccurate conclusions to completely unwarranted illusions.

Before discussing the possible meaning of a hierarchy of factor orders, notably of the third—tapering oblique—and only surviving sense of a hierarchy, one must ask whether it exists in nature at all. For what is commonly overlooked is that the pyramid structure, tapering to fewer factors at higher orders, could be an inevitable artefact of normal mathematical-statistical rules and need have no real existence at all in nature. One mathematical rule, when communalities are used, is that one cannot take out as many common factors as there are variables. Consequently, a hundred variables may define, say, only twenty primaries, and twenty primaries must yield fewer second order factors, and so on. But the fact that a number of higher order factors as great as the number of variables or lower order factors cannot be mathematically defined for lack of a sufficiency of variables is no proof that they do not exist. Indeed, the onus of proof that there *is* only this smaller number of higher factors at work lies on the psychologist who chooses to assert that the real structure of nature is a pyramidal hierarchy. Actually, I have given elsewhere (Cattell[16]) ample reasons for believing that the number of factors operating on a set of variables is normally decidedly greater than the number commonly taken out from *n* variables (provided we count real influences of small variance). These apply as much to the transition from primaries to secondaries as to the transition from variables to primaries. Recognition of this follows from acceptance of the fact that the number of real influences in a situation is one thing, and the number of factors we may take out, in accordance with mathematical and statistical restrictions, is quite another.

As a result of these considerations I shall propound a view of the influence pattern of the entities sought in factorization quite different from that hitherto accepted. Briefly, this theory is that the influences interact in what may be described in the most general terms as a *network* or reticule, and that the so-called hierarchy is an arbitrary piece chopped out of the network, which converges on fewer points at higher orders merely because of the mathematical rules which govern the 'cutting' of the network. This is illustrated in Fig. 4, where the only assumptions made are: 1. that causal effects operate in one direction only—that required by our definition of simple structure—from factors to variables and from higher order to lower order factors; 2. that there is no difference in frequency between factors and variables; and 3. that each variable is accounted for by more than one factor and each factor influences more than one variable.

In such a structure as that given in Fig. 4—which has the essential qualities implied by the terms network or reticule—the application of the usual procedures of factor analysis to four variables, v_2, v_3, v_4, and v_5, would result in the discovery of two primary com-

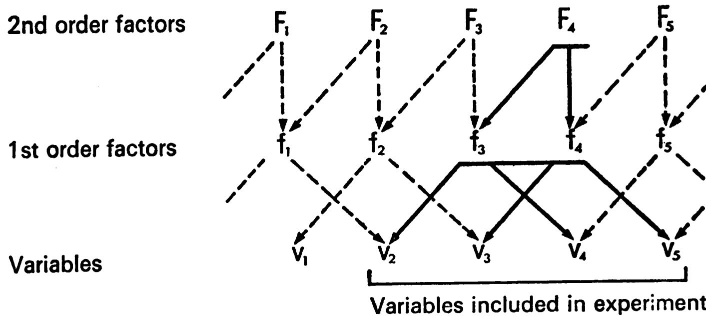

FIG. 4

False hierarchy in an essentially reticular structure

mon factors, f_3 and f_4, and the factoring of these would in turn result in the discovery of only one second order factor, F_4. These connections are shown by firm lines, whereas the undiscovered connections are left as interrupted lines. The firm lines clearly yield a hierarchy, yet the total structure is obviously not a hierarchy, there being as many higher order as lower order influences. Although I do not deny that tapered oblique hierarchies (or even monarchical ones) may at times be present, it would appear that most of the claims to demonstrate a hierarchy rest on nothing but the illusory effect just described. For example, the inclusion of variables v_1 and v_6 in an experiment would at once show that the four variables really operate under the influence of four first order factors, f_2, f_3, f_4 and f_5. This knowledge would result in the addition of f_2 and f_5 at the second order factoring, and this in turn would demonstrate that three second order factors, F_3, F_4 and F_5, are really operative on f_3 and f_4.

If, *on broadening the base of the variables* (or first order factors) one does *not* find more higher order factors, then one can conclude that a true hierarchy is present. The hierarchy of which we then speak would be different from the three types of hierarchy discussed above—dominant general factor, monarchical and tapering oblique—which have no proof of reality. It would be definable simply as: 1. a series of orders of factors, oblique at each level, and normally mutually overlapping in influence within each level; and, 2. one within which each order has fewer factors than the next lower order. The result would then be that higher order factors in such an *oblique pyramid hierarchy* will affect more of the initial sample of variables than lower order factors. But this will happen even in an ordinary network, for *regardless of which direction one* *moves in a network* (and of the separate existence of any real, excised pyramid) this broadening of influence necessarily occurs.

It may be that the structure of the mind in certain areas, notably the cognitive area, *does* correspond to a pyramid hierarchy, in the way that Burt,[4] Humphreys,[31] Vernon[41] and others have claimed. Even if wider search fails to broaden the basis of primary factor (and thus shift the top of the pyramid above the monarchical factor supposed to sit there), the status of the monarchical factor is still theoretically unsound. For if it affects all of the penultimate stratum of factors it is unrotatable —there is no hyperplane of unaffected factors by which its meaning can be uniquely determined. Of course, by some *other* method of factor analysis, in which additional penultimate stratum factors are deliberately introduced (Cattell[13]), a unique resolution of the last factor could be obtained. But this has not been done by those writers who propound a monarchical theory of general ability.

On the other hand, notably in the more complex fields of personality, motivation and learning, I would argue that a demonstrated hierarchy in any of these senses simply does not exist. And even in the cognitive area, recent evidence (Cattell[13]) that there is not *one* general ability factor but *two*—fluid and crystallized intelligence—upsets the monarchical hierarchy theory. The higher order structure which probably exists in the personality realm, and in many scientific realms, when examined by factor analysis, is decidedly more complex. I have paid this attention to pseudo-hierarchies because their true complexity will get due attention only if psychologists recognize that the traditional hierarchical notion is a fiction, created by the artificial limitations of calculation imposed by the rules in the factor analytic textbook.

OPERATIONAL IMPLICATIONS OF DEFINING A FACTOR AS A DETERMINER

The statement that the model assumed in testing hypotheses by properly-designed factor analytic experiments is more than a mathematical model, will now be clearer. Our model involves influences and causality, and actually defines independent and dependent variables, of which mathematics has no knowledge. Furthermore, interaction of influences may take place in all possible directions and connections. The only general model we can initially accept is, therefore, a reticular one, in which different orders of factors may interact in all kinds of ways, the hierarchy being a special case of the reticule, requiring special proof. The problem now before us is: 'By what means, in factor analysis, or with other experimental methods in addition to factor analysis, can we infer the particular system of causal connections—the structure—existing in any such reticule?' The postulated connections are quite general, put in this abstract way, but Fig. 5 will help to summarize the situation. In this diagram the assumption is made that every factor or variable acts on every other in all mathematically possible ways, including positive and negative feedback. (The positive and negative 'loadings' on each 'influence' arrow are not shown.)

