RATIONAL DECISION MAKING IN PORTFOLIO MANAGEMENT

by

Henry Allen Latané

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Approved by:

Clarence E. Philbrook

Adviser

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CHAPTER I

RATIONAL DECISION MAKING IN PORTFOLIO MANAGEMENT: A BRIEF SURVEY

All investment necessarily involves the future and, therefore, uncertainty. The extent of this uncertainty is a major factor in many investment decisions including the choice among available portfolios. The theory of the firm deals with the problems faced by business men in an environment characterized by change and uncertainty. The problems of portfolio management are similar in many respects. The rational portfolio manager has goals similar to those of the rational entrepreneur and is guided by similar criteria in making decisions.

This dissertation is a study in applied decision theory. It is an attempt to analyze how a rational portfolio manager, with specified probability beliefs, should choose one portfolio to hold out of all available portfolios. It does not deal with the problem of how to allocate a portfolio on the basis of probability beliefs about returns from individual stocks and bonds. Rather it deals with the related but different problem of how to choose among portfolios on the basis of probability beliefs about returns from portfolios. It is assumed that the portfolio manager has formed probability beliefs, not necessarily about returns from individual stocks and bonds, but about returns from portfolios consisting of groups of stocks and bonds, and wishes to choose among the portfolios on the basis of these beliefs. For example, portfolio A in Table 1.1 may be allocated .5 to General Motors Common Stock, .2 to U.S. Steel, and .3 to Pacific Gas and Electric. Portfolio B may consist of the same securities but allocated in different proportions or may consist of entirely different securities including bonds. In either case, the problem is to choose between the two portfolios, not among the individual stocks and bonds making up the portfolios.

The portfolio manager who has a group of stocks and bonds[1] with a market price of $100,000 can choose to continue to hold this portfolio or to substitute any other portfolio available to him at that price. In real life this choice is influenced by such factors as income and inheritance taxes, inertia, and lack of knowledge about available portfolios. Even so, the rational portfolio manager must ask himself how he should choose among portfolios aside from these influences. The present dissertation is an attempt to answer this question; the objective is to analyze how his choice among portfolios should be made, not how it is made usually.

Rational choice among portfolios involves two steps: (a) forming

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1 For the purpose of this study it is assumed that the portfolio manager has interests exactly the same as those of the wealth-holder or portfolio owner. A portfolio consists of a combination of stock and bonds and cash which may be considered to be equivalent to high-grade, short-term, non-interest bearing bonds. Such combinations include holding stock on margin. More than 100 percent of the net value of the portfolio may be invested in stock. The funds borrowed are classed as negative bond holding. Negative stock holding consists of selling stock short and holding the proceeds in cash. In symbolic form, let b be the proportion of the net value of the portfolio held in bonds or cash and q be the proportion held in stock. Then either b or q can be positive or negative but always $b + q = 1$.

probability beliefs about returns[2] from portfolios as these returns will be affected by future occurrences; and (b) choosing among portfolios on the basis of these beliefs. The two steps may be illustrated by Table 1.1 which shows the probability of occurrence of two future events labeled "prosperity" and "depression" in the last row and the estimated effects of prosperity and depression on the returns from Portfolio A and Portfolio B is the first two rows. Each such return is here called a payout, and a table showing the payout from each available portfolio for each relevant future occurrence is here called a payout matrix[3] of returns.

Table 1.1

Payout Matrix of Returns

	Future Occurrence	
	Prosperity	Depression
Portfolio A	1.10	.90
Portfolio B	1.06	1.02
Probability of Occurrence	.8	.2

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2 Let R be the return and i be the gain (or loss) per investment period per dollar of principal. Then $R = 1 + i$. If a wealth-holder holds a portfolio which costs $1,000 at the beginning of the investment period, pays $40 in dividends and interest during the period, and is sold for $1,060 at the end of the period, his return is 1.10.

3 This is the simplest possible payout matrix. Payout matrices may well show the combined effects of more than two types of future occurrences on the one hand and more than two portfolios on the other. For example, there might be three columns headed "increasing business," "falling business," and "stable business," each with an estimated probability of occurrence. Returns from five different portfolios might then be estimated for each of these conditions.

In terms of the payout matrix the portfolio manager has two problems which correspond to the two steps above: (a) filling in the payout matrix of returns; and (b) choosing among portfolios on the basis of this filled-in matrix. In real life, the first step--deciding upon the size, measured by the number of columns and rows, of the payout matrix and filling in the matrix with reasonable estimates of payouts and probabilities--is by far the most difficult part of the portfolio manager's job. This dissertation has little to say about this problem.[4] It deals with the second of the two steps in portfolio management: the problem of how to choose among portfolios on the basis of a filled-in payout matrix. In Table 1.1, for example, this would be the problem of choosing between Portfolio A and Portfolio B.

The uncertain consequences of choices among portfolios may be expressed in terms of a payout matrix such as that of Table 1.1. Such matrices will be important tools in the analysis of rational decision making. They will be used to define and describe the combined effects of strategies (i.e., courses of action) and future occurrences--combined effects which are involved in all choices with uncertain outcomes, including gambling and portfolio management. The matrix in Table 1.2, for example, illustrates the problem of a man who has $1.00 in hand and

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4 Probability beliefs about returns from portfolios may be derived in various ways. For example, the probability distribution of returns from a specified portfolio may be built up by first making estimates of returns from the individual securities making up the portfolio, as these returns will be affected by future occurrences, and then combining these estimates. Probability beliefs about returns from a specified portfolio may also be derived by starting from estimated returns from broad groups of securities such as the Dow Jones Industrial Average.

has the option to bet or not to bet $1.00 on whether or not heads occurs on the next toss of a fair coin.

Table 1.2

Matrix of Money in Hand After One Toss of a Fair Coin

Strategy	Outcome of Toss Heads	Tails
Bet	2.00	0
No Bet	1.00	1.00
Probability of Occurrence	.5	.5

Payout matrices in general, like that of Table 1.2, show the probabilities of all relevant future occurrences and the payouts resulting from the combined effects of each possible strategy on the one hand and each relevant future occurrence on the other. All this information is needed to reach a rational decision on the proper choice of a strategy. Consideration of the entire matrix is here taken to be an essential requirement of rational decision making. It would be impossible for a gambler to make a rational choice among strategies if he disregarded either the probability of the relevant future occurrences or any of the possible payouts.

The central problem of rational portfolio management is that of making repeated choices among portfolios. The portfolio manager does not make an irrevocable decision to hold indefinitely a particular group of stocks and bonds; instead, he is able to readjust his portfolio at the end of any individual investment period. Therefore, choice of the portfolio to hold during each separate investment period[5] must be considered a separate decision.[6] The consequences of these choices are uncertain. Furthermore, the effects of his reiterated choices among portfolios must be cumulative unless all gains are withdrawn from the portfolio and all losses are replaced at the end of each year.

Choosing among portfolios on the basis of a filled-in payout matrix involves the selection of a criterion (that is, measure or standard to be maximized or minimized) to be used as a guide in making rational choices among strategies. The remainder of this chapter will be devoted to a brief survey of the hierarchy of goals underlying such a criterion. For convenience of presentation, the goals underlying a rational criterion will be defined first without reference to the reiterative character of the choices, and then the discussion will be widened to include repeated choices with cumulative effects.

The Hierarchy of Goals and Guides

Rational choice among strategies under conditions of uncertainty involves a hierarchy of goals, and of guides for reaching these goals. This hierarchy consists of: (1) a goal; (2) a subgoal; (3) a criterion for choosing among strategies to reach the subgoal (i.e., a measure which must be maximized to attain the subgoal); and, finally, (4) methods for devising strategies which maximize the criterion. Methods

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5 Individual investment periods may be days, weeks, months, or years, but are hereafter called years.

6 Compare the old Wall Street saying: "To hold a stock is to buy a stock."

of devising strategies are the subject matter of later chapters of this dissertation and will not be discussed at this time. The subgoal is a crucial member of this hierarchy. It will form the basis of later chapters entitled "Subgoals and Criteria" and "Subgoals and Subjective Utility." This together with the following section is a brief overview of goals, subgoals, and criteria.

The goal in rational decision making is the maximization of some measure of value. Each decision is made for the sake of the difference the choice will make in terms of this objective. The measure of value to be maximized, which will be referred to as the maximand, may be either a subjective utility measure such as utiles, or an objective measure such as money or bushels of wheat. The decision maker is confronted with a payout matrix expressed in terms of a maximand and wishes to select that one from among all available strategies which will enable him to reach his goal.

The goal can be reached only in the future. It cannot be used as a basis for choosing among strategies with uncertain outcomes since what strategy will lead to achievement of the goal depends on future events. For example, consider the gambler faced with the payout matrix shown in Table 1.2. This gambler has \$1 in hand and has the option to bet \$1 on the toss of a fair coin. In the event of heads he will have \$2 if he bets and \$1 if he does not bet. In the event of tails he will have 0 if he bets and \$1 if he does not bet. The mere fact that this gambler wishes to maximize his money in hand at the end of the toss (his goal) does not give him a rational basis for deciding whether to bet or not to bet.

Since the goal cannot be used as the basis for choosing among strategies with uncertain outcomes, a subgoal is necessary. The subgoal is an objective which can be reached at the time of making the choice by the decision-maker who has a filled-in payout matrix. Subgoals give bases for choosing among courses of action with uncertain outcomes. In terms of the payout matrix in Table 1.2, one subgoal is the maximization of the mathematical expectation of the probability distribution[7] of cash in hand at the end of the toss of a coin. This is called the expected-value subgoal. Another subgoal is the maximization of the cash in hand after the toss, assuming the most unfavorable outcome of the toss (called the minimax subgoal).

A subgoal is necessary whenever the outcome of the choice is uncertain, whether the maximand is expressed in terms of subjective utility or of an objective measure of value. Consider the payout matrix in Table 1.3 which is expressed in terms of subjective utility.

Table 1.3

Payout Matrix of Utility of Money in Hand at End of Toss

Strategy	Outcome of Toss Heads	Tails
Bet	3	0
No Bet	1	1
Probability of Occurrence	.5	.5

7 The probability distribution of a set of payouts is the array of all possible payouts together with their probabilities of occurrence. The mathematical expectation of the probability distribution of the set

The matrix in Table 1.3 corresponds to the matrix shown in Table 1.2 when the subjective utility of having \$2 in hand is three times as great as the subjective utility of having \$1 in hand. The goal of maximum utility at the end of the toss still is not a sufficient basis for choosing in a rational manner whether to bet or not to bet. If heads occurs, the goal will be reached by betting; if tails occurs, the goal will be reached by not betting. In this case, too, a subgoal is necessary for rational choice among strategies.

The choice of a rational subgoal is at the heart of rational decision making under conditions of uncertainty. To be rational, a subgoal must be based on consideration of the whole payout matrix and must be coupled with the goal in a logical manner. Later it will be shown that one clearly defined subgoal—the maximization of the probability, P', of having a larger payout than from any other specified strategy, hereafter called the maximum chance subgoal—is a rational subgoal when choices are repetitive and effects of these choices are cumulative. Results from adopting this subgoal will be compared with results from adopting alternative subgoals.

After a rational subgoal has been selected, it is necessary to adopt a criterion to use as a guide in choosing among strategies to reach the subgoal. Some criteria are obvious. For example, the man

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of payouts from a strategy is computed by multiplying all possible payouts from that strategy by their respective probabilities, and then summing the products. The term "arithmetic mean" of a probability distribution here has exactly the same meaning as the term "mathematical expectation" of that distribution. One term is used here to identify the criterion and the other to identify the subgoal.

who has adopted the expected-value subgoal would use the arithmetic means of the probability distributions of the payouts from the various strategies as his criterion. The man who has adopted the minimax subgoal would use the smallest payouts from the available strategies as his criterion. When maximization of P', the probability of having a bigger payout than that yielded by any other specified strategy, is taken as a subgoal, however, the standard is not so obvious. It will be proved in the next chapter that P' will be maximized when the geometric mean, G, of the probability distribution of payouts is maximized. Consequently G is the criterion for those decision makers who wish to maximize P'.

