The Correlation between Targets and Instruments[1]

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The question of what can be learned from the correlation between targets and instruments has been discussed recently by Prest [4] and Worswick [5].[2] Prest endeavoured to argue that from the correlation between public expenditure and the degree of capacity utilization something could be inferred about the effectiveness of stabilization policy. Worswick took the contrary view that little or nothing could be inferred. Earlier, the point had been made by Kareken and Solow [3], in criticism of Friedman, that a perfectly successful monetary policy would show zero correlation between monetary change and the level of economic activity. Earlier still Friedman [2] had discussed the problem of the stabilization effect of public policy and had placed the burden of his argument on the correlation between the effects of that policy and the uncontrolled level of national income. The present exposition is essentially supplementary to Worswick's approach by viewing it explicitly in a context of control, and taking note of the existence of random variables.

We use the following notation and make the following elementary assumptions: Z_t is the target variable; U_t is the instrument variable (Z_t and U_t may be interpreted as gross domestic product and government expenditure if it is so desired); e_t is a normally distributed random shock with zero mean and unit variance, that is

$$E(e_t)=0; \qquad E(e_t e_{t-\theta})=1 \text{ if } \theta=0. \quad =0 \text{ if } \theta\neq 0.$$

Z_t and U_t are measured as deviations from their means.

We assume that we are dealing with a stationary process, and that the system is represented by the following equation:

$$(1) \qquad Z_t=aZ_{t-1}+U_{t-1}+ce_{t-1}+e_t, \quad |a|<1; \quad |c|<1.$$

We have deliberately chosen a simple example to make our analysis clear to the general reader not acquainted with stochastic control theory. For the sake of expositional simplicity the coefficient of U_t is also made unity. Our conclusions are affected only in points of detail, and the general results we arrive at would not change if we considered the model in the form

[1] I am indebted to the referee and Mr. R. Allard for valuable comments which have much improved this paper.

[2] References in square brackets are listed on p. 431, below.

(1A) $\phi_1(B)Z_t = \phi_2(B)U_t + \phi_3(B)e_t,$

where ϕ_1, ϕ_2, ϕ_3 are polynomials in the backward-shift operator B.

It is also worth noting that, while we have concentrated on a stable model, one role of economic policy may be to stabilize an otherwise unstable system.

If the system is uncontrolled and U_t is constant, the variance of Z_t is given by

(2) $E(Z_t^2) = (1+2ac+c^2)/(1-a^2).$

To determine the optimal control law, consider the variance of Z_t more explicitly. From (1) we have:

(3) $E(Z_t^2) = E(aZ_{t-1} + U_{t-1} + ce_{t-1})^2 + E(e_t^2).$

There are no cross product terms between e_t and e_{t-1}, U_{t-1}, or Z_{t-1} because of our assumption concerning the independence and randomness of e_t. In determining the optimal control law for U_{t-1}, no account can be taken of e_t which occurs later and cannot be predicted. Assume, however, that Z_{t-1} is observed early enough in time to help set U_{t-1}. This again is a simplification which does not affect our conclusions.

The least variance of Z_t is achieved when the value of the first expression on the right-hand side of (3) is put equal to zero:

(4) $U_{t-1} = -aZ_{t-1} - ce_{t-1}.$

If this value is inserted into (1) we have

(5) $Z_t = e_t.$

It follows that (4) may be re-written to give us the following control law:

(6) $U_t = -(a+c)Z_t.$

The variance of Z_t will be unity and that of U_t will be $(a+c)^2$. The covariance of U_t and Z_t is given by

(7) $E(U_tZ_t) = -(a+c)E(Z_t^2) = -(a+c).$

The covariance of U_t and Z_{t+1}, the variable that U_t affects, is given by

(8) $E(U_tZ_{t+1}) = -(a+c)E(Z_tZ_{t+1}) = -(a+c)E(e_te_{t+1}) = 0.$

Thus, the optimal control law implies that the instrument and target are perfectly correlated [positively or negatively depending on the sign of $(a+c)$], and the instrument and the target one period later are not correlated at all.

Assume now that we use a control law relating U_t to Z_t but not necessarily optimally:

(9) $U_t = hZ_t.$

Inserting this in (1), we have

(10) $Z_t = (a+h)Z_{t-1} + ce_{t-1} + e_t.$

Assume the value of h is such that the system remains stable, that is $|a+h|<1$. In fact, of course, mistaken policy by choosing h large enough could make the system unstable. (Note, therefore, that there are two senses in which policy may be harmful. It may increase the variance of the target variable compared with a no-policy situation, or it may make a stable system unstable. In the literature both of these phenomena appear under the general heading of "destabilizing policy", but, technically speaking, they are far from identical.)