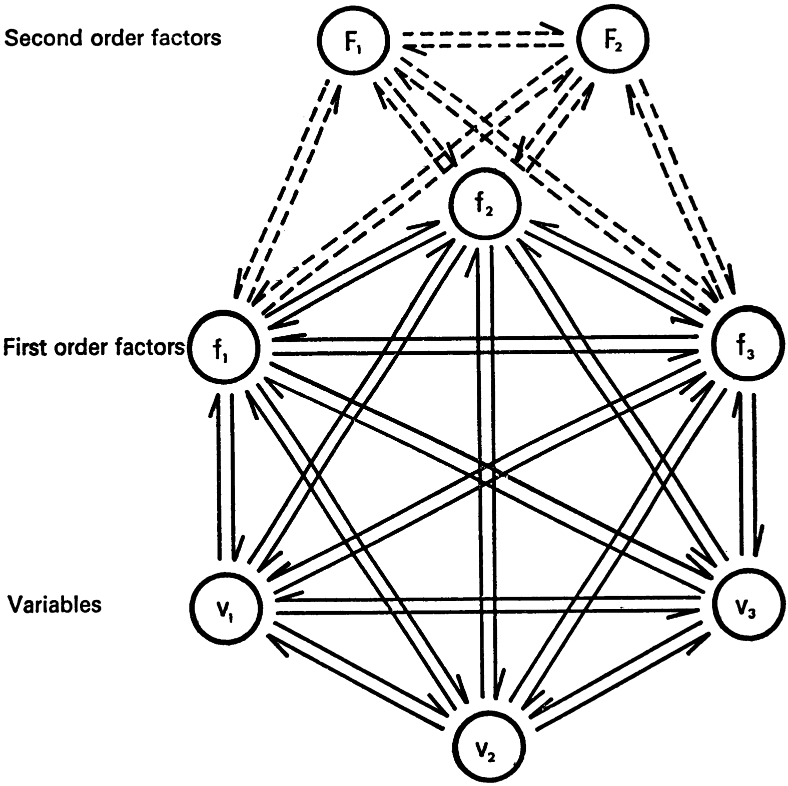

FIG. 5

Possible interactions of a set of eight distinct influences

To avoid overcrowding the sketch (*a*) interactions of second order factors directly with variables are omitted, and (*b*) only three variables are set out of the larger number which would be necessary to define three factors.

No systematic treatment of the general problem of defining the inferences about causal entities and directions of influence from factor analytic operations appears yet to have been attempted. Experimental studies, however, are replete with inferences on an inexplicit basis, particularly on that facet (one of three) which concerns the interaction of factors and tests. (The remaining facets concern interactions of factors with factors and variables with variables.) The schema which have actually been presented, namely SIPs such as the bipolar pattern, the simplex, the multiplex, etc., have usually connoted nothing more than a particular mosaic of correlations or loadings, with an occasional causal assumption that the effect is due to some real influences in a scientific model. Indeed, as they stand, such idealized mosaics as the bipolar, bifactor, simplex, radex and circumplex patterns are of no more than descriptive use, for the inferences that could be made from them to a scientific model have not been stated. Even descriptively most are frequently as misleading as useful, for real matrices can rarely be made to correspond exactly with them.

Nevertheless, it would be a service to factor analysis to have a taxonomy of ideal, stereotyped influence patterns, or SIPs, more systematically worked out than has yet been done. As I pointed out when I introduced this topic, one can vary the number of factors, the coverage of lower order variables, and the pattern of overlap with other factors, etc. When this is extended to include higher order factors the number of possible SIPs is very great. By way of a beginning, what appear to be the twelve most important have been systematically set out in Fig. 6. In this case the representation is at the level of stereotyped *influence* patterns, which go beyond the descriptive mosaics, though they imply quite specific mosaics. However, since the authors of some mosaics have never stated their assumptions in terms of an underlying scientific model, the use of older titles, like bifactor and circumplex, may be debated.

Let it be said forthwith that beyond SIP III no methods have yet been demonstrated whereby factor analysis could go directly to the scientific model from the data. One can infer what mosaics the later SIPs would imply in the actual matrices, but not, conversely, what the matrix mosaics would unambiguously imply in terms of SIPs. Parenthetically, SIP V, and others at that level, destroy any possibility of categorization as factors and variables. Any measurement, or estimated measurement, could be either or both.

It behoves us, at this point, to define more closely the status and meaning of the term factor. The epistemological status of a simple structure factor is that of an empirical *construct*,[10] but it commonly has

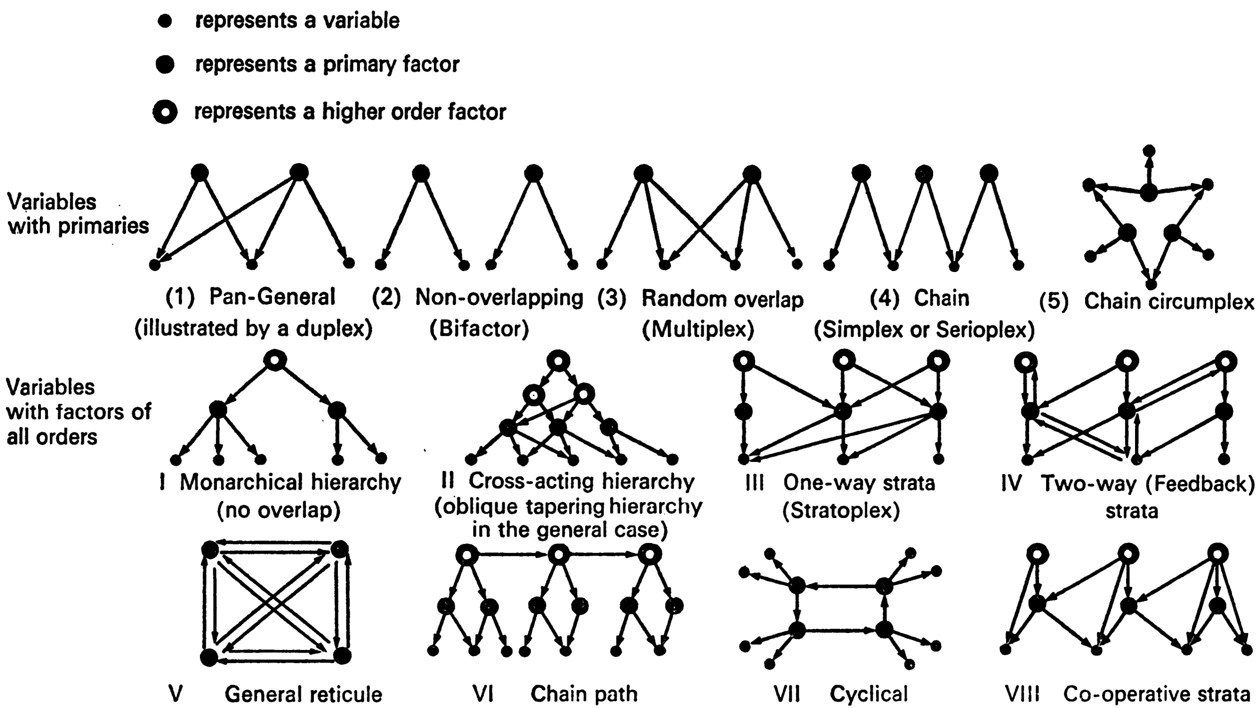

FIG. 6

Some outstanding Stereotyped Influence Patterns (SIPs) for describing factor and variable interrelations degrees of 'surplus meaning', borrowed from beyond the immediate system, which can turn it into a theoretical concept (see also Henrysson[29]). However, for such uniquely determined common factors in general, our view is that their only surplus meaning, beyond what is given by the properties of a *mathematical factor*, is that they are *influences*. Operationally, this additional meaning is derived from simple structure and confactor rotation operations.

The expression 'influence' seeks to define a broader concept of which both 'condition' and 'cause' are sub-species. In the Thurstone box problem,[39] for example, one would scarcely, in normal semantics, call 'length' a cause, and, in the Cattell-Dickman ball plasmode,[18] the weight factor is again a debatable cause—by some uses of 'cause'. The required generic expression is perhaps *determiner* rather than *influence*; for the variables loaded by length could not exist if a box had no length, while the change of velocity which one colliding ball will impart to another is determined if not caused (partly) by the weight of the ball. A *cause* is thus reserved for that special determiner in which we have additional *temporal sequence data*, which justifies the idea of one determiner acting upon another. To go beyond this would land one in metaphysics.