Methods for maximizing the criteria fall into the fourth order of the hierarchy of goals and guides for making rational choices. The man who has adopted the expected-value subgoal would choose that strategy which has the probability distribution of payouts with the highest arithmetic mean. The man who adopts the minimax subgoal would choose that strategy with the highest payout assuming that the most unfavorable event occurs. It is necessary to devise strategies or choose among strategies so as to maximize the standard. In terms of portfolio management, it is necessary to allocate a portfolio between groups of stocks and bonds in such a manner as to maximize the criterion. Portfolios so allocated reach the subgoal of the portfolio manager. Whether they will reach his goal will depend on future occurrences.

Goals and Guides for Repeated Choices with Cumulative Effects

Portfolio management has two characteristics which have not yet been given the emphasis they deserve. In the first place, the portfolio manager must repeat choices among portfolios year after year and, secondly, the effects of these choices are usually cumulative. For purposes of clarity in dealing with these characteristics of portfolio management it is specified that: (1) the portfolio manager is confronted year after year with the same payout matrix of returns; and (2) all returns are reinvested.[8] The purpose of this section is to survey briefly, in the light of these two specifications, the goal, subgoals, and criteria used in making rational choices among portfolios. The findings are not limited to portfolio management but are applicable to many other problems involving recurrent choices among strategies with uncertain outcomes and cumulative effects.

The goal of portfolio management is taken to be the maximization of portfolio value at the end of a period of time.[9] This period of time extends to the investment horizon of the portfolio manager and tends to remain constant from year to year. It is broken up into a large number, n, of individual investment periods called years. For example, a floor trader may think in terms of day-to-day fluctuations

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8 Both of these specifications will be relaxed and modified later. In real life, the payout matrix will presumably change from year to year, especially since probability beliefs about returns are influenced by beliefs concerning the stage of the business cycle and the general level of the stock market. This matter is discussed in the section entitled "Recurrent Risks" beginning on page 96.

9 See Friederich and Vera Lutz, The Theory of Investment of the Firm (Princeton: Princeton University Press, 1951), p. 16. The ultimate goal postulated above corresponds with the goal which the Professors Lutz assume to underlie all entrepreneurial profit maximizing behavior. They say: "We shall suppose that under all circumstances the entrepreneur will want to maximize the rate of return on his own capital over whatever period he has in view; this procedure will obviously give him the maximum capital sum at the end of the relevant period."

in prices and have an investment horizon extending over only one calendar year. In this case the calendar day would correspond to what is called a year. At the other extreme, many institutional investors, such as life insurance companies, have an investment horizon extending fifty or more years into the future. The individual investment period here may be as long as one calendar year but, in practice, portfolios are almost always evaluated and readjusted much more frequently. In either case, the investment horizon recedes as time passes, so that n tends to remain constant from year to year.[10] The portfolio manager at both the beginning of any year (year i) and at the beginning of year i + 1 will wish to maximize his portfolio at the end of, say, 100 years.

In the final analysis the rational man selects a goal because he believes that achieving this goal will maximize his subjective utility. There is no necessary conflict between maximizing subjective utility and maximizing objective portfolio value (or profit) except in the unlikely event that the decision maker prefers less wealth to more wealth, other things being equal. It will be shown later, however, that one portfolio will have the greatest probability, P', of being more valuable than any other specified portfolio at the end of n years, n being large, and that P' for that one portfolio will approach 1 as n approaches infinity.[11] Either the portfolio manager who maximizes

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10 It is assumed that n is large, whether or not it remains constant. If it does remain constant over time it is, in effect, infinitely large. In this case, the portfolio with the largest G is almost certain to produce a larger return than any other specified portfolio.

11 P' is defined as the probability of having a larger return than any other specified portfolio. The portfolio which produces the largest return over n years also is the most valuable portfolio at the end of n years. Thus P' also is the probability of being more valuable than any other specified portfolio at the end of n years.

subjective utility will select the portfolio with the maximum P', called the maximum chance portfolio, or he will select another portfolio which is almost certain to be less valuable in the long run. If the utility of a small gain or loss varies inversely with the wealth already possessed, the wealth-holder who bases his actions on subjective utility will choose the portfolio with maximum P'.[12]

The necessity for subgoals and the relation of these subgoals to the goal has been indicated. It has been noted that rational subgoals must be based on consideration of the whole payout matrix and must be logically related to the goal. There are two well known bases for choosing among portfolios (here called subgoals) which involve consideration of the whole payout matrix and which are coupled logically with the goal. These subgoals are: (1) maximization of the mathematical expectation of the probability distribution of portfolio returns expressed in money terms; and (2) maximization of the mathematical expectation of the probability distribution of the utilities of the portfolio returns. These two subgoals will hereafter be called the expected-value subgoal and the expected-utility subgoal. In this study a third subgoal is proposed—the maximization of P'. This subgoal will be called the maximum chance subgoal.

The choice of the expected-utility subgoal, that is, the choice of that portfolio which has the greatest mathematical expectation of the utilities of returns, has great intuitive appeal. For example, consider

12 The relationships presented in this brief overview are developed in Chapters II and III.

the payout matrix in Table 1.3.[13] Here the gambler will receive a payout with a utility of 3 if heads come up and a utility of 0 if tails come up, as compared with a utility of 1 if he does not bet. The mathematical expectation of the utility of the payout after betting is 1.50 (i.e., .5 x 3 + .5 x 0) as compared with 1.00 if the gambler does not bet. A gambler faced with such a matrix probably would be highly tempted to bet. However, when returns are reinvested, as is specified, the portfolio which has the greatest mathematical expectation of utilities will not be necessarily the portfolio most likely to be the most valuable at the end of n years. This relationship may be illustrated also by the payout matrix in Table 1.3 assuming that the gambler is faced with such a matrix for n consecutive tosses.[14] Such a gambler would maximize the mathematical expectation of the utility of his payout at the end of n tosses by betting all of his payouts on each toss, but he also would reduce his chances of having any payout at the end of n tosses.[15] If he bet on one toss of the coin, the mathematical expectation of the utility of his payout would be 1.50 but his chance of having any payout would be

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13 Although the example involves gambling rather than portfolio management, the principle is the same.

14 This assumption may not be realistic, as the utility of winning relative to the utility of holding cash in hand would presumably change after each toss. The example does, however, illustrate the point made here.

15 In this example the gambler can play only one game at a time. If he had the option to play many independent games at once, he could maximize the mathematical expectation of the utilities of the returns without great risk of total ruin. The problem of diversification of risks is discussed on page 47 . Risks which cannot be eliminated by diversification are of primary interest here.

only 1 out of 2. He would have a payout only if heads occurred. At the end of n tosses the mathematical expectation of the utility of his payout would be $(1.50)^n$, but his probability of having any payout at all would be only 1 in 2^n. He would have a payout only in the event of n heads and no tails in n tosses.

Thus, in this example, the portfolio with the highest mathematical expectation of utility does not have the highest probability, P', of being the most valuable at the end of n years; instead, an alternative portfolio will have the greatest P'. It will be proved that the portfolio having the probability distribution of returns with the highest geometric mean, called G,[16] also will have the greatest P' under the conditions now under discussion—that is, when n is large and when all returns are reinvested. As n becomes larger P' increases, so that when n becomes very large it becomes almost certain (i.e., P' approaches 1) that the portfolio with the highest G will be more valuable than any different portfolio. Selection of the portfolio with the maximum P' is accepted as a rational way to reach the goal of maximum portfolio value. There may be other subgoals for choosing among portfolios. These other subgoals must lead to the choice of either the portfolio with maximum P' or to different portfolios; these different portfolios will almost

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16 It should be noted that both the probability distribution of returns and G are expressed in terms of an objective measure of value. If payouts are expressed in terms of subjective utility, there is no proof that the strategy having the probability distribution of payouts with the highest geometric mean is also the strategy which has the highest P'. Later it will be shown that, if the utility of money varies in accordance with Bernoulli's utility function, maximization of G expressed in money also will maximize the mathematical expectation of utility.

certainly be less valuable at the end of a long series of years than the portfolio with maximum P'.

In summary, the following are accepted as rational goals and guides for a portfolio manager faced with repeated choices having cumulative effects:

Goal.—Maximization of portfolio value at the end of n years, n being large, assuming reinvestment of returns.

Subgoal.—Maximization of P', the probability of being more valuable than any other specified portfolio at the end of n years.

Criterion.—The geometric mean, G, of the probability distribution of portfolio returns.

Method.—Allocate that proportion of the portfolio to stock which will maximize G.

CHAPTER II

SUBGOALS AND CRITERIA

In this chapter a gambling model will be used to illustrate the necessity for a subgoal, the necessity for consideration of the whole payout matrix, and the relationship between the maximization of the mathematical expectation of value of the portfolio and the maximization of P', the probability of being more valuable than any other specified portfolio. It will be proved that the portfolio having the probability distribution of returns with the largest geometric mean, G, also has the greatest P' at the end of a long series of years (n years) assuming reinvestment of returns, and that P' approaches 1 as n approaches infinity.

Many problems involving probability, including rational decision making, can be clarified by the use of gambling situations where the odds are known. The following game was designed to be analogous to the problem of choosing among portfolios, and the payouts in the game were chosen to illustrate various subgoals and criteria used as guides in such choices.

Gambling Model

Let a gambler be given an opportunity to buy tickets which will cost $1.00 each and which he believes will surely pay off as shown in Table 2.1. All wealth must be bet on one color on every one of a large number, n, of tosses of a coin. The gambler wants to choose that color which will maximize his wealth at the end of the game (i.e., at the end of n tosses).

Table 2.1

Payout Matrix of Returns

Strategy	Outcome of Toss Heads	Tails	Criteria A	G
Red	2.50	0	1.25	0
Blue	2.25	.50	1.37	1.06
Green	1.75	.75	1.25	1.145
Black	1.02	1.01	1.015	1.014
No Bet	1.00	1.00	1.00	1.00
Probability of Occurrence	.5	.5		

The first column of Table 2.1 shows the returns for each color in the event of heads and the probability of heads occurring. The second column shows the returns in the event of tails and the probability of tails occurring. A is the arithmetic mean or, in other words, the mathematical expectation of the probability distribution of returns. For example, $A_{red} = .5 \times 2.50 + .5 \times 0 = 1.25$ where $.5 \times 2.50$ is the probability of heads occurring multiplied by the return if heads occur, and the second term is the corresponding figure if tails occur. In similar fashion G is the geometric mean of the probability distribution of returns. $G_{red} = 2.50^{.5} \times 0^{.5} = 0$ and $G_{blue} = 2.25^{.5} \times .50^{.5} = 1.06$.

The terms of Table 2.1 can be adapted to fit not only the general problem of choosing among courses of action but also the particular problem of portfolio management. The gambler is faced with the choice among five strategies (i.e., possible courses of action): he can bet on one of the four colors and he can refuse to bet. The portfolio manager who has a portfolio with a market price of $1,000 is faced with the choice among all portfolios (i.e., groups of stocks and bonds, and cash) available to him at that price. There are two relevant outcomes on each toss of a coin: heads and tails. These outcomes correspond to relevant future occurrences in portfolio management. For example, portfolio A, in Table 1.1, gives a return of 1.10 if business is prosperous in the forthcoming year (i.e., investment period) and a return of only .90 if business is depressed. In these terms "prosperity" and "depressions" are relevant future occurrences. If these are the only relevant future occurrences, there will be only two payouts for each portfolio, but often more than two must be considered. For example, the matrix may contain a column of payouts for the possible occurrence of depression in the steel industry concurrent with prosperity in textiles.

