Probability of a tossed coin landing on edge

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An experiment is reported in which an object which can rest in multiple stable configurations is dropped with randomized initial conditions from a height onto a flat surface. The effect of varying the object's shape on the probability of landing in the less stable configuration is measured. A dynamical model of the experiment is introduced and solved by numerical simulations. Results of the experiments and simulations are in good agreement, confirming that the model incorporates the essential features of the dynamics of the tossing experiment. Extrapolations based on the model suggest that the probability of an American nickel landing on edge is approximately 1 in 6000 tosses.

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I. INTRODUCTION

The sensitivity to initial conditions of the motion of a rigid body bouncing dissipatively on a surface ensures that coin tosses and die throws are practically satisfactory methods of generating random numbers [1]. Although in principle the motions of coins and dice are affected by thermal and quantum-mechanical fluctuations, a purely classical and deterministic description of the motion illustrates the reason for the sensitivity to initial conditions. To achieve randomness of outcomes, however, some randomness must be assumed in the initial conditions although the sensitivity ensures that practical methods of coin tossing and die throwing will give the expected results [2–7].

For fair coins and honest dice, the probabilities of the various final-state outcomes (heads and tails in the case of a tossed coin) can be predicted from the symmetry of the problem. When the object does not have an exact symmetry, the probability of the various final outcomes is unlikely to depend on geometrical features alone and surely involves the details of the dynamics of the bouncing motion. Some qualitative observations can be made: a weighted die is more likely to land with the weighted face resting on the table; a thin coin is less likely to land on its edge than a thick one. It is not clear at the outset how other parameters of the problem should affect the results.

This paper will focus mostly on the quantitative issue of what the probability is that a tossed coin will land (not just bounce, but come to rest) on its edge. This is approached in two ways, yielding results which are in close agreement.

The first approach involves a somewhat simplified dynamical model of the coin toss. This model includes a parameter β which corresponds to the thickness of the tossed object. This model is solved through numerical integration of the equations of motion, given the initial conditions of the toss. To estimate the probability of landing on edge, separate runs are carried out for many trials with randomly chosen initial conditions.

The second approach has been to carry out experiments in which objects of varying thickness are dropped onto a table. The probability of landing on edge is estimated by the frequency of edge landings observed in a large ensemble of independent trials. The dependence of the probability of landing on edge on the thickness of the objects has been found in this way.

Despite the simple dynamics of the model, the probability of landing on edge is in striking agreement between the experiments and the numerical simulations. This suggests that among the dynamical ingredients of the model are the essential processes which govern the likelihood of landing on edge.

Solution of the model reveals an unexpected dependence of the probability of landing on edge on the rate of energy loss of the bouncing coin, quantified in the model by the coefficient of restitution parameter γ. This has implications, including possibly quite practical–if unethical—applications to the die toss in situations where the apparent symmetry of the die does not lead to the probability distribution that the player might be expecting.

II. MODEL OF A COIN TOSS

The model [8], which will now be described, is admittedly an oversimplification of the actual motion of a tossed coin, but is introduced for the purpose of identifying the essential dynamical processes which govern the probability that a tossed coin will land on edge. The essential dynamical processes are those which, if absent from the model, lead to qualitatively incorrect predictions of the probabilities. Additional refinements to the model which may lead to slight improvements in agreement with experiments are not of primary interest here. While this distinction is crude, it is offered in case the reader is surprised at the absence of consideration of many aspects of the motion of actual tossed coins.

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The object which is being tossed will be referred to as "the coin" even though it could represent a tossed coin or a rolled die. The coin is assumed to be a rigid object in the form of a right circular cylinder of height s_1 and diameter s_2 as shown in Fig. 1. The ratio of these dimensions is related to the tip-over angle β defined by

$$\beta = \arctan\left[\frac{s_1}{s_2}\right] . \tag{1}$$

The mass of the coin is m and the radius of gyration of the coin through an axis passing through the center of the coin and parallel to a diameter is k.