A factor can only be recognized as a cause when, in addition to the correlational evidence from *simultaneous* measurements, we possess evidence of an invariable *sequence* of the two in the relation which resulted in the correlation, for any correlation between A and B can mean that A affected B, that changes in B caused changes in A, or that some third cause produced changes in both of them. However, we argue that in factor analysis inferences about causal direction can be made *even when no actual time observations are available in our experimental data*. They can be made from *indirect* evidence at a high level of probability.

This evidence is of the same general nature as that invoked by the archaeologist, the astronomer, the geologist and other scientists denied the advantage of actually being present when the causal actions of interest to them occur. It depends on the fact that time sequences may be translated or preserved in other media, i.e. in space or temperature differences. Thus an archaeologist infers that Troy IV followed Troy III, because its deposits are spatially above those of the latter, and an astronomer infers that a red star was formed before a blue one. So here, we use the structural pattern of the mathematical relationships to infer that one factor is a determiner or causal influence operating upon another. On closer examination it will be seen that the two main independent principles proposed for uniquely fixing rotation and the resolution pattern, namely, simple structure and confactor congruence, operate strictly on the assumption that a factor is a determiner. Thus in the former what is typically a cause will affect only a minority of a random sample of variables, while in the latter it is assumed that a unitary influence, as it becomes more powerful, will affect *all* the things it normally affects with proportionately greater variance contribution.

The crowning purpose of a fully developed factor analytic technique should be, by such devices, to trace the causal connections among the factors and variables. Our aim, then, is to seek the basis for such inferences in the general case where no ulterior information is available.

The problem is, therefore, 'What inferences can be made about directions of causal action among variables and higher and lower order factors *in simple, R-technique* factor analysis?' The nearest approach to a systematic attack on this problem is Sewall Wright's[42] development of the *path coefficient* in relation to ordinary correlations. However, Wright's method *assumes* the direction of causal action entirely from ulterior evidence, and therefore contributes nothing to the problem of inferring such action retrospectively from the structure of the correlations.

THE EVIDENCE OF STRUCTURE OBTAINABLE FROM DV MATRICES

No solution is attempted here for the full possibilities of mutual interaction depicted in Fig. 5, or the later SIPs of Fig. 6. Instead we propose the less ambitious problem which arises if we assume an influence structure such as is found in the simpler scientific models given in Fig. 4, and in the examples (1) to (4) in Fig. 6. That is, a system in which the factors are the independent and the variables the dependent variables. If we restrict discussion of method to simple structure, then what kind of simple structure pattern must appear in order for us to infer the particular number of factors, overlaps, etc., which occur in such a relation?

The postulate that a factor is a cause *when it affects only a minority of variables* in a widely representative selection, leads to the conclusion that the rotation which locates the factors as causes will be one in which every column has a maximum number of zeros. This leads to the inevitable arithmetical consequence that in seeking a solution which maximizes zeros in the *columns* we are automatically maximizing the zeros in the matrix as a whole and therefore maximizing zeros in the *rows* of the same matrix.

The effect of this conclusion upon our attempt to infer causal direction is disastrous, for whereas a predominance of zeros in *columns* argues for factors being causes, a predominance in the *rows* argues for the *factors* being dependent and the variables being the influences. This difficulty can be resolved, however, if we follow the logic of oblique solution to its ultimate conclusions. Indeed, the failure of simple structure when thus applied to the orthogonal case is only one more proof that the orthogonal resolution of mathematical factor analysis is wrong in the scientific sense, i.e., that it is not a model which can fit even the most general requirements of a causal system.

The next step in our argument depends upon the general relations among the six main dimension-variable (DV) matrices (Cattell[11]). These variants of the dimension-variable relation matrix, possible in the oblique case, are as follows:

1. The reference vector structure matrix, written V_{rs}. This is the usual matrix obtained by rotation for simple structure, yielding *correlations* between variables and reference vectors.

2. The factor pattern matrix, V_{fp}, which gives the *loadings* of factors on variables. It is proportional by columns to V_{rs} and retains the same simple structure:

i $$V_{fp} = V_{rs}D$$

3. The factor structure matrix, V_{fs}, namely, the *correlations* between factors and variables:

ii $$V_{fs} = V_{fp}C_f$$

(where C_f is the matrix of correlations among factors).

4. The reference vector pattern matrix, $V_{rp} = V_{fs}D^{-1}$.

5. The factor estimation matrix, V_{fe}, which sets out the weights to be given to the variables to obtain the best estimate of each factor. This can be calculated either in the usual way,

iii $$V_{fe} = V'_{fs}R^{-1}$$

(where R is the correlation matrix among variables) or in Tucker's fashion—

iv $$V_{fe} = (V'_{fp} V_{fp})^{-1} V'_{fp}$$

6. The dissociated factor pattern matrix, V_{dfc}, the contribution made by a factor to the variance of a variable in *dissociation from the effect of higher order factors.*

Now, as we suggested above, an r of, say, 0·8, may mean that 64 per cent of the variance of a will disappear when b is held constant or that 64 per cent of the variance of b will disappear if a is held constant. The former is nonsense if a is the cause of b, since nothing we do about 'holding' the consequence b need have any effect on a. With this in mind we may note first that the V_{rs} and V_{fs} matrices (as well as V_{afc} Cattell[11] not here described) are non-committal statistical figures, simply stating correlations, but *a theory of causal action* is implicitly written into the formulae V_{fp}, V_{fe} and V_{dfc} if they are derived by simple structure.

The factor pattern matrix, V_{fp}, tells us how much each factor contributes to the variance of each variable, considering how much is also contributed by other oblique factors. V_{fe} tells us how we can weight the variables to maximize the multiple correlation of the estimate of the factor with the true factor—assuming the factor position to be that settled upon in V_{fp} rotations. There is thus no mathematical principle whatever which requires that simple structure shall appear in V_{fe}. This matrix gives the contributions of the variables to the factor assuming, not that they are causal, but that they are manifestations or constituents from which the factor can be estimated.

However, we occasionally meet instances where we may strongly suspect on psychological grounds that a certain first order factor is a causal contributor to the second order.* That is to say, one of the alternative possibilities already written into Fig. 5 (such as $f_1 \rightarrow F_1$) is suspected, in which the direction of causal action is opposite to the usual direction, either simultaneously with or as a substitute to it. A general solution of the reticular influence model is beyond the scope of this paper, but we suggest that a solution may be profitably pursued by making comparisons among the six DV matrices (Cattell[11]), though this will obviously be extremely complicated. Some awareness of and reference to these matrices is necessary even in the solution of the 'one-way strata' model on which we shall now concentrate.

Such a 'stratoplex' model appears likely to have a good fit, at least as a first approximation, in the field of ability and in the field of motiva-

* As in the case of the anxiety factor considered later in this paper.

tion where later acquired habit systems will subsidiate to (or be reinforced by) earlier systems, as in the concept of the dynamic lattice (Cattell[10]). It also has application to group dynamics and to other areas of social psychology. In short, although compared to the general network model, it makes simplified assumptions, the methodology for handling it is well worth investigating.

THE CONCEPT AND OPERATIONAL RECOGNITION OF FACTOR STRATA

On the assumption that only the reticular model is fitted to the interpretive use of factor analysis in the universal case, we have rejected the monarchical hierarchical model for anything but investigations which claim information from ulterior sources. However, we recognize that the general reticular solution is too difficult, and we suggest thorough treatment of the one way strata SIP. Examples (1) and (2), in Fig. 6, show that the monarchical and the tapering hierarchy are special forms within the general one-way strata model III, but thereafter all SIPs include either a two-way (feedback) action, or an action of factors upon others of the same level, or a leap-frogging in which higher orders set directly on more than one lower order.