It is specified that the gambler believes that the coin is fair. Consequently, he believes that there is a probability of occurrence of .5 for heads, of .5 for tails, and of 1.0 for either heads or tails. If h be the number of heads which may occur in n trials, the gambler believes that .5 is the most likely value of h/n when n is an even number, and that h/n will approach .5 as n increases.[1] In like fashion, the

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1 This is not to say that he believes that the absolute difference between the most likely value of h, that is, $n/2$, and the actual value

portfolio manager may believe that there is a probability of .8 that business will be prosperous, of .2 that it will be depressed, and of 1 that either one or the other condition will prevail. Whether two or more relevant future occurrences are included in the matrix, the sum of the probabilities must add to 1. In other words, the matrix must contain the payouts for all of the relevant future occurrences. The portfolio manager may hold the same probability beliefs as to each of a long series of forthcoming years. If so, the asymptotic properties of the probabilities of business conditions are similar to those of the probabilities in coin tossing. If b is the number of years of good business in n years, the portfolio manager believes that the most likely value of b/n is .8, and that b/n will approach .8 as n increases.

The matrix (in Table 2.1) showing the payouts from each strategy for each future occurrence is expressed in terms of returns which are defined as payouts per dollar bet (i.e., per dollar committed to a strategy) per toss of the coin. For example, the return is 1.02 if black is selected and heads come up. This represents the principal (1.00) plus the gain (.02) and is equivalent to 1 plus the yield. In similar fashion, the return from a portfolio consisting entirely of high grade bonds bought to yield 2 percent and maturing at the end of the year would be 1.02. A gambler can lose all of the money he has bet, and the portfolio manager can lose his entire portfolio but never more than this amount; consequently, the return is always equal to or greater than 0.

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of h will tend to become smaller and smaller as n increases. On the contrary, the absolute difference, n/2 - h, tends to become larger and larger as n increases.

It is specified that the gambler must bet all of his wealth on one color on each toss of the coin. This specification is included in the model in order to make the choices among colors analogous to the choice among portfolios and the returns from a color analogous to the returns from a portfolio. For example, the portfolio manager may be faced with the choice between a portfolio consisting entirely of speculative stocks and a more conservative portfolio consisting of part stocks and part bonds. In similar fashion the gambler is faced with the choice of blue tickets or the more conservative green tickets.

The problem of portfolio management may be stated again in terms of the payout matrix. It is the problem of the decision maker who is faced with a payout matrix for n years and wants to choose in a rational manner one from all available portfolios in each of the n years. Construction of a payout matrix giving the outcomes of the strategies as affected by the relevant future occurrences along with the probability of each future occurrence is implicit in all rational decision making, including rational portfolio management. However, it is not the construction of such matrices but the choice of one from all of the strategies after the matrix has been constructed which is the problem under discussion.

The goal of the gambler faced with the choice among colors is to maximize his wealth at the end of n tosses of a coin assuming that he bets all of his wealth on each toss. The goal of the portfolio manager is assumed to be maximization of wealth at the end of n years, n being large, assuming reinvestment of all returns. When there is no uncertainty, the goal itself is a sufficient guide in choosing among courses of action.

In the case of the gambler confronted with payout matrix 2.1, the goal itself would form the basis for deciding rationally whether to bet or not to bet. The returns from black tickets are greater than 1.00 whether heads or tails occur, so the gambler can certainly gain by betting. When there is certainty of what will happen next, the gambler merely chooses the color which will maximize his payout. If he is certain that heads are going to come up next, he will bet on red, the color with the largest payout when heads occur. He will ignore the consequences of tails occurring. If he is certain that tails are going to come up next, he will bet on black, which has the highest payout when tails occur.

The Subgoal

When the decision maker cannot identify the strategy which will enable him to achieve his goal, a subgoal is needed. The decision maker who adopts a subgoal does not forego his goal. He merely chooses the subgoal as the best available landmark on the road to the goal. They are landmarks which can surely be reached by the decision maker who is confronted with a filled-in matrix such as that in Table 2.1, which shows the probability of each relevant future occurrence and all combined effects of strategies and future occurrences. For example, the gambler cannot choose the particular color, or series of colors, which will certainly maximize his wealth at the end of 100 tosses of the coin. The series of payouts depends not only on color but also on the outcome of events about which he has only probability beliefs. The gambler can, however, choose that color which would produce the highest mathematical expectation of value at the end of 100 tosses, and he might make it his subgoal to do so.

A rational decision maker must adopt a subgoal which is: (a) based on a balanced consideration of the payout matrix; and (b) logically coupled with the goal. No attempt will be made to define "logically coupled" in rigorous terms. Instead, two subgoals will be presented which are logically coupled with the goal, and conditions will be stated under which one of these two subgoals might be preferred to the other.[2] It is not assumed that there can be no other subgoals logically coupled with the goal. Subgoals fall into two classes depending on whether or not they involve balanced consideration of the whole payout matrix. The first class consists of subgoals which arise from a biased evaluation of the true probabilities and are therefore irrational. It includes the minimax and maximax subgoals to be described in the next paragraph. The second class consists of subgoals which give due weight to the true probabilities and are not necessarily irrational--nor are they necessarily rational. This class includes all subgoals based on measures of central tendency, on dispersion, and on higher moments of the distributions of payouts from the various strategies.

When a decision maker attempts only to minimize his losses and gives no weight to possible favorable occurrences, he is said to have a minimax[3] subgoal. A gambler adopting the minimax subgoal would examine

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2 A third logically coupled subgoal, the expected-utility subgoal, will be discussed in the next chapter.

3 The minimax subgoal is so named because the decision maker who adopts this subgoal attempts to minimize the maximum possible losses.

the payouts in the tails column in Table 2.1, and only those in the tails column. In other words, he would use the payouts in the tails column as his standard for choosing among strategies. He would then choose that color (black) which would give him the greatest return if the unfavorable event (i.e., tails) occurs. The minimax subgoal is of special interest in game theory. In game theory, it is assumed that the gambler is playing against an opponent who can choose among opposing strategies (i.e., future occurrences) in such manner as to do the gambler as much damage as possible. In Table 2.1 these opposing strategies are the occurrence of heads and tails. If the gambler were convinced that he was playing against an opponent who wanted to win from him and who could control the outcome of each toss of the coin, he would be well advised to look for the worst and to guide himself accordingly. Under these circumstances it would be rational to adopt the minimax subgoal. But the minimax subgoal is irrational in the gambling model here under discussion and in portfolio management. There is no opponent who controls the relevant future occurrences; but rather the probability of each occurrence is known. It is not rational under these conditions to disregard the possibility of favorable payouts in making choices among strategies.

The subgoal of the gambler who attempts to maximize his winnings if the most favorable combination of events occurs is called the maximax subgoal. This subgoal may be adopted by the gambler who believes that luck is on his side and wants to take full advantage of his luck. He considers only the most favorable payouts (the heads column in Table 2.1) and chooses that strategy (red) which gives him the maximum return when the most favorable event (heads) occurs. This choice, like the minimax choice, obviously does not give balanced consideration to the probabilities of the relevant occurrences and is, therefore, irrational.

In the gambling model it was specified that the probability of heads is .5. In other words, the probabilities are independent and future tosses are not affected by past occurrences. Under these circumstances, it would be irrational for the gambler to take as his subgoal the choice of that strategy which might seem to have the greatest possibility of a favorable payoff judged by the past pattern of tosses. It would be irrational for such a gambler to attempt to improve the odds in his favor by adopting such a strategy as "pick red after tails have come up five times in a row." In similar fashion, it is specified that the portfolio manager is dealing with the problem of repeated choice among strategies when faced with the same payout matrix time after time. In real life, past performance undoubtedly has a marked influence on constructing the payout matrix,[4] but given the matrix, it has no bearing on choices among portfolios.

Subgoals which are based on consideration of the whole payout matrix are not biased but they are not necessarily rational. A strategy with a probability distribution of payouts which has a small variance usually is preferred to one which has a large variance. This wish to avoid uncertainty about returns cannot be described as irrational, but it is not logically coupled with the goal. Minimizing variance must be rejected as a rational subgoal because it often leads to strategies which

4 This problem is discussed further in Chapter V.

cannot possibly reach the goal. In the example shown in Table 2.1 the gambler who wished to minimize variance would not bet on black even though it pays 1.02 if heads occur and 1.01 if tails come up. The rational gambler, on the contrary, would clearly prefer black to not betting even though the distribution of returns from black has more variance than the distribution of returns from not betting.

Two other unbiased subgoals already have been identified: the expected-value subgoal and the maximum chance subgoal. In terms of the gambling model, the first of these subgoals is the choice of that color which maximizes the mathematical expectation of returns at the end of the game. This color would be blue, which has the probability distribution of payouts with the largest arithmetic mean ($A_{blue} = 1.37$). The arithmetic mean of the probability distribution of payouts is the criterion when the expected-value subgoal is adopted. This criterion is maximized when the color blue is chosen. The mathematically expected return for the gambler who repeatedly bet all of his wealth on blue would be 1.37 at the end of one toss and $(1.37)^n$ at the end of n tosses. The latter return is the highest possible mathematical expectation of returns at the end of n tosses. Any single bet on any other color during the whole series of n tosses would reduce it.

The Maximum Chance Subgoal

The second unbiased subgoal, already identified, is the choice of that color which maximizes the probability, P', of having a higher payout than from any other specified color at the end of n tosses, n being large.

On an ex post facto basis, each of the colors included in Table 2.1 would prove to be the best color to have selected for some combination of heads and tails. For example, the color red, which has the highest payout (2.50) when heads occur, would be the best color to choose if the next n tosses were all heads and no tails occurred. The probability of occurrence of this combination of heads and tails can be calculated exactly by using the binomial expansion. It becomes smaller and smaller as n increases. When there is only one toss the probability of occurrence of all heads and no tails is .5. Under these conditions (i.e., when $n = 1$), $P'_{red} = .5$. When $n = 2$ the probability of occurrence of all heads and no tails is .25 so $P'_{red} = .25$. When $n = 100$, P'_{red} is 1 in 2^{100}. The P' for each color for any n can be calculated in similar fashion.

The probability, P', of having a higher payout than any other specified color at the end of n tosses, assuming that all returns are bet on every toss, depends not only on the payouts from the various colors but also on n. This is shown in Table 2.2 which shows the payout matrix of returns at the end of n tosses with $n = 1, 2, 3,$ and 4.

The first four rows of Table 2.2 correspond to the payout matrix in Table 2.1 with the rows and columns transposed. The fifth row shows the proportion of the possible occurrences (i.e., the probability, P') in which each color gives a larger payout than any other color when $n = 1$. The red tickets give a larger return than any other tickets when heads occur and the black tickets give a larger return when tails occur. Consequently, $P'_{red} = .5$ and $P'_{black} = .5$ when $n = 1$. In no outcome does blue or green give a greater payout than any other color; so P' for each of these is zero.

Table 2.2

Payout Matrix of Returns at End of n Tosses

Outcome of Toss	Probability of Occurrence	Red	Blue	Green	Black
		Color of Ticket			
		$n = 1$			
h_0t_1	.5	0	.50	.75	1.01
h_1t_0	.5	2.50	2.25	1.75	1.02
Math. Expectation (A)		1.25	1.37	1.25	1.015
Geometric Mean (G)		0	1.06	1.145	1.014
p'		.5	0	0	.5
		$n = 2$			
h_0t_2	.25	0	.25	.56	1.02
h_1t_1	.50	0	1.12	1.31	1.03
h_2t_0	.25	6.25	5.06	3.06	1.04
Math. Expectation (A^2)		1.56	1.89	1.56	1.03
Geometric Mean (G^2)		0	1.12	1.31	1.03
p'		.25	0	.50	.25
		$n = 3$			
h_0t_3	.12	0	.12	.42	1.03
h_1t_2	.38	0	.56	.98	1.04
h_2t_1	.38	0	2.53	2.30	1.05
h_3t_0	.12	15.65	11.40	5.35	1.06
Math. Expectation (A^3)		1.95	2.60	1.95	1.05
Geometric Mean (G^3)		0	1.19	1.50	1.05
P'		.12	.38	0	.50
		$n = 4$			
h_0t_4	.06	0	.06	.32	1.04
h_1t_3	.25	0	.28	.74	1.05
h_2t_2	.38	0	1.27	1.72	1.06

(Table 2.2--Payout Matrix of Returns at End of n Tosses--continued)

Outcome of Toss	Probability of Occurrence	Red	Blue	Green	Black
		($n = 4$)			
h_3t_1	.25	0	5.69	4.03	1.07
h_4t_0	.06	39.10	25.65	9.35	1.08
Math. Expectation (A^4)		2.44	3.58	2.44	1.06
Geometric Mean (G^4)		0	1.27	1.73	1.06
P'		.06	.25	.38	.31

The possible outcomes of the tosses when $n = 2$ are two tails (h_0t_2), tail-head, head-tail, and two heads. The table shows all possible payouts when $n = 2$ when all returns are bet on each toss. Red gives a larger payout than any other color when two heads occur (i.e., $2.50 \times 2.50 = 6.25$). The probability of this is .25. Consequently $P'_{red} = .25$ when $n = 2$. Correspondingly, green gives a larger return than any other color when tail-head or head-tail occurs and thus has a P' of .50 when $n = 2$.