The surface onto which the coin is dropped will be referred to as "the floor." The floor is assumed to be rigid, level, and flat. The floor is the surface $z=0$ in Cartesian coordinates. The coin is not able to penetrate the floor. The floor is assumed to be frictionless, so that the x and y components of the coin's momentum and the z component of the coin's angular momentum are constants of the motion. The x and y parts of the motion decouple from the other phase-space coordinates and the initial x and y components of both momentum and position can, without loss of generality, be set to zero. The translational motion of the coin can be described by the height of the center of the coin $Z(t)$ as shown in Fig. 2, and the vertical component of velocity $V(t)$.

If the initial angular velocity of the coin is zero when it is dropped, and if the mass of the coin is uniformly distributed throughout its volume, then in subsequent motions the angular velocity will always lie along the same direction in the xy plane. Furthermore, this angular velocity will be a constant of the motion between instants when the coin collides with the floor. Let the direction of the angular velocity be supposed to be parallel to the x axis. In this case the rotational motion of the coin can be described by a single angular velocity ω and an angular coordinate θ, the latter being shown in Fig. 2. In practice, the assumptions of uniform mass distribution and zero initial angular velocity could be relaxed. For what follows it will be assumed that the mass distribution is such that the x axis remains a principle axis of the moment of inertia tensor. This allows the "coin" to have a hole in its center as Japanese coins do. The angular velocity could also be allowed to be nonzero at the start of the toss as long as it is parallel to the x axis.

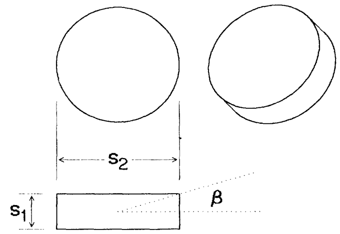

FIG. 1. The dimensions of the idealized "coin" are illustrated. The tip-over angle β is related to the diameter s_2 and the thickness s_1 through $\tan(\beta)=s_1/s_2$.

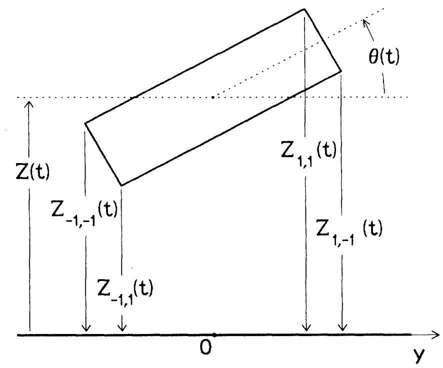

FIG. 2. The idealized coin is shown in cross section. The center of the coin lies a distance Z directly above the origin O. The orientation of the coin is specified by angle θ. The distances of the four corners above the floor are denoted by Z_{j_1,j_2}, where j_1 and j_2 take the values ± 1.

At the outset of a toss, the coin is dropped with initial phase space coordinates Z_d, V_d, θ_d, and ω_d. For simplicity, only tosses with $V_d=\omega_d=0$ have been considered. The initial orientation θ_d is selected by a random number generator in the range from 0 to π. The initial height Z_d is always chosen to be large enough that the motion of the coin is effectively randomized. The number of collisions with the floor made by the coin during the important part of the motion is never less than 10 and usually 20 or more. In addition, if the basins of attraction of the outcomes can be seen in a plot of the initial angles of a large number of tosses then Z_d is considered to be too small. The value of Z_d required depends on the coefficient of restitution parameter defined below. It is not advantageous to choose Z_d overly large because of increases in computing time.

When the coin is not in contact with the floor, the only force on it is via the gravitational acceleration g, which goes along the minus z axis. Air friction is ignored. To make the parameters of the problem dimensionless, units are chosen such that $s_1^2+s_2^2=4$, $g=1$ and $m=1$.

The mechanical energy of the coin is

$$E=Z(t)+\tfrac{1}{2}V^2(t)+\tfrac{1}{2}k^2\omega^2(t) \tag{2}$$

using the dimensionless units.

Consider a phase of the motion in which the coin does not come into contact with the floor. Let the time t be set to zero at some moment at which the phase-space coordinates of the coin are Z_0, V_0, θ_0, and ω. The angular velocity ω remains a constant of the motion and the other coordinates are given at time t by

$$Z(t)=Z_0+V_0t-\tfrac{1}{2}t^2 , \tag{3}$$

$$V(t)=V_0-t , \tag{4}$$

$$\theta(t)=\theta_0+\omega t . \tag{5}$$

The coin collides with the floor when one of the four corners, as illustrated in Fig. 2, comes into contact with the floor. The possibility of two corners being simultaneously in contact with the floor does not have zero probability as explained below. However, the present discussion assumes that only one corner of the coin contacts the floor during a collision. In addition, the temporal duration of the collision is assumed to be zero, so that the contact occurs at a single point on the floor.