In the strata model we assume an indefinite number of factors operating at each of a number of operationally-defined levels. Theoretically there could be influence in both directions, making a two-way as well as a one-way model possible, but we shall consider only a one-way influence model (higher order affecting only lower) which for brevity we will call the stratoplex model. The 'peer status' of factors in one and the same stratum is defined initially by the functional condition that all members of the same stratum influence only members of the next 'lower' stratum. They do not influence each other or members of the peer group above them.

In investigating the operational steps necessary to obtain evidence in terms of the stratoplex, one must begin by recognizing that factor 'order' is not the same thing as factor 'stratum'. Order is given immediately in terms of operations: the factors, say A to D, obtained by factoring entities a to k, are one order higher. Stratum indicates an influence relation in a model, and probably cannot be safely inferred from a single factoring operation, though it is true that operational proof of order is a large part of the evidence.*

* The difference becomes evident if one thinks of order studies in a network which happens to be a circumplex, where successive orders are not strata, but rings.

The main reason why order operations and strata steps are not the same is that in the initial sampling of variables, at any level, one cannot be sure that one has started with variables that are themselves all in the same stratum.

There is not space here to deal with all the alternative possibilities that may befall the investigator. Suffice it to say that if the sample of variables chosen does indeed contain nothing but members from one and the same basic level, then the first order factors will be all on one stratum higher. In the other cases, four in number (see Fig. 7), it can be shown that i. accidental inclusion of one higher stratum measure in the factoring of a set from a lower level will result in the appearance of the former as a specific, i.e. as a factor with only one variable loaded on it, whereas *two or more* will result in the appearance of a set of factors some of which are from one stratum level and some from another; ii. the factors themselves, as individual factors, will not be mixed, and iii. what appears operationally is either a second stratum or a third stratum factor; though a considerable problem remains in recognizing (*a*) when this heterogeneity of strata in the single order of a single experiment has happened, and (*b*) *which* factor is at which stratum level.

Elsewhere (Cattell[10]) the problem of defining the stratum to which a factor belongs has been decided on the basis of the *density* of representation occurring in the sample of variables, and thus rests on our ability objectively to define a *population of variables* and a *density of representation* of such variables, in a factor analysis. If the variables are close together, highly similar, and with a high 'density of representation' they will factor largely into lower stratum factors, whereas if widely separated and diverse, i.e. of sparse density, the first factor analysis will in the main go direct to higher stratum-order factors. It is known, for example, in some instances (Cattell,[10] Cattell and Scheier[20]), that second order factors in questionnaires become first order factors in objective personality tests, possibly because questionnaire items make fine distinctions in behaviour which in a miniature situational response is covered by a single variable.

However, the concept of distance between variables in the density definition must be quite independent of correlational operations, *which yield distance only in the sense used in Mahalanobis' generalized distance function, D, or Cattell's pattern similarity coefficient,* r_p. Without this, no independent check would be gained. The advantage of an exact density concept and measurement operation would be that one could from the

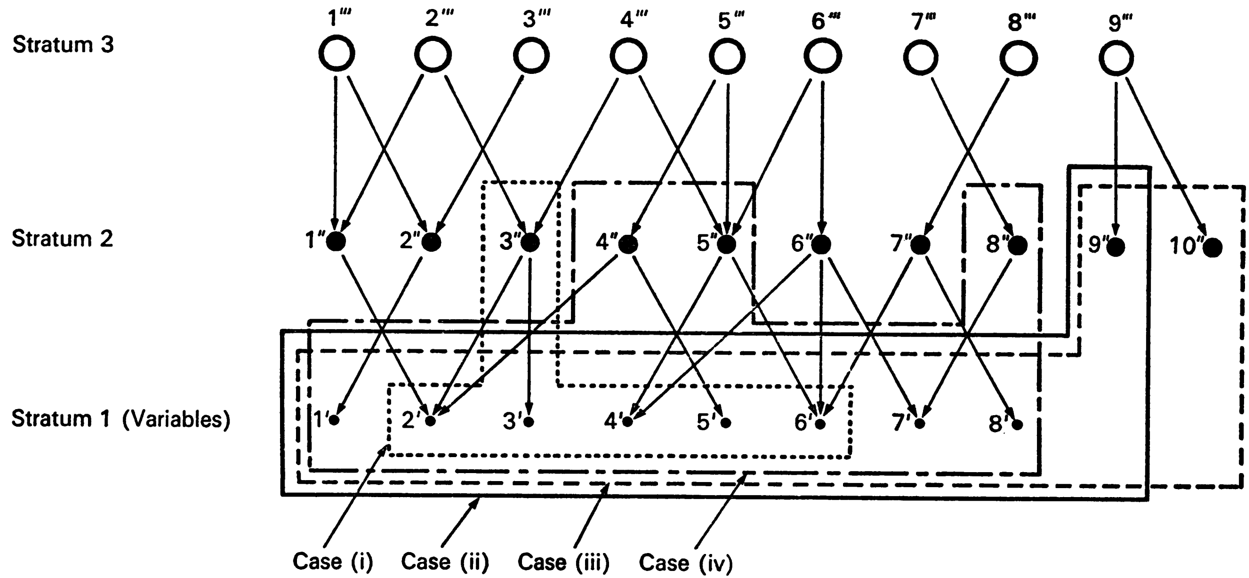

FIG. 7

Relation to inferences about factor strata of operation revealing factor order

Case (i) One primary factor from stratum 2 is accidentally included with variables from stratum 1, and it has psychological representation in the other first stratum variables. No problem arises.

Case (ii) One primary factor from 2 included in 1 (as above), and it has *no* psychological representation in stratum 1 variables.

Case (iii) Two or more primaries from 1 accidentally included in 2, with no psychological representation in first stratum variables.

Case (iv) Two or more primaries from 1 included in 2, and they *have* psychological representation in first stratum variables.

beginning, without evidence from factoring, predict what order of factors one would be obtaining. Suggestions for independent operations for defining density of representation of variables have been made elsewhere (Cattell[10]). However, as these operations are different, the strata concepts reached might not necessarily rest on the notion of *influence* direction. Indeed if the criterion of strata order is that variables in stratum *X* influence variables in stratum *Y*, and we can depend upon simple structure to witness this, then the present approach, though difficult, is sufficient.

The stratoplex model is only a specialized and simplified form of the reticular model and there is no more than a probability that it will hold in psychological experimental situations. If it holds there are methods, as we have seen, of establishing, by examination of successive factor analyses, what the strata structure in a given set of variables really is. The simple procedure of saying that the strata are identical with the operational orders is insufficient and incorrect. A more subtle manner of inference is required, from planned and re-planned experiments, comparing operational orders in each.* The evidence that the model itself is fitting must come from the consistency of the strata picture it yields in different samples of variables.

While investigation and formulation of the problem of stratum order may challenge some of the substantive psychological concepts already formulated, yet it is nevertheless reassuring in more general terms. For it shows that, granted strata, our factor analytic operations will not merely yield sets which are *mixtures* of first and second strata factor variables, neither will it produce individual factors which remain conceptually neither at one level nor the other.† In short, it does *not* produce strata results which are purely at the mercy of accidents of sampling, though factors from different strata will sometimes

* The past confusion between order and stratum suggests that a number of conclusions about order need to be re-examined, for it is likely that some factors now thought to be of the same order are actually at different strata levels. A probable instance is that factors B (intelligence) and G (superego strength), considered as primaries in the 16 Personality Factor Questionnaire, tend to stand out as specifics in the second-order factoring (Gorsuch,[25]) and are therefore probably cognate with the second stratum factors of exvia-invia, anxiety, cortetia, etc., found in questionnaires and objective tests.