In the illustrative game the gambler wishes to maximize his wealth at the end of n tosses of a coin, n being large. The possible future occurrences when $n = 100$ consist of all 101 possible combinations of heads and tails in 100 tosses. The probability of each occurrence and the payouts in the event that any one of the six colors is chosen is stated in the form of a payout matrix of returns after 100 tosses in Table 2.3.

Table 2.3

Payout Matrix of Returns After 100 Tosses

Outcome of Toss	Probability of Occurrence	Red	Blue	Green	Black
h_0t_{100}	$(1/2)^{100}$	0	$.50^{100}$	$.75^{100}$	1.01^{100}
h_1t_{99}	$100 \times (1/2)^{100}$	0	$2.25 \times .50^{99}$	$1.75 \times .75^{99}$	1.02×1.01^{99}
⋮	⋮	⋮	⋮	⋮	⋮
$h_{100}t_0$	$(1/2)^{100}$	2.50^{100}	2.25^{100}	1.75^{100}	1.02^{100}
Mathematical Expectation	(A^{100})	1.25^{100}	1.37^{100}	1.25^{100}	1.015^{100}
Geometric Mean (G^{100})		0	1.06^{100}	1.145^{100}	1.014^{100}
P'		$(1/2)^{100}$	.010	.988	.002

In Table 2.3, h_i, with $i = 0, 1, \ldots, n$, represents the number of heads which may occur in n tosses of a coin, and t_i represents the number of tails. The 101 possible combinations range from h_0t_{100} to $h_{100}t_0$ when $n = 100$. The probability of each of these occurrences may be calculated from the binomial expansion by computing $\binom{n}{i}(h/n)^{100}$. These probabilities are shown in the first column of Table 2.3. The first row of Table 2.3 shows the probability (first column) of 100 tails and no heads in 100 tosses and the payouts from each strategy if this combination of heads and tails occurs. In Table 2.3, A represents the arithmetic mean of the probability distribution of returns after 1 toss and A^{100} is the arithmetic mean (mathematical expectation) of the distribution of returns after 100 tosses. In similar fashion, G represents the geometric mean of the probability distribution of returns after 1 toss and G^{100} is the geometric mean of the distribution after 100 tosses.

The payout matrix of returns for each of the 101 possible occurrences for each of the five strategies (i.e., Table 2.3) includes a column showing the distribution of all possible payouts from holding blue tickets on each of the 100 tosses. Letting h be the number of heads which may occur in n tosses, the distribution of payouts from the blue tickets is $2.25^h \times .50^{n-h}$, since blue pays 2.25 when heads occur and .50 when tails occur and all returns are reinvested. The corresponding distribution of payouts from the green tickets is $1.75^h \times .75^{n-h}$. The returns from the blue tickets are equal to or greater than the returns from the green tickets when

$$2.25^h \times .50^{n-h} \geq 1.75^h \times .75^{n-h} \tag{2.1}$$

or, expressed in logs, when

$$h(\log 2.25) + (n-h)(\log .50) \geq h(\log 1.75) + (n-h)(\log .75)$$

that is, when

$$h/n \geq (\log 2.25 + \log .75 - \log .50 - \log 1.75)/(\log .75 - \log .50) \geq .615\,. \tag{2.2}$$

Equation (2.2) indicates that when 62 or more heads out of 100 tosses occur, the blue ticket gives a greater return than the green ticket. The probability of 62 or more heads out of 100 tosses of a fair coin is only .01049 (as obtained from binomial tables[5]), so green produces greater

5 See, for example, Computation Laboratory of Harvard University, Tables of the Cumulative Binomial Probability Distribution (Cambridge, Mass.: Harvard University Press, 1955).

returns than blue in nearly 99 percent of the possible combinations when $n = 100$. When $n = 1000$ blue is the better choice in much less than 1 out of a million possible combinations. The return from this 1 in a million outcome is so large, however, that the mathematical expectation of the distribution of returns from blue is greater than the mathematical expectation of the returns from green.

Similarly, the returns from the black tickets, which return 1.01 when tails occur and 1.02 when heads occur, are equal to or greater than the returns from the green tickets, which return only .75 when tails occur but 1.75 when heads occur, when $h/n \leq .355$. When 35 or fewer heads come up in 100 tosses the black tickets give a larger return than the green tickets. The probability of this is .002. The green tickets give a larger return in 998 out of 1000 possible combinations of 100 heads and tails. The red tickets, which return 0 when tails occur and 2.50 when heads occur, are the best tickets only when heads come up on every toss. The probability of this is 1 in 2^{100} but the theoretical return is so great, if this extremely unlikely event occurs, that the mathematical expectation of returns from the red tickets is equal to the mathematical expectation of the green tickets.

To summarize, when $n = 100$, betting on green gives a larger return than betting on any other color (or not betting at all) when not less than 36 and not more than 61 heads occur out of 100 tosses. The probability of heads occurring in this range of frequencies is approximately .988. Thus betting on green gives a higher return than betting on any other specified color in well over 98 percent of the possible future occurrences.

When $n = 1{,}000$, the chance that one of the other colors will produce a larger return is on the order of 1 in a million.

In the particular example shown in Table 2.1, the color with the largest G (green) has the largest P' when n is large and P' approaches 1 as n approaches infinity. As will be shown below, this relationship between G and P' holds for all cases where the probability of occurrence and the size of the returns after n trials can be calculated by using the binomial expansion.

The color green is selected because it has the highest G. It gives the highest possible return when $h/n = .5$. Suppose a gambler is given the option to choose another color, say pink, which gives an equal or greater return than the return from green when

$$h/n \geq .5 + z \tag{2.3}$$

with $0 < z \leq .5$, and not otherwise. As n increases indefinitely it becomes more and more unlikely that a combination of heads and tails will occur such that $h/n \geq .5 + z$. So pink will be less and less likely to produce a higher return than green. Further, no matter how small a z is specified and how large a P' is specified, with $P' < 1$, it is always possible to choose an n large enough so that the probability of occurrence of $h/n \geq .5 + z$ is less than $(1 - P')$. Therefore, in the long run, that color (green) wheich has the probability distribution of returns with the highest G will have a greater probability (P') of giving a higher return than any other specified color (pink), even though pink gives a higher return when over half of the tosses are heads. Further, P' will approach 1 as n becomes very large. Analogous statements can be made when pink gives a greater return than green when less than half of the tosses are heads. These statements can be generalized to cover any binomial distribution by substituting p_p for .5 in equation (2.3) where p_p is the postulated probability of occurrence.

Algebraic Statement of Rational Portfolio Management

In previous sections of this chapter the problems of portfolio management have been discussed in terms of choices among portfolios when there were only two relevant occurrences in each year. When there are only two relevant occurrences in each year, the probability of occurrence and the size of the returns after n years (i.e., the payout matrix after n years) can be calculated by using the binomial expansion. This and the following section are designed to generalize the problem to include more than two relevant occurrences in each year. The problem of choice among portfolios may be stated in terms of the payout matrix in Table 2.4. It is the problem of the decision maker who is faced with such

Table 2.4

Payout Matrix of Portfolio Returns

Portfolio	Relevant Future Occurrences $1, \ldots, j, \ldots, k$	Criteria A	G
1	$a_{11}, \ldots, a_{1j}, \ldots, a_{1k}$	$\sum_{j=1}^{k} p_j a_{1j}$	G_1
⋮	⋮ ⋮ ⋮	⋮	⋮
i	$a_{i1}, \ldots, a_{ij}, \ldots, a_{ik}$	$\Sigma\ p_j a_{ij}$	G_i
⋮	⋮ ⋮ ⋮	⋮	⋮
t	$a_{t1}, \ldots, a_{tj}, \ldots, a_{tk}$	$\Sigma\ p_j a_{tj}$	G_t
Probability of Occurrence	$p_1, \ldots, p_j, \ldots, p_k$		

a payout matrix for n years and wants to choose in a rational manner one portfolio from all available portfolios in each of the n years.

In Table 2.4, p_j represents the probability of the j^{th} occurrence, with $\Sigma p_j = 1$, and a_{ij} represents the return from the i^{th} portfolio, with $i = 1, \ldots, t$, if the j^{th} occurrence takes place, with $j = 1, \ldots, k$. A return is the payout per dollar of portfolio value per investment period (year). Returns cannot be negative, so $a_{ij} \geq 0$. A_i is the mathematical expectation (i.e., arithmetic mean) of the probability distribution of returns from the i^{th} portfolio, so $A_i = \sum_{j=1}^{k} p_j a_{ij}$. G_i is the geometric mean of the probability distribution of returns from the i^{th} portfolio. G_i also is the antilog of the mathematical expectation of the probability distribution of the logs of returns from the i^{th} portfolio so $G_i = \text{antilog}(\Sigma p_j \log a_{ij})$.

It is assumed that the portfolio manager is faced with a payout matrix such as Table 2.4 for n years. The portfolio manager who wishes to maximize his wealth at the end of n years would wish to maximize the product of the n individual returns.[6] This product is maximized if the return in each individual year is maximized. As long as the investment horizon and the payout matrix remain unchanged, proper maximizing action in one year would also be proper in the next year. If portfolio i, for example, is the rational choice in year j it also is the rational choice in year j + 1. Consequently, rational portfolio management involves

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6 He also would wish to maximize the geometric mean return over n years. This is the nth root of the product of the individual returns.

selecting one portfolio and holding it as long as probability beliefs about returns remain unchanged.[7]

The portfolio manager who wishes to choose a portfolio to maximize the geometric mean return over n years is faced with a payout matrix of returns derived from that shown in Table 2.4. This matrix (Table 2.5) is expressed in terms of possible combinations of occurrences and geometric mean returns for n years.

Table 2.5

Payout Matrix of Geometric Mean Returns for n Years

Portfolio	Combinations of Occurrences $1, \ldots, k, \ldots, r$	Criteria A	Criteria G
1	$g_{11}, \ldots, g_{1k}, \ldots, g_{1r}$	A_1	G_1
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots$	$\vdots$
i	$g_{i1}, \ldots, g_{ik}, \ldots, g_{ir}$	A_i	G_i
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots$	$\vdots$
t	$g_{t1}, \ldots, g_{tk}, \ldots, g_{tr}$	A_s	G_s
Probability of Occurrence	$p_1 \quad \ldots \quad p_k \quad \ldots \quad p_r$		

In Table 2.5, p_k is the probability of the k^{th} combination of occurrences and the g_{ik} is the geometric mean return over n years if the

7 This does not mean that the individual portfolios must contain identical securities from year to year. It means only that the payouts from portfolio i, for example, in year (j + 1) have the same probability distribution as in the year j. The problem of proportionate allocation of the portfolio in order to maintain continuity in the payout matrix for the special case where the portfolio is divided between one risk asset and one safe asset is discussed on page 99.

i^{th} portfolio is chosen in each of the n years and the k^{th} combination of occurrences takes place. The combinations of occurrences in this matrix represent all possible combinations of the occurrences reported in Table 2.4 taken n at a time. It is directly comparable to Table 2.3 except that returns after 100 years in the former table are expressed in terms of products rather than as geometric means. The value of these g's and the likelihood of their occurrence (i.e., the p's) may be calculated directly from the matrix in Table 2.4. The A's and G's in Table 2.5 equal the corresponding figures in Table 2.4—that is, the arithmetic and geometric means of the probability distributions of the g's equal the arithmetic and geometric means of the corresponding distributions of a's.