The four corners of the coin are labeled by integers j_1 and j_2, which take the values ± 1. The height of corner (j_1, j_2) at time t is $Z_{j_1,j_2}(t)$, and, during a time interval in which the coin does not contact the floor is given by

$$Z_{j_1,j_2}(t) = Z_0 + V_0 t - \tfrac{1}{2}t^2 + j_1 \sin(\theta_0 + \omega t + j_2 \beta) \ . \tag{6}$$

The time at which the coin will next hit the floor is the smallest time t_c such that for some choice of j_1 and j_2, $Z_{j_1,j_2}(t_c) = 0$. The method by which t_c can be systematically and efficiently calculated has not been reported elsewhere, so it will be described here in some detail.

For any choice of j_1 and j_2, the smallest positive root of Eq. (6) is not less than the smallest positive root of the equation

$$\begin{aligned} 0 = Z_0 + V_0 t - \tfrac{1}{2}t^2 + j_1 \sin(\theta_0 + j_2 \beta) \\ + j_1 \omega \cos(\theta_0 + j_2 \beta) t - \tfrac{1}{2}\omega^2 t^2 \ , \end{aligned} \tag{7}$$

where use has been made of the inequality

$$\frac{d^2}{dx^2} \sin(ax) \le a^2 \ . \tag{8}$$

Suppose that at some point in the coin's motion, its phase space coordinates are Z_0, V_0, θ_0, and ω and we want to know the next time t_c (starting at 0) at which the coin will contact the floor. There are two cases, based on whether Z_0 is above or below 1.

Case 1. If $Z_0 > 1$ then find the (single) positive root of the equation

$$1 = Z_0 + V_0 t - \tfrac{1}{2}t^2 \ . \tag{9}$$

A positive root, denoted t_1, must always exist to this equation, so long as $Z_0 > 1$. Next, replace Z_0, V_0, θ_0, and ω with the new values of these phase-space coordinates at time t_1, making use of Eqs. (3), (4), and (5). Finally, proceed to case 2.

Case 2. If $Z_0 \le 1$ then for each j_1 and j_2 find the smallest positive root of Eq. (7). A positive root must exist for each case since the coin has not yet penetrated the floor. Let the smallest of these four roots be denoted t_b. Replace Z_0, V_0, θ_0, and ω with the new values at t_b using Eqs. (3), (4), and (5). Finally, proceed to either case 1 or case 2 based on the new value of Z_0.

This process is repeated until the additional changes in the phase-space coordinates become comparable to machine precision, which was double precision for this work. The phase-space coordinates of the coin give its position and orientation upon collision with the floor and give its velocity and angular velocity immediately before the collision. These latter two quantities will change discontinuously during the collision. There is no possibility either that the iterations will not converge or that the algorithm will miss the smallest root and converge to the wrong root.

As mentioned previously, the collision with the floor has negligible duration. The conventional assumption is that there is a coefficient of restitution parameter γ ($0 \le \gamma \le 1$). The case $\gamma = 1$ corresponds to no energy loss during the collision. Let V' and ω' refer to just before and let V'' and ω'' refer to just after the collision. Then

$$V'' = V' - (1+\gamma) k^2 \frac{(V' + y\omega')}{(k^2 + y^2)} \ , \tag{10}$$

$$\omega'' = \omega' - (1+\gamma) y \frac{(V' + y\omega')}{(k^2 + y^2)} \ , \tag{11}$$

where y is the position of the contact point, given by

$$y = j_1 \cos(\theta + j_2 \beta) \ , \tag{12}$$

using those values of j_1 and j_2 which gave the smallest root of Eq. (7).

Equivalently, one can consider the velocity of the corner of the coin which is colliding with the floor just before and just after the collision. This velocity has components U'_y, U'_z just before and U''_y, U''_z just after. It follows from Eqs. (10) and (11) that $U''_z = -\gamma U'_z$. This makes the connection with the usual interpretation of the coefficient of restitution.