† There could, of course, be an *intermediate* stratum running part way between two strata, like a half floor in the contemporary 'split level' house. This, however, involves problems of the general reticular model and is best considered in that context.

initially be produced together in one order. Indeed, simple structure criteria, pursued with alert co-ordination among successive studies, and with an eye to the possibility of an initially heterogeneous bag at any one order, seems fully capable of yielding definitive knowledge of strata relationships.

FORMULAE FOR CONTRIBUTIONS OF FACTOR VARIANCE ACROSS A NETWORK

The first aim of basic research in any scientific area must be to determine the reticular structure of functional relationships. Only when the pattern and sequence of interaction among such factors is accurately mapped and understood is it possible to seek laws or check quantitative relations assumed to obtain between varying influences by holding constant this or that factor. Still more, perhaps, in applied research, it becomes necessary to estimate how much an influence X will contribute to a dependent variable Y when it acts through several intervening variables.

A complete situation for the stratoplex is given below.*

R_0, R_1 and R_2 have been used respectively for the successive matrices of correlations between variables, factors and still higher orders, as the emphasis here *is* on an orderly succession. V_{fp} and V_{fs} indicate factor pattern and factor structure variable matrices respectively.[16] First stratum, second stratum, etc., factors are called primaries, secondaries, tertiaries, etc.

Since it is our purpose to deal with variance contributions via the mutual, variable-to-factor and factor-to-variable V matrices (V_1, V_2, V_3 for the ascending orders), and with results in terms of the score matrices and the estimation matrices for factor and variable scores, we shall use the symbols Z_0, Z_1, Z_2, Z_3, etc., for the standard score matrices (for N people), the subscripts indicating that the scores are respectively for variables, primaries, secondaries, tertiaries and so on.

The basic formula v below, which relates factor loadings to correlations among variables, derives its orthogonal origins from the analysis of a correlation matrix into latent roots and vectors. Although this extraction of orthogonal factors is the initial step in factor analysis, we deal with the final oblique resolution which the psychologist is accus-

* Wright's path coefficient is also useful (Wright[42]). But the investigator using the stratoplex model is not simply asking about correlation, but about variance contribution acting in a specified direction.

tomed to use once simple structure rotation conditions have been met. The orthogonal position can always be readily handled as a special case of this. For example, in formula v, R_1 reduces to an identity matrix (in effect disappears) when factors are orthogonal, and V_{fp} then becomes identical with V_{fs}.

v $$R_{ro} = V_{fp\cdot 1} R_1 V'_{fp\cdot 1},$$

where R_{ro} is the *reduced* correlation matrix, and the subscript r stands for 'reduced'.

To use the scores on these factors to restore scores on the original variables we proceed with:

vi $$Z'_0 = V_{fp1} Z'_1 + U_1 Z'_{u1},$$

where Z_0 is the test-score matrix, Z_1 the common factor score matrix, U_1 the matrix of orthogonal, specific (unique) factors for the n variables in the rows of the V_{fp} common factor matrix, and Z_{u1} is the score matrix of N people on the n unique factors (specific plus error). (Parenthetically, Z is an $N \times n$ matrix (N people, n variables) and Z_1 is an $N \times k$ (k factors).)

The successive steps carrying correlation and score relations only to the third order (see equations vii and viii below) suffice to establish the generalization up to any order. The successive (primary and secondary) correlation matrices can be analysed just like the variable correlation matrix in v, thus:

vii $$(a)\ R_{r1} = V_{fp2} R_2 V'_{fp2}$$
$$(b)\ R_{r2} = V_{fp3} R_3 V'_{fp3}.$$

The successive factor score matrices beyond vi above will relate as in viii:

viii $$(a)\ Z'_1 = V_{fp2} Z'_2 + U_2 Z'_{u2}$$
$$(b)\ Z'_2 = V_{fp3} Z'_3 + U_3 Z'_{u3}.$$

From these we can formulate the restoration of a correlation matrix among variables or factors at any one stratum from the factor loadings and the correlations for an immediately higher stratum by the recursion formula:

ix $$R_{rx} = V_{fp(x+1)} R_{(x+1)} V'_{fp(x+1)} = R_x - U^2_x,$$

where R stands for the correlation matrix with ones in the diagonal, and R_r for the reduced matrix.

Similarly, the relations between scores of any two successive strata can be generalized, as in x:

$$\text{x} \qquad Z'_x = V_{fp(x+1)}Z'_{(x+1)} + U_{(x+1)}Z'_{u(x+1)}.$$

From these simple, single-stage recursion formulae we can proceed to the formulation of a transformation over any number of stages, i.e. to express the contribution to a set of variables or factors of a factor remote from it in the stratoplex. In accordance with our general model we use the V_{fp} matrix because of the assumption (Cattell[11]) that influence is in the direction of factors upon variables. By the statistical properties of factoring we are also compelled to assume that our formulae deal with the usual tapered-off hierarchy—a chopped-out triangular wedge from the total strata.

The general formula for restoring the lower order correlation matrix, when beginning with factors n orders higher, now becomes:

$$\text{xi} \qquad R_{ro} = V_{fp1}V_{fp2} \ldots V_{fpn}R_nV'_{fpn} \ldots V'_{fp2}V'_{fp2}.$$

In the case where the general reticulum happens to present a true tapering pyramidal hierarchy, finishing at the nth factor extraction with a *single*, massive, general factor (as some believe to be true in the general cognitive field) then the highest matrix, R_n, becomes unity (one factor identity matrix) and drops out of the formula. It must be stressed, however, that unless one has ulterior evidence about the notation of this final single factor its character is fictional. A general factor exists, but it is indeterminate. Fortunately this does not affect the determinacy of the lower order factors, even in the tapering hierarchy, but it affects the true monarchical hierarchy if an attempt is made to rotate the specifics of the lower order factor into a special conformity to it.

To estimate *scores* on lower order from scores on higher order factors we use formula xiii below.

From vi we can proceed, by substitution for Z'_1 as in viii (a) above, to:

$$\text{xii} \qquad Z'_0 = V_{fp1}V_{fp2}Z'_2 + V_{fp1}U_2Z'_{u2} + U_1Z'_{u1},$$

and from vi, xii and x, it will be evident that a general formula can be written for restoration of a lower stratum score matrix from the data of higher strata, at any degree of distance, $d = n$, as in xiii below:

$$\text{xiii} \; Z'_0 = V_{fp1} \cdots V_{fpd}Z'_d + \sum_{d=1}^{d=n} V_{fp1} \cdots V_{fp(d-1)}U_dZ'_{ud} + UZ'_{u1}$$

If desired, the notation of supermatrices can be used (see Horst, 1963[43]), thus:

$$V = V_{fp1}V_{fp2} \cdots V_{fpd}{:}V_{fp1}V_{fp2} \cdots V_{fp(d-1)}U_d \cdots {:}V_{fp1}U_2{:}\ U_1,$$

and

$$Z' = Z'_d{:}\ Z'_{ud}{:} \ldots Z'_{u2}{:}\ Z'_{u1}.$$

Then xiii can be written:

$$Z' = VZ'. \qquad \text{xiv}$$

Here the observable scores, Z, are written as the product of a factor pattern and a factor score supermatrix.