The portfolio manager can reach his goal if he can select a portfolio which will yield the maximum return in each year. When the relevant future is known so that, for example, $p_j = 1$ in Table 2.4, the choice is simple—the portfolio manager merely chooses that portfolio, say the i^{th} portfolio, which maximizes a_{ij}. When no occurrence is certain and no portfolio is superior to all other portfolios for every possible future occurrence, however, it is impossible to pick a portfolio which will maximize with certainty the payout either for one year or for n years. The portfolio manager has to choose a portfolio yielding a distribution of payouts based on probability beliefs about the possible combinations of occurrences.

In order to make a rational choice among portfolios, it is necessary to consider the whole payout matrix. In this consideration, the probability that any one portfolio, say the i^{th} portfolio in Table 2.5, will give the best possible return is important. When g_{ik} is the highest return in the column representing all possible returns when the k^{th} combination of occurrences takes place, portfolio i gives the highest possible return in at least p_k proportion of the possible combinations of future occurrences. The sum of all the probabilities of the combinations of occurrences in which portfolio i gives the highest geometric mean return for n years is called P'_{in}. If upon examination of the whole payout matrix it is found that $P'_i = 1$, it is clear that portfolio i should be chosen. Choice of that portfolio with the maximum P' is taken as a rational subgoal in choosing among portfolios.

Proof of the Maximum Chance Theorem

It is a fundamental theorem of this dissertation that, when n is large and returns are reinvested, the portfolio having the probability distribution of returns with the highest geometric mean, G, also has the greatest probability, P', of producing a higher return than any other specified portfolio if n is sufficiently large; and P' approaches 1 as n approaches infinity. For this reason, G is here accepted as a rational criterion for choosing among portfolios.

Let P'_{in}, with $i = 1, \ldots, t$, be the proportion of the possible combinations of occurrences in which portfolio i produces a bigger return than any other available portfolio when exposed to the same risks for n years, and let G_i be the geometric mean of the probability distribution of returns. Then the maximum chance theorem states that when portfolio m is the portfolio with the geometric mean return G_{max}, and portfolio i is any other portfolio, so that $G_{max} > G_i$ then

$$P'_{mn} > P'_{in}$$

when n is sufficiently large—and P'_{mn} approaches 1 as a limit, while P'_{in} approaches 0 as a limit, as n approaches infinity. In other words, portfolio m will almost certainly produce a higher return than any other specified portfolio in the long run.

In order to give general proof it is necessary to cover the following cases: (a) $G_{max} = 0$; (b) $G_{max} > G_i = 0$; and (c) $G_{max} > G_i > 0$.

Case (a): $G_{max} = 0$.—If $G_{max} = 0$, it means that for each available portfolio there is some combination of occurrences which will result in a zero return, so that the portfolio becomes worthless if this combination occurs. In other words, if $G_{max} = 0$, there is some chance of ruin in whatever course of action may be adopted. Specifically, it must mean that the wealth-holder does not have the option of holding a proportion of his portfolio in cash or other safe asset. Under these unrealistic[8] conditions, there can be no $G_{max} > G_i$ and consequently the maximum chance subgoal does not apply.

Case (b): $G_{max} > G_i = 0$.—The portfolio with a positive G_{max} clearly will dominate the portfolio in which $G_i = 0$ in a larger and larger

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8 It is unrealistic to assume that $G_{max} = 0$ when the portfolio manager has the option of holding part or all of his portfolio in cash or other safe asset. The minimum G_{max} for such a portfolio is 1.0—which can be obtained by holding all cash.

proportion of the time as n increases. Assume that the return from the i^{th} portfolio equals 0 when the j^{th} combination occurs; that the probability of this is p_j when $n = 1$; and that the return from portfolio m is greater than the return from portfolio i only when the j^{th} combination occurs. The probability that the j^{th} combination will not occur in the first year is $(1 - p_j)$ and the probability that the j^{th} combination will not occur in n years is $(1 - p_j)^n$. This probability becomes smaller and smaller as n increases, so that the probability of portfolio i giving a return of 0 becomes greater and greater and approaches 1 as n becomes very large. When portfolio i gives a 0 return, portfolio m gives a greater return. Therefore, it becomes more and more likely that portfolio m will give a higher return than portfolio i and P'_{mn} approaches 1 as a limit as n increases without bound.

Case (c): $G_{max} > G_i > 0$.—Proof that $P'_{mn} > P'_{in}$ when n is large and approaches 1 as a limit when n approaches infinity depends on the fact that the arithmetic mean of a random sample of n items from a population with finite variance tends to approach the population mean as n becomes large. It can be proved that the probability, α, that samples of n items from a population with finite variance will have a sample mean differing from the population by more than a specified amount, z, depends on n and can be made smaller than any specified number by choosing a sufficiently large n. Chebyshev's inequality forms the basis for the above statement.[9]

To use Chebyshev's inequality it is necessary to convert the probability distributions of returns in the rows of Tables 2.4 and 2.5 to distributions of logs. All portfolios with a geometric mean of 0 are excluded (see Case (b)), so all distributions under consideration have a finite variance. In Table 2.5, g_{ik}, with $k = 1, \ldots, r$, is the geometric mean of a random sample of n returns from the probability distribution of returns of the i^{th} portfolio shown in Table 2.4. That is, g_{ik} is the geometric mean of a sample of n items from the probability distribution of a_{ij}, with $j = 1, \ldots, k$. Correspondingly, $\log g_{ik}$ is the arithmetic mean of a sample of n logs from the probability distribution of $\log a_{ij}$. Further, the population mean of the distribution of $\log a_{ij}$ is $\log G_i$. In other words, $\log G_i = \sum_{j=1}^{k} p_j \log a_{ij}$.

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9 A small variance indicates that large deviations from the mean are improbable. This statement is made more precise by Chebyshev's inequality. Let X be a random variable with mean $\mu = E(X)$ and variance $\sigma^2 = Var(X)$, and let z be any number greater than 0. Then, according to Chebyshev's inequality,

$$Pr\ \{X - \mu\ | \geq z\} \leq\ \sigma^2/z^2\ = \alpha\ .$$

But the random variable X may be the sum of n random variables. In this case σ^2 (and also $\sigma^2/z^2 = \alpha$) varies inversely with n and approaches 0 as n approaches infinity. Consequently, the probability (α) of the sample mean ($\bar{X}$) differing from the population mean (μ) by more than a specified amount (z), that is $Pr\{\bar{X} - \mu\ | \geq z\}$, can be made smaller than any specified number by choosing a sufficiently large n. See, for example, William Feller, An Introduction to Probability Theory and Its Applications (New York: John Wiley and Sons, Inc., 1950), page 182, for a statement and proof of Chebyshev's inequality.

Let α_i be the probability that the arithmetic mean of any sample of n logs from the probability distribution of logs of returns from the i^{th} portfolio will differ from log G_i by z or more. And let α_m be the corresponding probability that the same mean of n logs from the probability distribution of log a_{mj} will differ from log G_m by z or more. Both α_i and α_m and their sum, $\alpha_i + \alpha_m$, depend on n and can be made smaller than any specified positive number by making n large enough.

Let $z \leq (\log G_m - \log G_i)/2$. Under these conditions returns from portfolio i can be larger than returns from portfolio m only when the sample mean of n logs from the probability distribution of log a_{ij} exceeds $\log G_i + z$ or when the sample mean of n logs from the probability distribution of log a_{mj} is smaller than $\log G_m - z$. As is shown above, the probability of either of these occurrences singly or together approaches 0 as a limit as n approaches infinity. Therefore, as n approaches infinity, the probability, P'_{mn}, that the return from the m^{th} portfolio will be larger than the return from portfolio i approaches 1 as a limit. Portfolio m is almost certain to give a higher return than portfolio i in the long run.

CHAPTER III

SUBGOALS AND SUBJECTIVE UTILITY

Rational portfolio management involves the problem of choice among strategies with uncertain outcomes. This is the ancient problem of the gambler who has the option to choose among bets. Classical writers on probability theory recommended that problems of this kind be solved by first computing the expected winnings (possibly negative) for each available bet and then by choosing that bet which has the highest mathematical expectation of winning. Their use of mathematical expectation was based on grounds of equity; that is, they were interested in which of two players, if either, would have the advantage in a hypothetical bet. In 1738, Daniel Bernoulli in four short paragraphs demonstrated that the use of the mathematical expectation of winnings did not always apply and proposed instead that gamblers should evaluate bets on the basis of the mathematical expectation of the utilities of winnings.[1]

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1 Because they bear directly on the problem in hand, the first four paragraphs of Bernoulli's article on the measurement of risk are quoted in full in the appendix at the end of this chapter. Bernoulli's example is as follows: "Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this men evaluate his chances of winning at ten thousand ducats? Would he not be ill advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble." Daniel Bernoulli, "Exposition of a New Theory on the Measurement of Risk," translation by Louise Sommer in <u>Econometrica</u>, 22, January 1954, pp. 23-24.

In terms of subgoals as defined in this study, Bernoulli showed that use of the expected-value subgoal did not always lead to choices which seemed rational to him and proposed instead the use of the expected-utility subgoal. Bernoulli's criticism of the expected-value subgoal is considered valid. The solution proposed here, however, is not recourse to subjective utility but rather the use of the maximum chance subgoal—i.e., the maximization of P'. Bernoulli's significant question concerns the utility of each possible payout. The significant question studied here is the long-term effects of repeated choices among portfolios.

Bernoulli's example is somewhat aside from the daily business of living but, when stripped of its gambling wrappings and expressed in terms of payouts and returns, it is seen to represent a major segment of economic decision making. The hypothetical market price of the ticket, which has an equal probability of paying 20,000 ducats or 0, is 9,000 ducats. This information may be stated in the form of a payout matrix (Table 3.1) showing actual payments in thousands of ducats and the payments per ducat risked.

Table 3.1

Payout Matrices for Bernoulli's Problem

Strategy	Future Occurrence Ticket Wins	Ticket Loses	Criteria A	G
(a) Payouts in thousands of ducats:				
hold ticket	20	0	10	0
not hold ticket	9	9	9	9
(b) Payouts per ducat risked:				
hold ticket	2.22	0	1.11	0
not hold ticket	1.0	1.0	1.0	1.0
Probability of occurrence	.5	.5		

Both the poor man and the rich man have the option either to hold the lottery ticket or to hold 9,000 ducats. Table 3.1 shows this option expressed in terms of thousands of ducats and in terms of payouts per ducat risked. Possible payouts range from 2.22 per ducat risked to 0. Payouts with ranges such as this--indeed, much greater ranges--are ordinary economic occurrences. Practically every business decision involves risks of this order or greater at the margin. As one example, the department store manager has to decide whether one more clerk will produce enough sales or savings to cover his pay or whether the added payroll will be a dead loss. The "poor man" today is also continually faced with implicit or explicit decisions as serious as that faced by Bernoulli's lottery ticket owner. He must decide whether to move to a new job, buy a new home, sign a second mortgage. He is continually offered the opportunity to undertake such risky ventures as buying his own truck, opening a restaurant, buying some uranium stock, some oil stock, some investment shares. Some of these options may be highly advantageous, and he must choose some one course of action in each case. The effects of these choices are cumulative, that is, the decision maker never comes back to exactly the same position that he occupied before making the choice. The major difference between Bernoulli's problem and other choices among courses of action is that the ticket owner's choice is clearly defined while the other opportunities are usually ignored or the choices muddled.

Thus Bernoulli's example is representative of a wide class of choices. The decision maker is being faced continually with such choices and the outcome of each decision affects his entire future. In the following discussion this example is stated in payout matrices constructed to illustrate choices based on: (a) classical mathematical expectation (i.e., the expected-value subgoal); (b) Bernoulli's subjective utility (i.e., the expected-utility subgoal); and (c) the maximum chance subgoal.

Expected-Value Subgoal

Table 3.2 shows the classical approach to choosing among risky ventures. The payout matrix, expressed in terms of thousands of ducats, shows the possible winnings of a poor man faced with the choice of holding or selling a lottery ticket which he found and the possible winnings or losings of a rich man faced with the choice of buying or not buying that same ticket.