The change of the coin's energy during the collision is

$$\Delta E = -\tfrac{1}{2}(1 - \gamma^2) \frac{(V' + y\omega')^2}{(1 + y^2/k^2)} \ . \tag{13}$$

When $\gamma < 1$, ΔE is negative, so the collisions are partly inelastic.

In an experimental situation, the value of γ depends on the material of the coin, the material of the floor, and the structure of the floor. In addition, dependence of the restitution on V', θ, and ω' is expected, but ignored in this simplified treatment. Reference [7] describes how γ (which they call β) can be estimated by replaying simulations of a toss in real time and coding the computer to emit a sound for each collision. Adjusting γ until the computer simulation sounds like a real toss, they took γ to be 0.5.

Since only the outcome of the toss is of interest, it is sufficient to stop the simulation once the energy E goes below 1. The coin is projected to land on edge if θ is in the range $n\pi + (\pi/2) - \beta < \theta < n\pi + (\pi/2) + \beta$ for any integer n. Otherwise, the coin is projected to land on a face—heads or tails.

The above discussion adequately describes most tosses. It sometimes happens that a toss requires an infinite number of collisions with the floor before the outcome is determined. This situation occurs only when $\gamma < 1$. This is illustrated in Fig. 3, for a coin with $\gamma = 0.8$. Because of this possibility, it is necessary to explain how this situation is handled.

The transition from bouncing to sliding motion is recognized in the computer program by the failure of the energy E to decrease more than a negligible amount over several bounces. If no provision is made in the computer program, the coin will become stuck at a collision point

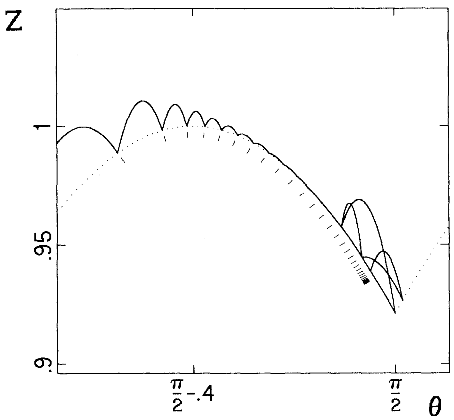

FIG. 3. The motion of the coin is shown as a solid line in a plot of Z vs θ. The parameters of the coin are $k=0.5$, $\beta=0.4$, and $\gamma=0.8$. The initial conditions of the toss are $Z_d=4$, $V_d=0$, $\theta_d=1.489$, and $\omega_d=0$. For purposes of illustration, the motion has been followed as the energy goes below 1. An example of the transition from bouncing to sliding motion can be seen as the coin settles down to a landing on its edge. The points of collisions with the floor are marked by light dashes. The accumulation point of bounces is indicated by the increasing density of dashes. The sliding motion continues until θ reaches $\pi/2$, where the motion reverts to bouncing as the energy continues to decrease.

and the energy will never go below 1.

The velocities V and ω correspond to motion in the $Z\theta$ plane which is tangential to the curve corresponding to contact of the coin with the floor. There is no sliding friction in this model, so the coin can move continuously along this curve with no loss of energy. Unlike the roller skater on a spherical roof, the normal force never goes to zero as the velocity increases, so the coin will stay in contact with the collision line, shown as a dotted line in Fig. 3. In addition, when $E>1$, the coin cannot reverse the direction of its motion along the collision line.

When the coin reaches a cusp in the collision line, which can occur at $\theta=n(\pi/2)$ for any integer n, the motion of the coin must change abruptly. The specification of the model does not unambiguously determine what will happen next. The choice made in the simulations reported here is to continue the motion as if the coin is colliding with the floor with contact only at the corner that is coming into new contact with the floor. The result is a transition back from sliding to bouncing motion, as shown in Fig. 3.

This now completes the specification of the model. Similar models have been reported elsewhere. In Ref. [3], the model has the same phase space, but stops the motion at the first collision with the floor. References [2] and [6] correspond to the present model, but with $\beta=0$. Reference [4] also has $\beta=0$, but incorporates sliding friction with the floor so that the phase space of the motion is larger. Reference [5] studies the present model with $\beta=0$ and $\gamma=1$. Reference [7] is identical to the present model for the special case $\beta=\pi/4$, except for a different continuation of sliding motion when a cusp in the collision line is reached.