FIG. 8

The higher order projection matrices: contrast of Cattell–White and Schmid–Leiman loadings of higher order factors on variables

By Cattell–White formula

	First order					*Second order*			*3rd order*	
	1′	2′	3′	4′	h^2	1″	2″	h^2		h^2
1	·00	·00	·00	·50	·25	·00	·30	·09	·21	·04
2	·00	·50	·00	·00	·25	·00	·00	·00	·00	·00
3	·00	— ·50	·60	·00	·61	·36	— ·30	·09	·09	·01
4	·50	·60	·00	·00	·61	·25	·00	·06	·21	·04
5	— ·50	·60	·00	·00	·61	— ·25	·00	·06	— ·21	·04
6	·60	·00	·50	·00	·70	·60	— ·25	·24	·33	·11
7	— ·60	·00	— ·50	·00	·70	— ·60	·25	·24	— ·33	·11
8	·00	·00	·60	— ·60	·79	·36	— ·66	·28	— ·16	·03
9	·00	·00	·00	·50	·25	·00	·30	·09	·21	·04
10	·00	·00	·00	·60	·36	·00	·36	·13	·26	·07

Inter-correlations of first order factors (Cattell–White)

	1′	2′	3′	4′
1′	100	00	15	18
2′		100	00	00
3′			100	— 08
4′				100

FIG. 8 (*continued*)

Inter-correlations of second order factors (Cattell–White)

	1″	2″
1″	100	60
2″		100

By Schmid–Leiman Formula

	First order					Second order			3rd order		
	1′	2′	3′	4′	h^2	1″	2″	h^2	1‴	h^2	Eh^2
1	·00	·00	·00	·40	·16	·00	·21	·04	·21	·04	·25
2	·00	·50	·00	·00	·25	·00	·00	·00	·00	·00	·25
3	·00	— ·50	·52	·00	·52	·19	— ·21	·08	·09	·01	·61
4	·43	·60	·00	·00	·55	·13	·00	·02	·21	·04	·61
5	— ·43	·60	·00	·00	·55	— ·13	·00	·02	— ·21	·04	·61
6	·52	·00	·43	·00	·46	·32	— ·18	·13	·33	·11	·70
7	— ·52	·00	— ·43	·00	·46	— ·32	·18	·13	— ·33	·11	·70
8	·00	·00	·52	— ·48	·50	·19	— ·47	·26	— ·16	·03	·79
9	·00	·00	·00	·40	·16	·00	·21	·04	·21	·04	·25
10	·00	·00	·00	·48	·23	·00	·25	·06	·26	·07	·36

For notes on Fig. 8 see Addendum on p. 266.

The formulae given above appear to contribute the first *general* exposition of lower and higher strata interrelationships in scores *and* correlations, although a well-known formula dealing with a special aspect of this problem has been devised by Schmid–Leiman.[37] They developed a 'procedure for rotating an oblique simple structure into a hierarchical factor solution' (*op. cit.*, p. 56). The procedure they describe is directed specifically to a pure hierarchical structure finishing in a single third-order factor. They derive a factor matrix, *B*, which differs from our formulae xii, xiii and xiv in disregarding U_1. They do not point out, as we have done here, that their orthogonalized factors correspond to the higher order primary factors along with the lower order unique factors, for their approach has a different purpose and conceptual framework.

The aim of the Schmid–Leiman formula was to determine the loadings of higher order factors directly upon the variables after the factors have been rotated into *orthogonal positions*. In these positions all but the monarchical factor are merely the truncated specifics of the original oblique factors they represent, i.e. specifics after the higher order variance is abstracted.

Here we propose to contrast the Schmid–Leiman version of what we may call the Higher-Order-Factor-To-Variable, or, Higher-Order-Projection matrix, with the Cattell–White alternative formulation.*

The main differences are as follows:

1. Although both normally *begin* with factors rotated to simple structure, the Cattell–White transformation retains at every stratum these unique oblique structures, whereas the Schmid–Leiman extracts at each stratum the higher order variance, and leaves at each lower level only the orthogonal residual specifics from the oblique factors, not the factors in their full variance. The final step in the Schmid–Leiman is to end in a single general factor or a set of orthogonal factors.

Thus the loadings of the second order factors on the variables, in the Higher-Order-Projection matrix, which, we will symbolize as V_{fpII}, become by the Cattell–White formula xvi below, equal to V_{fp1} times V_{fp2}, whereas in the Schmid–Leiman it is V_{fp1} times V_{fp2} times U_2. In the latter, the loadings on the variables are not those of the true secondaries but only the projections of the truncated remains of these factors after much of their variance has been taken into factors of a still higher order.

2. The Schmid–Leiman HH Projection matrix cannot reach stability and completeness unless and until the successive factorings end either in a set of factors whose simple structure is naturally orthogonal—an extremely rare condition in our experience—or in a single general factor. The corresponding Cattell–White HH Projection matrix has a functional completeness when terminated at any stratum.

3. In the Cattell–White HH Projection matrix the summed squares of loadings of factors on a variable will *not* sum to the communality of the original unrotated matrix whereas in the Schmid–Leiman it will. This contrast is part of the conceptual difference that the V_{cs} is a matrix of loadings only, not correlations, in the C–W case, but of correlations with orthogonal factors in the S–L case.

The h^2 values are smaller in the S–L, since they are confined to one stratum with all higher order variance taken out, whereas in the C–W

* This formulation was developed while Owen White was a research assistant to the writer.

they include the variance for the given stratum in addition to that which would go into all higher strata. Contrast of two successive h^2 columns in the C–W shows how much predictive power would be lost by dealing only with the factors at a higher order.

4. The loadings in the S–L matrix, *as far as the first order factors* are concerned, will correspond to those defined as the dissociated factor matrix, V_{dfc}, above. The simple structure on these will be the same as for the ordinary oblique factors, but the significant loadings will all be reduced by the same constant ratio, V_j, on any one factor. Thus the S–L factors will have smaller loadings than those in the C–W, and will yield, too, a different rank order of contribution to the variables. A row in the S–L composite will thus look quite different from, and have no simple relation to, a row in the C–W, except that zeros will appear in the same places.

The derivation of the Cattell–White formula is clearest, perhaps, if one considers the general formula xi above, for restoring the reduced correlation matrix from factor pattern matrices. If we symbolize the Cattell–White loadings of the nth order factors *directly on the variables* by $V_{fp \cdot N}$, this too will restore the correlation matrix and we can write:

$$\text{xv} \quad R_{r0} = V_{fp1} \ldots V_{fpn} R_n V'_{fpn} \ldots V'_{fp1} = V_{fpN} R_n V'_{fpN},$$

whence

$$\text{xvi} \qquad V_{fpN} = V_{fp1} \ldots V_{fpn} \text{ (C–W formula)},$$

which forms the systematic basis of calculation for all the values in the Cattell–White Higher-Order-Projection matrix, which differentiates it from the corresponding Schmid-Leiman matrix.

THE SIMPLE STRUCTURE OF HIGHER ORDER FACTORS AND THE PROBLEM OF PSEUDO-SECOND ORDER FACTORS

The main uses of these two higher order projection matrices would appear to be: (*a*) to predict variable scores from factors; (*b*) to calculate formulae (V_{fe}'s) for estimating higher order factors directly from variables; (*c*) to decide when it is worth while—in terms of accuracy lost for economy gained—to shift from measuring several first order to fewer higher order factors; (*d*) to provide an alternative basis for the rotation and identification of higher order factors; and (*e*) to provide an alternative basis for interpreting higher order factors.

The last two uses—(*d*) and (*e*) above—require us to fish in rather deep theoretical waters. Several writers have wanted to use the C–W or S–L matrices to provide a new basis for rotating higher order factors by rotating them for simple structure *directly on the variables*. The logic of this seems faulty. If the factors influence directly only *the next lower* stratum of factors, then they should be explicitly rotated to *simple structure on that stratum*, an assumption underlying formula xvi. But does this not also imply, as many psychologists seem to have tacitly assumed, that when the corresponding loadings on variables are examined these will *also* show a simple structure pattern? In other words, is simple structure on V_{fpn} automatically a simple structure on $V_{fp(n-1)}$?