Table 3.2

Payout Matrix of Gains and Losses

Strategy	Future Occurrence: Ticket Wins	Future Occurrence: Ticket Loses	Criterion A
(a) Poor Man			
Hold ticket	20	0	10
Sell ticket	9	9	9
(b) Rich Man			
Buy ticket	11	-9	1
Not buy ticket	0	0	0
Probability of Occurrence	.5	.5	

Table 3.2 shows the probability of the lottery ticket paying off or not and the net payout to the poor man and to the rich man for each of

two courses of action. The classical writers would calculate the mathematical expectation, A, of the net payouts and choose that strategy which maximizes A. In this case they would recommend that the rich man buy the ticket, and that the poor man refuse to sell the ticket for 9,000 ducats.

The mathematical expectation of the probability distributions of the payouts was recommended as a basis for reaching this decision as a matter of equity. If a great number of tickets for independent drawings were sold at a price equal to the mathematically expected payout, neither the buyer nor the seller would be likely to benefit greatly from the transaction. Presumably both would end up about where they started. If large numbers of tickets were exchanged at a significantly different price either the buyer, or the seller, probably would gain at the expense of the other party to the transaction.

Stated in other terms, when a decision maker can surely bet the same small amount on a large number of independent trials, he can maximize the expected value of his gain, and also the likelihood of having more gain than can be obtained by any other plan, by choosing that set of bets which gives him the greatest mathematically expected payout. For example, if Bernoulli's poor man had found 10,000 tickets involving 10,000 independent drawings, each with a payout equally likely to be 2 ducats or 0, he clearly would be unwise to sell his block of tickets for 9,000 ducats. His winnings on 10,000 different trials would be almost certainly very close to 10,000 ducats, the mathematical expectation of the value of the set of tickets, and the advice of the classical writers would be sound.

The arithmetic mean, as is indicated above, is a good criterion when there are large numbers of independent trials. Even decision makers who make repeated choices with cumulative effects, for example the operators of roulette wheels and insurance companies, are rightly interested in this average when each risk is small in relation to total wealth. There is little or no conflict between the use of the arithmetic mean as a criterion and the use of the geometric mean of the probability distributions of payouts per dollar of wealth (i.e., G) as a criterion under these conditions. This is indicated in Table 3.3 which shows the contrast between returns when each risk involves 100 percent of wealth and when it involves only 1 percent of wealth. Table 3.3 is based on the gambling model shown in Chapter II.

Table 3.3

Payout Matrix of Returns

Strategy	Possible Occurrence Heads	Tails	Criteria A	G
Game I—100 percent of wealth bet on each toss				
Red	2.50	0	1.25	0
Blue	2.25	.50	1.37	1.06
Green	1.75	.75	1.25	1.145
Black	1.02	1.01	1.015	1.014
No Bet	1.00	1.00	1.00	1.00
Game II—1 percent of wealth bet on each toss				
Red	1.025	.99	1.0075	1.0073
Blue	1.0225	.995	1.0087	1.0086
Green	1.0175	.9975	1.0075	1.0074
Black	1.0002	1.0001	1.0001	1.0001
No Bet	1.00	1.00	1.00	1.00
Probability of Occurrence	.5	.5		

In Game II the payout per dollar bet is exactly the same as in Game I but the payout per dollar of wealth is much lower, as only 1 percent of wealth is risked on each toss. Under the latter condition, the blue tickets have the highest G and P' and would be selected by the gambler who has adopted maximization of P' as his subgoal, even though these tickets do not have the highest G when 100 percent of wealth is risked as in Game I. In this case the choice of that ticket which has the highest arithmetic mean payout per dollar risked (i.e., per dollar of wealth when 100 percent of wealth is risked) would be a good guide to maximization of P'. This is true in general when the amount risked is small in proportion to total wealth.

Expected-Utility Subgoal

Bernoulli used the lottery ticket example to show that the mathematical expectations of the probability distributions of returns are not good guides in making choices involving large risks. He proposed, instead, that the mathematical expectations of the probability distributions of the utilities of the returns be used as guides. Table 3.4 expresses hypothetical utilities of the poor man who is faced with the choice of selling or not selling the ticket and the rich man who may buy the ticket.

The utilities shown in Table 3.4 are purely hypothetical. The underlying assumptions are: (1) the poor man has 1,000 ducats plus the lottery ticket; (2) the rich man has 100,000 ducats; and (3) the utilities of the payouts (i.e., wealth at the end of the lottery) vary directly as

Table 3.4

Payout Matrix of Utility of Wealth at End of Lottery

Strategy	Future Occurrence Ticket Wins	Ticket Loses	Criterion A
(a) Poor Man			
Hold ticket	1.32	0	.66
Sell ticket	1.00	1.00	1.00
(b) Rich Man			
Buy ticket	2.05	1.96	2.005
Not buy ticket	2.00	2.00	2.00
Probability of Occurrence	.5	.5	

the logarithms of the payouts.[2] If the poor man sold the lottery ticket for 9,000 ducats he would have total wealth of 10,000 ducats whether or not the ticket wins. The utility of this wealth is taken to be 1.00 (i.e., log 10). If, on the other hand, he holds the ticket, he would have total wealth of 21,000 ducats with a utility of 1.32 (i.e., log 21) if the ticket wins and a total of 1,000 with a utility of 0 if the ticket loses. The mathematical expectation of the utilities of holding (i.e., .66) is lower than the mathematical expectation of the utilities of selling the ticket, so Bernoulli would advise the poor man to sell his ticket. The total utility of the rich man's wealth of 100,000 ducats is taken to be 2.00. This wealth would be reduced to 91,000 ducats with a

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2 This assumption as to utilities also is made by Bernoulli. See page 55.

utility of 1.96 if a losing ticket were bought for 9,000 ducats and raised to 111,000 with a utility of 2.05 through the purchase of a winning ticket. Since purchase of the ticket increases the mathematical expectation of the utilities of the payouts, Bernoulli would advise the rich man to purchase the ticket.

Whether or not particular payout matrices, such as Table 3.4, expressed in terms of subjective utility are realistic is not a problem here. But Bernoulli's procedure is very much at issue. He defines the "mean utility" of a course of action as the mathematical expectation of the probability distribution of the possible utilities from that course of action. He then merely states, with no discussion, that this mean utility (now called moral expectation) can be used as a basis for valuing risks, that is, as a basis for choosing among courses of action.[3] In other words, he explains why he expresses his profits (or losses) in terms of subjective utility, but does not give any justification for maximizing the mathematical expectation of these utilities. Bernoulli's use of subjective utility has had wide recognition, but his use of mathematical expectation has not been adequately analyzed.[4]

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3 See paragraph 4 in the appendix to Chapter III.

4 The use of the mathematical expectation of the probability distribution of the utilities of the payouts has not been questioned. Rather mathematical expectation now is used as a basis for defining utility. The present emphasis on the axiomatic approach to utility is largely derived from John von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior (Revised ed.; Princeton: Princeton University Press, 1953). On page 28 they say: "We have practically defined numerical utility as being that thing for which the calculus of mathematical expectations is legitimate."

The Maximum Chance Subgoal

Bernoulli's problem also can be solved by the use of the maximum chance subgoal. Table 3.5 shows the payout matrix of returns for a poor man, who is assumed to have a wealth of 1,000 ducats aside from his lottery ticket, and a rich man, who is assumed to have wealth of 100,000 ducats.

Table 3.5

Payout Matrix of Returns

Strategy	Future Occurrence: Ticket Wins	Ticket Loses	Criteria: A	G
(a) Poor Man				
Hold ticket	2.1	.1	1.1	.46
Sell ticket	1.0	1.0	1.0	1.0
(b) Rich Man				
Buy ticket	1.11	.91	1.01	1.005
Not buy ticket	1.00	1.00	1.00	1.00
Probability of Occurrence	.5	.5		

The payout matrix of returns in Table 3.5 shows each possible return from the various strategies. The poor man has an initial wealth of 1,000 ducats plus a lottery ticket which he has an option to sell at 9,000 ducats, giving him a total initial wealth of 10,000 ducats. If he sells the ticket he will get a return of 1.0 on this amount whether the ticket wins or loses. If he holds the ticket and wins he will have 21,000 ducats or a return of 2.1. If he holds and loses his wealth will be only 1,000 ducats, giving him a return of .1. In similar fashion, the rich man has initial wealth of 100,000 ducats and the opportunity to buy the lottery ticket for 9,000 ducats. His wealth will either increase to 111,000 or decline to 91,000 ducats if he buys the lottery ticket, thus giving him a return of either 1.11 or .91. The arithmetic mean, A, of the probability distribution of payouts is higher for the poor man when he holds the ticket and for the rich man when he buys the ticket. The geometric mean, G, of returns for the poor man is higher when the ticket is sold, however, and the G for the rich man is higher when he buys the ticket.

Over a long enough period of time many economic choices involving returns of the same order of magnitude repeat themselves. Bernoulli's poor man may never find another lottery ticket, but he probably will have many options among courses of action with as wide, or wider, a range of returns. It is assumed here that both the rich man and the poor man will have many opportunities to risk the same proportions of their respective fortunes on approximately the same terms and that both men prefer more wealth to less wealth, everything else being equal. If these assumptions are valid, the maximization of P', the probability of having more wealth at the end of a long series of such choices than can be obtained by any other specified course of action, is a rational subgoal and G is a rational criterion. The use of the maximum chance subgoal results in courses of action for the rich man and for the poor man which seemed rational to Bernoulli.

The decision maker who is interested in maximizing his wealth at the end of a long series of choices should ask himself how he would come out in the long run if he made the same choice on the same terms over and over again. It is not necessary for him to ask himself what is his individual subjective utility of winning. This is not to say that other goals, rather than the goal of maximum wealth at the end of a long series of choices, are irrational. Indeed, the use of subgoals based on the goal of maximum wealth often may be irrational. For example, the man who desperately needs $10 to escape a jail sentence and who has only $1 may well be justified in taking a gamble to get his money even though this gamble would not stand the maximum chance subgoal test. Even under these conditions, however, it would be useful for the man to know that he shoud not often act in such a manner, if he wants to build up his fortune so as to avoid like predicaments in the future.

Bernoulli's Utility Function

In his paper Bernoulli reaches the conclusion that, in general, the utility resulting from any small increase in wealth will be inversely proportional to the quantity of goods previously possessed. This is generally credited with being the first use of a utility function. Through a combination of graphic analysis and the calculus he then develops a rule for estimating the value of a risky proposition.[5]

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5 See Harold T. Davis, The Theory of Econometrics (Bloomington, Indiana: The Principia Press, 1941), pp. 56-58, for a derivation of Bernoulli's formula from his postulates in modern mathematical terms.

Bernoulli's rule is as follows: "Any gain must be added to the fortune previously possessed, then this sum must be raised to the power given by the number of possible ways in which the gain may be obtained; these terms should then be multiplied together. Then of this product a root must be extracted the degree of which is given by the number of all possible cases, and finally the value of the possessions must be subtracted therefrom; what then remains indicates the value of the risky proposition in question."[6] It is apparent that Bernoulli's "gain" plus "the fortune previously possessed" corresponds to portfolio payout and that Bernoulli is saying in effect that the value of the risky proposition is measured by the geometric mean of the probability distribution of portfolio payouts less the original cost. In his paper Bernoulli gives two measures of the value of a risky venture. These measures are: (1) the geometric mean of the payouts (see above), and (2) the mathematical expectation of the utilities of the payouts.[7] There is no conflict between these two measures when the utilities of the payouts vary as their logarithms, as is assumed by Bernoulli. Both measures then lead to the same choices among risky ventures since the geometric mean of a probability distribution of returns is maximized when the arithmetic mean of the logs of the returns is maximized.

Bernoulli gives a number of applications of his formula to gambling and to insurance. In each instance he is able to give a

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6 Bernoulli, op. cit., p. 28.