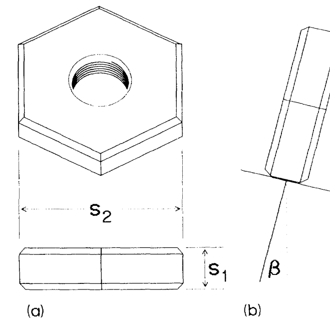

FIG. 4. (a) The points of measurement of the dimensions of the brass nuts given in Table I are illustrated. (b) The method of measuring the tip-over angle β is shown. The angle of tilt of a support is slowly increased until the object tips over.

III. EXPERIMENT

The experiment that has been carried out involves dropping machined pieces of metal of varying shapes into a hard, flat, smooth table. The table surface was a layer of smooth plastic sheet glued to a sheet of plywood. The surface was nearly level as judged from the straightness of the path of a ball bearing rolling across the surface. The available area measured 40×60 cm^2. Objects that fell off the table or struck the wall on one side were not counted. The initial height was limited since too large a height led to most of the tosses being discounted, making the experiment too tedious. In practice a height of 15 cm was used. The method for dropping the objects involved placing two or three similarly prepared nuts on a horizontal card, and manually sweeping them off with a slowly moving stick.

The objects used were brass nuts which were machined down to various thicknesses. The general shape of the objects is shown in Fig. 4(a). The grinding procedure required care, since there was a tendency for the machined sides to become out of right angles with the edges. The six sides of the nuts were not altered.

The detailed characterization of the geometrical properties of the nuts will be described after the model has been introduced, since the description of the model will make it clear which characteristics of the nuts are important to their probability of landing on edge.

The results of the experiments with the brass nuts are shown in Table I. The probability of landing on edge, P_e, is the ratio of the number of landings on edge to the number of tosses that were not discounted.

TABLE I. Results of experiments with brass nuts.

Nut label	s_2 (mm)	s_1 (mm)	Counted tosses	Landed on edge	P_e
A,B	12.55±0.02	5.76±0.02	373	44	0.118
D,E	12.52±0.01	4.63±0.05	810	51	0.063
G,H	12.55±0.02	3.51±0.04	739	17	0.023
J,K,L	12.52±0.02	2.93±0.04	1500	20	0.013

IV. DISCUSSION

The probability of landing on edge is P_e. If Z_d is sufficiently large and θ_d has a smooth random distribution [6], then P_e depends only on k, β, and γ. Only simulations with the radius of gyration parameter $k=\frac{1}{2}$ are reported here, since that corresponds to a uniform mass distribution in the case that β is small.

The numerical results of simulations to estimate P_e are shown in Fig. 5 for coins with $\gamma=0.4$, 0.6, and 0.8. By symmetry, P_e must equal $\frac{1}{2}$ when $\beta=\pi/4$. There is an obvious trend that P_e gets small as β approaches zero. Furthermore, when β is held constant, P_e is greater for small values of γ. The statistical uncertainties (one standard deviation) are never more than $\pm 11\%$, which is comparable to the symbol size in the plot. The uncertainties for larger values of β are on the order of 2%.

At this point, the model calculation will be related to the experiment that was performed. Since, as illustrated schematically in Fig. 4(a), the brass nuts do not have a simple shape, the identification of a value of β to correspond with a model calculation requires some care. The choice made here is based on thinking of β as the maximum angle by which the object can be tilted without tipping over. The method for measuring the tip-over angle is shown in Fig. 4(b). Tip-over angles for the brass nuts are shown in Table II. For the thicker nuts, this method could not be applied since the objects began to slide before they tipped over. For these cases, β is simply obtained from the dimensions of the nut, given in Table I, as measured with a micrometer across flat faces of the nut. For either method, there is an uncertainty of the value of β since the tip-over angle can depend on the initial orientation of the nut. The average of all tip-over angles is used.

For simplicity, the radius of gyration k has been set to 0.5 for all the nuts. Detailed estimates of k based on the mass distribution differ by only a few percent, and are not used here.