In this case we have

(11) $$E(Z_t^2)=[1+2(a+h)c+c^2]/[1-(a+h)^2]$$

(12) $$E(Z_tZ_{t-1})=[1+(a+h)c](a+h+c)/[1-(a+h)^2]$$

(13) $$E(U_t^2)=h^2[1+2(a+h)c+c^2]/[1-(a+h)^2]$$

(14) $$E(U_tZ_t)=h[1+2(a+h)c+c^2]/[1-(a+h)^2]$$

(15) $$E(U_tZ_{t+1})=h[1+(a+h)c](a+h+c)/[1-(a+h)^2].$$

Consider now the correlation between U_t and Z_{t+1}. Define $R(U_tZ_{t+1})$ as the correlation coefficient between U_t and Z_{t+1}:

(16) $$R(U_tZ_{t+1})=[1+(a+h)c](a+h+c)/[1+2(a+h)c+c^2].$$

We now wish to consider the behaviour of this expression. To do this it will be convenient to define $b\equiv a+h$, and to rewrite (16) as

(17) $$R(U_tZ_{t+1})=(1+bc)(b+c)/(1+2bc+c^2).$$

We know that this is equal to zero at $b=-c$. It is equal to $+1$ when $b=+1$, and -1 when $b=-1$. How does it behave as b varies?

(18) $$\frac{\partial R(U_tZ_{t+1})}{\partial b}=\frac{(1+c^2)(1+2bc+c^2)-2c^2(1-b^2)}{(1+2bc+c^2)^2}$$

$$=\frac{2c^2b^2+2c(1+c^2)b+(1+c^2)^2-2c^2}{(1+2bc+c^2)^2}$$

$$=\frac{2c^2b^2+2c(1+c^2)b+1+c^4}{(1+2bc+c^2)^2}.$$

An examination of the numerator of this expression shows that, viewed as a quadratic in b, it has no real roots. In other words, $\partial R(U_tZ_{t+1})/\partial b$ is not zero in the relevant range. Since it is a continuous function of b, it must not change its sign in this range. Since its slope is positive at $b=0$, it is positive throughout the relevant range.

Interpreting all our results so far, Eq. (2) gives us an expression for the variance of the target variable when no policy is used. Eq. (9) assumes that policy is a function of the target variable, but may be imperfect (as a result, for example, of mis-estimation of the model). Even in these circumstances it is possible that sub-optimal control will lower the variance of the target variable compared with no control.

Since we have assumed the control rule to relate the current control to the current value of the target variable, these two are perfectly correlated. The effect of the control is, however, on the value of the target variable in the subsequent period. We examine in (16), therefore, the value of this correlation.

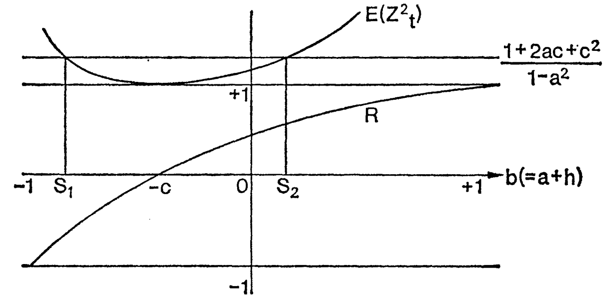

FIGURE 1

In Figure 1 we find it convenient to work in terms of $(a+h)$, which is the coefficient of Z_{t-1} in the system equation consequent on choosing a value of h as the coefficient of Z_t in the control equation. The minimum variance strategy occurs where $h=-a-c$, that is, the expression for $E(Z_t^2)$ given by (11) is at a minimum at that point. This expression is then plotted for other values of $(a+h)$ between $+1$ and -1. We also plot a line equal to the variance of Z_t in the no-control situation. The intersection of these lines gives the range, S_1S_2, in which control reduces the variance of the target variable.

Next we plot the line R which gives the correlation between U_t and Z_{t+1}. It has a value of zero at the minimum variance position, and its slope is positive there as elsewhere in the range -1 to $+1$. The diagram has four ranges: (a) -1 to S_1, where R is negative and policy is destabilizing; (b) S_1 to $-c$ where R is negative and policy is stabilizing (c) $-c$ to S_2, where R is positve and policy is stabilizing; (d) S_2 to $+1$, where R is positive and policy is destabilizing. The important conclusion follows, therefore, that the correlation between the instrument and target throws no light on the effectiveness of policy; and this also throws considerable doubt on the value of regression studies between the two variables.[1] (This is precisely the point that Worswick makes, and it is one which Caves [1] in his comment on Worswick fails to understand. The issue is not "correct" versus "perverse" signs, but that signs are no help at all.) All that can be said is that when the minimum variance strategy is used, correlation is zero between U_t and Z_{t+1}, and unity between U_t and Z_t, while the variance of U_t itself is positive. (The first

[1] In our first-order model, if the disturbances are not auto-correlated, it is, of course, true that the variance of Z_t is an increasing function of $|a+h|$, and the correlation between Z_t and Z_{t-1} is $(a+h)$. Such a simple result, however, does not follow for higher-order models or in the presence of auto-correlated disturbances.

condition, of course, will hold when U_t is constant, but the other two will not.)

It may be repeated, in conclusion, that while we have set the argument out in simple terms, the conclusion follows *a fortiori* in more complicated models.

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REFERENCES

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[5] Worswick, G. D. N., "Fiscal Policy and Stabilization in Britain", in Cairncross (ed.), *op. cit.*