7 See last sentence on page 67.

specific answer. He says that everyone who bets any part of his fortune on a mathematically fair game of chance is acting irrationally, and he then determines what odds a gambler, with a specified fortune, must obtain to break even in the long run. Most of his problems still are interesting in their own right and many have a bearing on proper portfolio management. For instance, he demonstrates, with numerical examples, the advantages of diversification among equally risky ventures and between risky and safe assets.

Bernoulli's approach to the valuation of risky ventures is not contradictory to the maximum chance approach. Not only do the two approaches lead to the same conclusion when they both can be applied but they tend to support one another. Wealth-holders may be divided into two groups. The first group contains those wealth-holders to whom each risk is a unique event either because they do not expect it to recur or because they keep its effects entirely separate from the results of other risks. For example, the man who each year sets aside a small sum to bet on the races during his vacation with the intention of living it up if he wins and writing it off to experience if he loses, presumably is not actuated by long-run profit maximizing motives. The effects of each risk are kept separate. Analysis based on maximum chance has nothing to offer this first class of wealth-holders. The choice between profit and safety or expected return and variance is a matter of subjective utility. Bernoulli's assumption that the satisfaction derived from a small gain tends to vary in inverse proportion to the initial wealth may or may not be a shrewd guess.

The second class of wealth-holders includes those who expect to be faced repeatedly with risks of the same general type and magnitude. This group includes those making most business and portfolio decisions and hence is of great importance. It includes, specifically, all those who want to maximize the values of their portfolio at the end of n years assuming reinvestment of all returns. Here there is a definite rule for choosing between risk and return, based on maximum chance principles. This class may be subdivided further into (a) those who undertake only one risky venture at a time, and (b) those who are able to diversify their risky ventures. Because so many economic phenomena, including yields on stocks, tend to fluctuate together over time, diversification among risky ventures cannot go as far towards eliminating risk as otherwise would be the case. Final choice among efficient portfolios for both groups (a) and (b) is based on maximization of G not because this maximizes subjective utility but because it maximizes P'.

Bernoulli states that the wealth-holder (here called portfolio manager) should ask himself whether the added satisfaction associated with the expected gain justifies undertaking the risky venture. He bases an exact rule of behavior on his assumption as to how the added satisfaction varies with the size of the potential gain or loss in relation to the size of the portfolio. The rule may or may not be empirically useful, but it is grounded on rather shaky evidence as to the exact shape of the utility function. According to maximum chance analysis the wealth-holder or portfdio manager should ask himself how he can maximize his chances of getting a better return than can be obtained with any other specified plan assumingthat he risks the same proportion of his portfolio on the same terms over and over again. It turns out that the formula which enables the portfolio manager to answer the maximum chance question is the same as that developed by Bernoulli on grounds of subjective utility.

In conclusion Bernoulli says:

> Though a person who is fairly judicious by natural instinct might have realized and spontaneously applied much of what I have here explained, hardly anyone believed it possible to define these problems with the precision we have employed in our examples. Since all of our propositions harmonize perfectly with experience it would be wrong to neglect them as abstractions resting upon precarious hypotheses. 8

Professor Stigler, in a review article,[9] gives considerable space to Bernoulli's hypothesis in reference to the slope of the wealth-holder's utility function even though the major emphasis of the article is on utility not affected by probability. He refers to the fact that LaPlace and Marshall, among others, have accepted the law as a realistic guide. He also points out the similarity of Bernoulli's law to the Weber-Fechner psychological hypothesis that the just noticeable increment to any stimulus is proportion to the stimulus. Stigler says, "Bernoulli was right in seeking the explanation[10] in utility and he was wrong only in making a special assumption with respect to the slope of

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8 Ibid., p. 31.

9 George J. Stigler, "The Development of Utility Theory," Journal of Political Economy, Vol. 58 (1950), 373-377.

10 Bernoulli is explaining the reason for the limited value of the game involved in the St. Petersburg paradox. This game is a type of risky venture with an infinitely large mathematically expected value but with an extremely small probability of winning.

the utility curve for which there was no evidence and which he submitted to no tests."[11]

More recently Professor L. J. Savage in a section of "Historical and Critical Comments on Utility" had this to say:

> Bernoulli went further than the law of diminishing marginal utility and suggested that the slope of utility as a function of wealth might, at least as a rule of thumb, be supposed, not only to decrease with, but to be inversely proportional to, the cash value of wealth. To this day, no other function has been suggested as a better prototype for Everyman's utility function. . . .Though it might be a reasonable approximation to a person's utility in a moderate range of wealth, it cannot be taken seriously over extreme ranges. 12

Individual Risk Preference

As indicated in the previous section, Bernoulli took the following steps to develop his utility function and to justify diversification among risky ventures and between risk assets and safe assets: (1) He showed—subject to the previously discussed implicit assumption as to subgoals—that the value of a risky venture to the individual wealth-holder is not the arithmetic mean of the probability distribution of returns (i.e., the mathematical expectation of returns) but may be taken to be the arithmetic mean of the probability distribution of the utilities of the returns. (2) He stated that, in the absence of the unusual, the gain in utility resulting from any small increase in wealth may be assumed to be inversely proportional to the quantity of goods previously possessed. (3) He developed a formula for calculating the utility of a

11 Stigler, op. cit., p. 375.

12 Leonard J. Savage, The Foundations of Statistics (New York: John Wiley and Sons, Inc., 1954), p. 94.

risk asset to the individual wealth-holder using as a criterion the utility function developed in step 2. According to Bernoulli the subjective utility of the wealth-holder's assets, including the risky venture, is measured by the geometric mean, G, of the probability distribution of payouts from such assets. (4) Using this formula, he was able to calculate exactly the utility of the wealth-holder's assets, including the risky venture, and to show that diversification among risky ventures increases this utility.[13]

Bernoulli's step 2 may be a reasonable assumption as to utility[14,15] but is subject to so many qualifications and exceptions (it does not explain gambling, for example) that it has not been accepted as a suitable basis for erecting the superstructure of steps 3 and 4. The valuation of risky ventures has been left to individual risk preference without any criterion as to what this preference is likely to be. For example, Marschak[16] presents a hypothetical table in which an asset's marginal contribution is determined by adding together its contribution

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13 Bernoulli, op. cit., pp. 24, 25, 28, 30.

14 Cf. Alfred Marshall, Principles of Economics (8th ed.; New York: The Macmillan Co., 1950), p. 135. Marshall says: "In accordance with a suggestion made by Daniel Bernoulli, we may regard the satisfaction which a person derives from his income as commencing when he has enough to support life, and afterwards as increasing by equal amounts with every equal successive percentage that is added to his income; and vice versa for loss of income."

15 See also L. J. Savage's comment quoted on page 59 of this dissertation.

16 Helen Makower and Jacob Marschak, "Assets, Prices and Marketing Theory," Economica, V (1938), 261-288. A.E.A. Readings in Price Theory (Chicago: Richard D. Irwin, 1952), Vol. VI, 301-302.

to "lucrativity" and safety measured in "lucrativity units" determined by the safety preference rate for a single individual. These individual safety preference rates, in turn, are a matter of taste and must be accepted as given. Friedman and Savage[17] build on Bernoulli's step 1 but modify step 2 by developing a doubly inflected curve comparing utility with income.

Markowitz starts off his analysis of portfolio selection by pointing out that "the portfolio with the maximum expected return is not necessarily the one with the minimum variance. There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return."[18] He assumes that the investor considers, or should consider, expected return a desirable thing and variance of return an undesirable thing, and he defines an efficient portfolio as a portfolio with minimum variance for a given expected return or more and a maximum expected return for a given variance or less. He develops a method for selecting efficient portfolios from the set of all possible portfolios but does not give any basis for choice among the efficient portfolios except the individual's safety preference rate. This dissertation sets forth an objective basis for choosing among efficient portfolios without the necessity of depending on individual risk preferences.

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17 Milton Friedman and L. J. Savage, "The Utility Analysis of Choices Involving Risk," Journal of Political Economy, 56 (1948), 279-304.

18 Harry Markowitz, "Portfolio Selection," Journal of Finance, VII (March 1952), page 79.

The Need for an Objective Criterion

The difficulty of evaluating subjective risk preference and the need of an objective criterion is well indicated in the following quotation from a recent journal article dealing with selection of an optimum combination of crops for a farmer:

> The introduction of risk into an economic model of a firm and consequently into a linear programming model of a firm has been accomplished by describing risky outcomes as probability distributions and choosing from among alternate possible distributions by the expected utility hypothesis.
>
> Two basic weaknesses have appeared in applying this method of incorporating risk. One difficulty arises in choosing a value for the constant α, which in this case is some sort of risk aversion indicator, and is, to some degree, governed by the personal characteristics of the entrepreneur. A large value for α indicates that the entrepreneur places great weight on the variance as a deciding factor and is consequently highly averse to risk, and vice versa. The estimation of such a constant to be used in a model is thus quite important; the wrong choice will invalidate any results obtained. The derivation of this constant is a delicate task beyond the scope of this paper.[19]

A major advantage of the criterion for choice among risky ventures developed in this dissertation is that it avoids the necessity for direct subjective determination of such factors as Marschak's "lucrativity units" or Freund's "risk aversion indicator." As Roy remarks: "A man who seeks advice about his actions will not be grateful for the suggestion that he maximize expected utility."[20]

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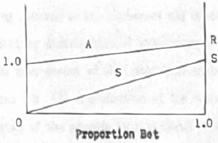
19 Rudolph J. Freund, "The Introduction of Risk into a Programming Model," Econometrica, 24 (July 1956), 253-263.

20 A. D. Roy, "Safety First and the Holding of Assets," Econometrica, 20 (1952), 433. Quoted in Cleon Harrell, "Formal Interrelationships Between Economics and Probability Theory," Southern Economic Association Meeting, November 17, 1956.

The criteria for choice between risk and safety in portfolio management can be illustrated by assuming that a gambler has the choice of holding his money in cash or of betting on a gambling device which, with equal probability, will return R - s on loss occasions and R + s on gain occasions with an expected return of R per dollar played, with R greater than 1 and R - s less than 1. The gambler's portfolio at any time consists of the proportion of his wealth held in cash plus the proportion bet on the gambling device. The expected returns and standard deviations of returns of all portfolios divided between the safe asset (i.e., cash) and the risk asset (i.e., bets on the gambling device) are shown in Figure 2.1.

Figure 2.1

Expected Portfolio Return (A) and Standard Deviation (S) of Portfolio Returns Distributed by Proportion Bet on a Gambling Device Equally Probably Paying R + s or R - s per $1.00 Bet



When the gambler bets 0 proportion of his wealth, the expected return from his portfolio is 1.0 and the standard deviation of returns is 0. As the proportion bet increases both the expected portfolio return and the standard deviation of returns increase. When he bets all of his wealth, the expected portfolio return is R and the expected standard deviation of returns is s. As long as R is greater than 1.0 and R - s is less than 1.0, all possible combinations of the two assets in this range are efficient portfolios in that any one of the combinations gives the maximum possible expected return for some standard deviation or variance and the minimum standard deviation or variance for some expected return. Neither Marschak, nor Friedman and Savage, nor Markowitz would be able to help the gambler in choosing among these effficient portfolios beyond telling him that he should gamble heavily if he has a high preference for risk, and should be very conservative in his betting if he has a high risk aversion factor. In this dissertation an attempt is made to give the gambler (and wealth-holders in general) an objective criterion for making this choice.

The wealth-holder who adopts the maximum chance subgoal can reach this subgoal by using the geometric mean, G, of the probability distribution of returns as his criterion and choose that portfolio which has the probability distribution of returns with the highest G. Bernoulli also has shown that choice of that portfolio with the highest G is a rational choice if: (1) maximization of the mathematical expectation of the utilities of the payouts is a rational subgoal; and (2) if the utility of a small gain or loss varies inversely with the amount of wealth already possessed.

Most economists recognize that the mathematical expectation and the variance of the probability distribution of returns, and the chance of ruin, are important to the wealth-holder--but they leave it to individual risk preference to balance one factor against the others. G depends on both the mathematical expectation and the variance of the probability distribution of returns, and when G is maximized there is no chance of ruin if the wealth-holder's probability beliefs are correct. Consequently, maximization of G falls within the generally accepted range of rational behavior. This is not to say that G is the only rational criterion for portfolio management; it is to say, however, that it is a useful criterion when dealing with a broad range of problems. When the portfolio with maximum G is not chosen, there must be justification for choosing to hold a portfolio which has little chance of being the most valuable portfolio in the long run.