Let us take a specific level, that of the second order factor projections on the variables. By virtue of the simple structure in the first order V_{fp1}, and in the second order V_{fp2}, will there be simple structure in the second order? Applying the Cattell–White formula we have:

$$\text{xvii} \qquad \underset{(n\times 1)}{V_{fp\cdot II}} = \underset{(n\times k)}{V_{fp1}} \cdot \underset{(k\times 1)}{V_{fp2}}.$$

If, on average, $1/p$th of the variables in a row of V_{fp1} and a column of V_{fp2} are zero, the chances of getting $1/p$th of the row-column products to be zero is less than unity. The fraction of *single* products which are zero is *greater* than p, namely, $\frac{2p(1-p)+p^2}{1} = 2p-p^2$ (where $p < 1$).

But in the matrix multiplication we take k products at a time and the chance of one of these containing nothing but zeros is far smaller. Except for some quite special and fortunate relation between the positive and negative values in the row and column, a zero will appear in V_{fpII} only when *every* product of a k row by a k column happens to be a zero. In practice the hyperplane entries will not be *exact* zeros, but if we set the same standard of hyperplane width in V_{fp1}, V_{fp2} and V_{fpII} this will not affect the issue. To demonstrate this concretely we have taken in Fig. 9 a hypothetical example, in which 60 per cent of the variables are in the hyperplane of the primaries, V_{fpI}, and 50 per cent in the hyperplane of the secondaries, V_{fpII}.

As the theoretical introduction would suggest, the outcome on the V_{fpII} proves to be much poorer than the hyperplane on the primaries and secondaries. There is a progressively poorer hyperplane *on the initial variables* as we move to higher order factors. It might be that a peculiar property of real data is that simple structure on the primary and secondary is also maximally simple on V_{fpII}. To test this we have taken a well-known 44 variables example ($N = 300$) consisting of 44 items from the Sixteen Personality Factor Questionnaire used in a number of cross-cultural researches (Cattell, Pichot and Rennes[19]) to check the cross-cultural constancy of factor structure on the eleven factors represented. The original primaries stand up well and have been factored to six second orders (already well known from other factor-

FIG. 9

Calculations of higher order loadings on variables by Cattell–White formula, showing effects on simple structure

$V_{fp1} \times V_{fp2} = V_{fpII}$

First order factors

Variables	1′	2′	3′	4′
1	0	0	0	5
2	0	5	0	0
3	0	− 5	6	0
4	5	6	0	0
5	− 5	6	0	0
6	6	0	5	0
7	− 6	0	− 5	0
8	0	0	6	− 6
9	0	0	0	5
10	0	0	0	6

Good (60 per cent) Hyperplane count

×

Second order factors

	1″	2″
1′	5	0
2′	0	0
3′	6	− 5
4′	0	6

Good (50 per cent) Hyperplane count

=

Second order factors

I	II
0	30
0	0
36	− 30
25	0
− 25	0
60	− 25
− 60	25
36	− 66
0	30
0	36

Poor (35 per cent) Hyperplane count

FIG. 9 (*continued*)

V_{fpII} × V_{fp3} = V_{fpIII}

	I	II
1	0	30
2	0	0
3	36	− 30
4	25	0
5	− 25	0
6	60	− 25
7	− 60	25
8	36	− 66
9	0	30
10	0	36

Poor (35 per cent) Hyperplane count

	1'''
1''	85
2''	71

	1'''
1	21
2	00
3	09
4	21
5	− 21
6	33
7	− 33
8	− 16
9	21
10	26

Still poorer (25 per cent) Hyperplane count

ings of the 16 P.F.). These represent Anxiety, Exvia–Invia (Extraversion–Introversion) and two other factors we need not discuss here. The loadings of these four on the items have been calculated by the Cattell–White formula and some have been set out in Fig. 10. From the full table we find that 60 out of 176 loadings stand in the $\pm \cdot 10$ hyperplane, a value of 34 per cent compared with 59 per cent in the same hyperplane width in the 484 loadings of the primary factors (Cattell, Pichot and Rennes[19]). The result again suggests that no simple structure exists on V_{fpII} comparable to that on V_{fp}.

FIG. 10

Loadings (Cattell–White formula) of 16 P.F. items directly on second order factors of Anxiety, Exvia—Invia, etc. (Data from Japanese sample)

	Anxiety F_1	Exvia F_2	Path. F_3	Unknown F_4
1 Would you rather (if salary, prestige, etc., were equal) do the work of (*a*) a physicist, or (*b*) a salesman for some invention? (An A Factor variable: loading ·81.)	30	43	43	— 52
2 Is your health unpredictable, forcing you frequently to alter your plans? (A C Factor variable: loading ·60.)	— 28	08	— 18	— 11
3 Are you considered a lively, enthusiastic (perhaps too lively and enthusiastic) person? (An F Factor variable: loading ·64.)	15	52	00	— 10
4 Would you like the kind of job that offers change, travel and variety (in spite of other drawbacks)? (An F Factor variable: loading ·31.)	31	23	— 07	— 18
5 Do you like generally (*a*) to assume that you can meet difficulties as they arise, (*b*) or to plan a piece of work to meet all difficulties? (A G Factor variable: loading ·31.)	05	17	06	16
6 Do you find it difficult to address or recite to a large group? (An H Factor variable: loading ·71.)	— 46	60	22	10
7 Are you inclined to worry without any reason for doing so? (An O Factor variable: loading ·69.)	40	10	— 13	— 07
8 Do you have periods of feeling grouchy when you just don't want to see anyone? (*a*) very rarely, (*b*) quite often. (An O Factor variable: loading ·26.)	22	07	43	32
9 Do you frequently get in a state of tension and turmoil when thinking of the day's happenings? (A Q4 Factor variable: loading ·30.)	18	— 04	33	— 12
10 Do you tend to get angry with people rather easily? (A Q4 Factor variable: loading ·13.)	35	05	— 20	01

The second order anxiety factor loads the items from the primaries $C(-)$, $H(-)$, O and $Q4$ just as one would expect from the loadings of the latter on the second order. However, the loadings are lower than on the first orders and the hyperplane of zero loadings one would expect on the extraversion contributors A and F fails to appear. The exvia (extraversion) factor also loads the items of its primaries, A, F and H—in this case as well as they load their primaries—but the hyperplane is again poorer than with primaries though still surprisingly good. The conclusion would seem to be that one might hope to infer the nature of second order factors from their projections directly on variables, but not with such accuracy as from the structure on primaries.

There are three alternative resolutions to second order structure and position—resolutions that various psychologists have entertained at different times by examining:

1. Simple structure of true second orders on the primaries;
2. Simple structure of true second orders on the variables;
3. Simple structure of pseudo-second orders on the primaries.

The second alternative is not necessarily the same as finding that degree of simple structure on the variables that corresponds to true simple structure on the primaries. It is possible that a *better* simple structure could be found *directly* on variables. We have rotated both examples to maximum simple structure, by the oblimax programme, with results shown in Fig. 11. The structure (Row 4), due largely to the unusually poor (36 per cent) count for secondaries on primaries in the second example, is a shade better (mean of 46 per cent versus 43 per cent) than that fixed by the simple structure on the primaries.

However, by the Bargmann test,[3] neither the 45 per cent nor the 47 per cent is significant.