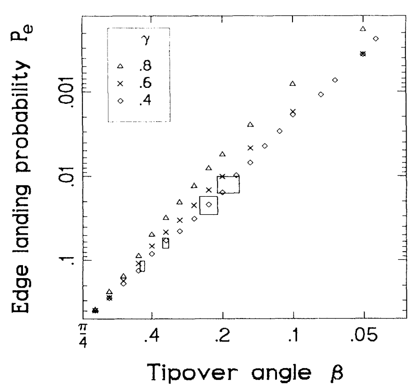

FIG. 5. The probability of landing on edge P_e is plotted vs the tip-over angle β for a range of values of the coefficient of restitution parameter γ. The value of k is $\frac{1}{2}$. Results of the experiment described in the text are plotted as open rectangles. The vertical dimension of the rectangles corresponds to one standard deviation above and below the observed edge landing frequency. The horizontal dimension of the rectangles reflects the variation in the tip-over angle among the 12 tipping modes of the brass nuts.

TABLE II. Tip-over angles for the brass nuts.

Nut label	Aspect ratio angle (rad)	Tip-over angle (rad)
A,B	0.430±0.002	slides first
D,E	0.354±0.004	slides first
G,H	0.273±0.003	0.23±0.02
J,K,L	0.230±0.003	0.19±0.02

With this identification of β, the probability of landing on edge P_e is shown for the four nut thicknesses in Fig. 5 as open rectangles. The vertical dimension of the rectangles comes from statistical uncertainty and one standard deviation above and below the mean is shown. The horizontal dimension shows the uncertainty in β.

The coefficient of restitution of the table was estimated by dropping one of the brass nuts from a height of 31 cm and observing the height of the first rebound. Rebounding nuts that were spinning rapidly were discounted. Twenty-five drops were required to obtain ten nearly vertical rebounds. A rebound height of 18 cm was typical. This corresponds to a value of $\gamma=0.76$. An alternate measurement of γ was obtained by dropping a brass washer which was constrained to drop directly on its edge. A drop of 31 cm resulted in a rebound of 18 cm, giving the same result for γ.

In Fig. 5, the experimentally obtained probabilities are all consistent with γ being 0.4. It is not clear whether the differences in the values of γ result from limitations of the measurement method or the approximations of the model.

Since the agreement between the model and experiment shows no systematic deviation as β gets smaller, it is tempting to extrapolate the curve to thicknesses of familiar coins. One example will be considered here, which is the American nickel (five American cents, 5¢). The diameter of this coin is 21.25 mm and the thickness is 1.96 mm at the rim. This gives an aspect angle of 0.092 rad. However when tip-over angles are measured, the results are 0.037 rad for tipping over with the President's head up and 0.051 rad for tipping over with the President's head down. In other words, a slight disturbance of a nickel which is set up on edge on a level table is more likely to result in the coin falling over "heads." The experiment to verify this is entertaining and easy to perform. The difference between tip-over angles between heads and tails is reproducible among several coins of this denomination. If $\beta=0.04$ is selected as a representative tip-over angle, the extrapolation of the model leads to a probability of landing on edge of 1 in 6000 tosses. This has not been tested experimentally.

As a check on whether the hexagonal shape of the brass nuts has an effect on P_e, we used a British one-pound coin (1£) for which β was found to be 0.14. In 1000 tosses of this coin, there were 6 landings on edge. Extrapolation of the model, using the same value of γ, 0.4, leads one to expect 5.5 landings on edge out of 1000 tosses. Evidently, the model also works for the one-pound coin.

The simple model of a coin toss introduced here does not take into account the full dynamics of the bouncing coin. It is not clear which of these effects are most important to the value of P_e. The floor has been assumed to be frictionless in this model. In an actual coin toss, there is a horizontal component of impulse during the collision with the floor. The relevance of the Amontons law of friction is not clear, although this is a reasonable phenomenological treatment [4]. Real "floors" are certainly not rigid and smooth and perhaps not even level. It is clear that the motion of the coin involves six degrees of freedom and not two. This does not necessarily make simulation of the motion much more difficult, but it increases the number of unknown phenomenological parameters in the model. Finally, the effect of internal vibrational degrees of freedom of the coin should not be discounted, based on the audible ringing during the toss.

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