The third alternative—resolution into pseudo-second orders—requires some description. In work published during the 40's and 50's, and even the 60's, the practice has been to take out only as many factors at the first order as one guessed there were factors at the second. It is then asserted that these, rotated in the greatly reduced space, have, for all practical purposes, the same meaning as second order factors. To take only those which have been used as a basis for more extensive theories, there are Eysenck's dysthymia and extraversion,[22] Peterson's general ego-super-ego-vs-delinquency dimension[36] and Becker's attempt to treat the 16 P.F. as a four factor scale.[2] Related, but not identical, are Spearman's general ability at a first order level, and, in

FIG. 11

Hyperplane counts on experimental matrices comparing possibilities of second and first order rotation

	Experiment 1, *artificial data*, 10 *variables*	*Experiment* 2 (44 *variables*) 16 *P.F. data* (*Japanese sample*)
	per cent	*per cent*
1 Primary factors rotated to simple structure on variables	60	50
2 Secondary factors rotated to simple structure on primaries	50	36
3 Secondary factors projected on variables after being rotated to simple structure on primaries	35	38
4 Secondary factors rotated directly to maximum simple structure on variables	45	47
5 Pseudo secondary or 'mongrel' factors rotated to maximum simple structure on variables	35	35

personality, the studies by Norman[34] and Tupes[40] which take out only five or six factors (still naming them as primaries, however) where, from similar ratings, Burt[5] and the present writer have preferred to take out a dozen or more.

These theoretical assertions, especially those of Becker and Eysenck, are the more misleading because there is, in fact, a general resemblance between the second order pattern and the first and second primaries when rotated in inadequate space. The overlap of these *space-deformed* factors with the true second orders is due to the fact that both explain the variance in, say, three or four factors, where perhaps fifteen may be necessary.

Even so, when closely examined (Fig. 12), the imitation, in loading pattern, of the true second orders by these pseudo-second orders is poor. This is because in one case the missing variance is a series of centroid factors, each a mixture of everything, while in the other it is the specific factor variance of the primaries, i.e. that part of the primaries which does not come into the second order common space. The hyperplanes to which one is most likely to rotate in the foreshortened, under-extracted space are those of the primary factors, which, however, are likely to be considerably blurred by being projected on the reduced space (see Diagram II, in Cattell[9]). The effect would be that of a primary confounded with and inflated by an approximate secondary.

Enough has been said, perhaps, to show that the guesswork involved in deciding how many second orders exist *before* one has taken out the primaries, and the inelegance of seeking a solution in short, deformed space, combine to make this 'pseudo-second order' approach scientifically indefensible. The cost is a wrong concept of the factors and a structure which, being composed as it were of rubble rather than fitted stone, is incapable of carrying us higher toward any dependable superstructure, e.g. of third order factors. It also prevents precise separation of such concepts as anxiety, introversion and neuroticism. Moreover, the hyperplanes are noticeably poorer than for the true primaries or the secondaries. This, and the illegitimate manner of reaching such resolutions, should suffice to warn factorists to avoid such mongrel concepts which are neither one thing nor the other.

We therefore tentatively conclude that: 1. The simple structure true second order factors on primaries is a little poorer than that of primaries upon variables; 2. the projection of the second orders, from their simple structure position found on primaries to their loadings on

FIG. 12

Effect of extracting at first order the same number of factors as exist at second order

Projection of true second order factors directly on variables (from Fig. 9)

	1″	2″
1	0	30
2	0	0
3	36	— 30
4	25	0
5	— 25	0
6	60	— 25
7	— 60	25
8	36	— 66
9	0	30
10	0	36

Correlation between factors ·60

Hyperplane count
35 per cent

Projection of pseudo-second order factors (oblimax rotation) directly on variables

	1″	2″
1	— 01	+ 51
2	— 10	+ 10
3	50	— 23
4	21	24
5	— 43	— 05
6	83	+ 08
7	— 83	— 08
8	40	— 58
9	— 01	+ 51
10	— 01	+ 59

Correlation between factors ·00

Hyperplane count
(accepting ± ·10)
40 per cent

variables, gives a distinctly poorer simple structure than either of the regular projections on an immediately adjacent stratum; 3. a better simple structure of secondaries on variables can be found, but it is still not good and the position is a poor approximation to the best second order rotation on primaries; 4. pseudo-second order factors have hyperplanes directly on variables about as good as those of true second orders. But the hyperplane count is poorer than for primaries on variables or second orders on primaries. They are neither good primaries nor good secondaries.

SUMMARY

1. Simple structure and confactor principles offer the most meaningful resolution of correlations, if factors are regarded as influences.

2. Many models are possible, both for (*a*) patterns of factor loadings on variables, and (*b*) loadings of higher order on lower order factors. The general reticular model, a network with unrestricted directions of influence, is the most generally acceptable solution to accommodate most scientific possibilities. The popular monarchic hierarchy is often a constantly recurring artefact arising from the statistical limits of any single factor analysis.

3. As there is so far no way of inferring the complex causal connections in the reticule directly from factor analytic evidence alone, discussion has been restricted to the simpler *one-way stratoplex* model.

4. The notion of strata belongs to a model, and that of orders to an operation. To find strata most efficiently, operations should begin with variables sampled evenly from a sphere of equal density. A succession of higher order factor analyses properly coordinated, suffices, however, to locate strata. Factors initially containing representatives from different levels become sorted out, and a factor lying 'between two strata' can be recognized as such.

5. Formulae can be developed to express the variance shared between two factors of different 'order' (i.e., not immediately contiguous in the reticule or strata model) and this can be viewed as the contribution of one to the variance of the other. The path coefficient is available for *general* reticular calculations *when relations become known*, and the Schmid-Leiman and Cattell-White formulae handle the problem in the one-way hierarchical or strata models. These latter are particularly concerned with finding the loadings of higher order factors directly on the variables. Assuming the one-way stratoplex model, formulae are developed here for the complete relations among (*a*) the correla-

tions between variables, (*b*) the scores on the variables, (*c*) the loadings of variables, in terms of various higher order factors, (*d*) the loadings of higher on lower order factors, (*e*) factor scores, and (*f*) the intercorrelations among factors.

6. The Cattell-White formula expresses loadings of higher order factors on variables in the general setting described, whereas the Schmid-Leiman expresses loadings for factors first set as orthogonal in a hierarchy. The two formulae have different statistical properties and are useful for different purposes.

7. Attention is called to misleading inferences and constructions in personality theory based on what have been called pseudo-second order factors. When only as many first order factors are extracted as one believes second order factors to exist, the resulting rotation commonly produces imitations of second order factors, inflated by overlap with primaries. These can be shown theoretically to be inadequate and unstable, and in practice fall short of the hyperplane count of true primaries. They are unresolved, space-distorted representations of factors.

8. The main purpose of this paper has been (i) to survey theoretical models and to present the advantages and the practical solutions for the stratoplex model, and (ii) to demonstrate that, at the practical level, the Cattell-White formula enables one to evaluate the percentage loss of criterion prediction through resorting to the economy of a battery measuring a few second order factors, instead of more first order factors. This loss can be appreciable and suggests that keeping to the first order battery is generally to be preferred.

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Notes to Fig. 8 (p. 251)

Note 1: The example used here is the same as in Fig. 9, where the single steps of obtaining higher order loadings by the Cattell-White formula are shown in detail.

Note 2: It will be seen that squaring the row values gives the h^2 (communality) directly in the Schmid-Leiman case but only after further calculation in the Cattell-White case.

Note 3: Strictly the Cattell-White resolution should offer *no* values for the third order factor because it cannot be rotated to simple structure, in the absence of enough hyperplane stuff. To show a complete series comparison, however, the third order factor is accepted as it comes from the centroid, unrotated, and with an arbitrary choice of correlations to give a single factor.