APPENDIX TO CHAPTER III

(From Econometrica, 22, January 1954, pp. 23-24)

Excerpt from

EXPOSITION OF A NEW THEORY ON THE MEASUREMENT OF RISK[1]

By Daniel Bernoulli

1. Ever since mathematicians first began to study the measurement of risk there has been general agreement on the following proposition: Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of possible cases where, in this theory, the consideration of cases which are all of the same probability is insisted upon. If this rule be accepted, what remains to be done within the framework of this theory amounts to the enumeration of all alternatives, their breakdown into equi-probable cases and, finally, their insertion into corresponding classifications.

2. Proper examination of the numerous demonstrations of this proposition that have come forth indicates that they all rest upon one hypothesis: since there is no reason to assume that of two persons encountering identical risks, either should expect to have his desires more closely fulfilled, the risks anticipated by each must be deemed equal in value. No characteristic of the persons themselves ought to be taken into consideration; only those matters should be weighed carefully that pertain to the terms of the risk. The relevant finding might then be made by the highest judges established by public authority. But really there is here no need for judgment but of deliberation, i.e., rules would be set up whereby anyone could estimate his prospects from any risky undertaking in light of one's specific financial circumstances.

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1 Translated from Latin into English by Dr. Louise Sommer, The American University, Washington, D. C., from "Specimen Theoriae Novae de Mensura Sortis," Commentarii Academiae Scientiarum Imperialis Petropolitanae, Tomus V (Papers of the Imperial Academy of Sciences in Petersburg, Vol. V), 1738, pp. 175-192.

3. To make this clear it is perhaps advisable to consider the following example: Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble. The rule established in 1. must, therefore, be discarded. But anyone who considers the problem with perspicacity and interest will ascertain that the concept of value which we have used in this rule may be defined in a way which renders the entire procedure universally acceptable without reservation. To do this the determination of the value of an item must not be based on its price, but rather on the utility it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.

4. The discussion has now been developed to a point where anyone may proceed with the investigation by the mere paraphrasing of one and the same principle. However, since the hypothesis is entirely new, it may nevertheless require some elucidation. I have, therefore, decided to explain by example what I have explored. Meanwhile, let us use this as a fundamental rule: If the utility of each possible profit expectation is multiplied by the number of ways in which it can occur, and we then divide the sum of these products by the total number of possible cases, a mean utility (moral expectation) will be obtained, and the profit which corresponds to this utility will equal the value of the risk in question.

. . .

CHAPTER IV

METHODS OF ALLOCATING PORTFOLIOS SO AS TO MAXIMIZE THE GEOMETRIC MEAN PORTFOLIO RETURN

In previous chapters reason has been given for accepting G, the geometric mean of the probability distribution of portfolio returns, as a rational criterion for choosing among portfolios. This measure may be used in choosing among all possible portfolios on the basis of probability beliefs about returns from these portfolios. The aim will then be to select the portfolio with the highest G. This chapter deals with the subject of how to do so, insofar as the problem is one of allocation of the portfolio value between a group of risk assets, on the one hand, and a group of safe assets, on the other.

In simplest form, the central problem of the first three chapters of this study has been the choice between portfolio A and portfolio B when faced with a payout matrix showing the probability distributions of returns from the two portfolios. Such a matrix is shown in Table 4.1.

Table 4.1

Payout Matrix of Returns

	Gain Year	Loss Year	Criteria A	Criteria G
Portfolio A	1.365	.825	1.095	1.061
Portfolio B	1.260	.900	1.08	1.065
Probability of Occurrence	.5	.5		

It has been shown that choice of that portfolio with maximum G (i.e., portfolio B in Table 4.1) will maximize the probability (P') of having a larger return than can be obtained from any other specified portfolio at the end of n years, n being large. For this reason G is accepted as a rational criterion for choosing among portfolios.

Portfolio A and portfolio B in Table 4.1 may consist of two groups of securities allocated in different proportions. For example, portfolio A may consist of a portfolio allocated 90 percent to common stocks and 10 percent to bonds, and portfolio B may consist of a portfolio allocated 60 percent to the same stocks and 40 percent to bonds (see Table 4.3). Changes in allocation between risk assets and safe assets affect both the arithmetic mean and the geometric mean of the probability distribution of portfolio returns. There is some one allocation which will maximize G.

The present chapter is devoted to the practical problem of how to allocate a portfolio between a safe asset (typically a group of high-grade bonds or cash) and a risk asset (typically a group of stocks) in such a manner as to maximize G. The portfolio manager is faced with a payout matrix showing the probability distributions of returns from stock and bonds and wishes to divide his portfolio between these two types of assets in such manner as to maximize the probability of getting a higher return than can be obtained from any other specified allocation between the two assets. Such a matrix is shown in Table 4.2.

The portfolio manager forms probability beliefs about returns from stock, such as those reflected in the payout matrix in Table 4.2, and wishes to allocate his portfolio between bonds and stock on the basis of his beliefs. Bond-stock allocations which maximize G probably will change during the course of a business cycle but this change will be brought about by changes in payout matrices—not by changes in proper maximizing action, given the payout matrices. The portfolio manager's opinion as to the general level of the stock market and the stage of the business cycle presumably will influence his probability beliefs about returns from stock and bonds. Consequently the probability distributions of payouts from stock and bonds, such as is shown in Table 4.2, presumably will change from year to year. However, for

Table 4.2

Payout Matrix of Returns

	Occurrence		Criteria	
	Gain Year	Loss Year	A	G
Stock	1.40	.80	1.10	1.058
Bonds	1.05	1.05	1.05	1.05
Probability of Occurrence	.5	.5		

any given payout matrix there is only one allocation which maximizes G.

Allocation When Returns are Binomially Distributed with $p = .5$

In this section the problem of proper allocation of a portfolio between a risk asset and a safe asset, in gambling and in portfolio management, will be stated for the simplest possible distribution of returns from the risk asset (i.e., the binomial distribution with the occurrence of each of two possible returns equally probable).

Let a gambler be given the option to bet any amount he wishes on each of n tosses of a fair coin with a return of \$3.00 per \$1.00 bet if heads occur and a return of 0 if tails occur. His problem is to maximize his wealth at the end of n tosses. It has already been established that choice of that course of action which produces the highest G is a rational choice under these conditions. It is now necessary to determine the proper procedure to maximize G.

Let q be the proportion of the total wealth bet on each toss of the coin. Then, if heads occur, the return on the proportion bet is 3.00 and on the proportion held in cash (i.e., 1 - q) is 1.00, so that the total return is $q \times 3 + 1 \times (1 - q) = 2q + 1$. If tails occur, the return is $1 - q$. In Table 4.3 these returns are stated in the form of a payout matrix.

Table 4.3

Payout Matrix of Returns

Strategy	Occurrence		Criteria	
	Heads	Tails	A	G^2
Bet 1.0 of wealth	3.00	0	1.50	0
Bet q of wealth	$1 + 2q$	$1 - q$	$1 + q/2$	$(1 + 2q)(1-q)$
Bet 0 of wealth	1.00	1.00	1.00	1.00
Probability of Occurrence	.5	.5		

The geometric mean of the probability distribution of returns, G, is maximized when G^2, which equals $(1 + 2q)(1 - q)$ is maximized. A necessary condition for this is for $dG/dq = 0$. The value of q which satisfies this condition, called q_{max}, is .25. The G from betting 25 percent of wealth on each toss is 1.061. This is the return when heads occur in exactly half of the tosses, and is the highest possible return for this combination of heads and tails for the gambler who bets the same proportion of his wealth on each toss. The relationship between q and G is illustrated in Table 4.4, which shows the payout matrix of returns from betting various proportions of total wealth on each toss in this game.

Table 4.4

Payout Matrix of Returns

Proportion of Total Wealth Bet	Occurrence		Criteria	
	Heads	Tails	A	G
1.00	3.00	0	1.50	0
.90	2.80	.10	1.45	.529
.80	2.60	.20	1.45	.721
.70	2.40	.30	1.35	.849
.60	2.20	.40	1.30	.938
.50	2.00	.50	1.25	1.000
.40	1.80	.60	1.20	1.039
.30	1.60	.70	1.15	1.058
.25	1.50	.75	1.125	1.061
.20	1.40	.80	1.10	1.058
.10	1.20	.90	1.05	1.039
.00	1.00	1.00	1.00	1.000
Probability of Occurrence	.5	.5		

In Table 4.5 a payout matrix, such as that in Table 4.3, is stated in terms of returns from speculative stocks and high-grade bonds in gain and loss years.

Table 4.5

Payout Matrix of Returns

Portfolio	Occurrence		Criteria	
	Gain Year	Loss Year	A	G
All stock	1.40	.80	1.10	1.058
60 percent stock	1.26	.90	1.08	1.065
All bond	1.05	1.05	1.05	1.050
Probability of Occurrence	.5	.5		

Table 4.5 shows the probability of occurrence and the returns from portfolios in gain years and loss years. When a portfolio manager believes that gain years and loss years are equally probable and that stock will return 1.40 in a gain year and .80 in a loss year, as against 1.05 from bonds in both years, he can maximize the geometric mean of the probability distribution of returns by placing 60 percent of his portfolio in stock at the beginning of each year. In the long run, such an allocation between stock and bonds is almost certain to result in a higher return than can be obtained by any different allocation.

Allocation With Any Distribution of Returns

The general problem of allocating a portfolio between a risk asset and a safe asset in order to maximize G may be stated in terms of the payout matrix of returns shown in Table 4.6.

Table 4.6

Payout Matrix of Returns

Portfolio	Relevant Future Occurrences $1, \ldots, j, \ldots, k$	A	G
All stock	$R_1, \ldots, R_j, \ldots, R_k$	$\bar{R}$	G_1
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots$	$\vdots$
q stock b bonds	$qR_1 + bC, \ldots, qR_j + bC, \ldots, qR_k + bC$	$q\bar{R} + bC$	G_q
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots$	$\vdots$
All bonds	$C, \ldots, C, \ldots, C$	C	C
Probability of Occurrence	$p_1, \ldots, p_j, \ldots, p_k$		

In Table 4.6, q represents the proportion of the portfolio put in stock at the beginning of each year and b represents the proportion put in bonds, with $p + b = 1$. The returns from an all-stock portfolio (i.e., $q = 1.0$) are represented by the probability distribution of returns from stock, R_j, with $j = 1, \ldots, k$. This distribution has an arithmetic mean of $\bar{R}$ and a geometric mean of G_1, where G_1 is the G when 1.0 of the portfolio is allocated to stock. The certain return from an all bond portfolio is C. The arithmetic mean return from a portfolio divided q in stock and b in bonds is $q\bar{R} + bC$ and the geometric mean is G_q. The problem here being considered is to allocate the portfolio between bonds and stock in such a manner as to maximize G_q. This is a simple maximizing problem which can be solved by stating G_q in terms of logs and differentiating with respect to q. Based on Table 4.6 it is apparent that

$$\log G_q = p_1 \log(qR_1 + bC) + \ldots + p_j(qR_j + bC) + \ldots + p_k \log(qR_k + bC)$$

but $b = 1 - q$ so

$$(4.1)\quad \log G_q = p_1 \log[C + q(R_1 - C)] + \ldots + p_j \log[C + q(R_j - C)] + \ldots + p_k \log[C + q(R_k - C)] .$$

G_q is maximized when the derivative of $\log G_q$ with respect to q is 0. In another form, $G_q = G_{max}$ when $q = q_{max}$ and $\frac{d \log G_q}{dq} = 0$. That is, when

$$(4.2)\quad p_1 \frac{R_1 - C}{C + (R_1 - C)q_{max}} + \ldots + p_j \frac{R_j - C}{C + (R_j - C)q_{max}} + \ldots + p_k \frac{R_k - C}{C + (R_k - C)q_{max}} = 0 .$$

Equation (4.2) may be solved directly, or by trial and error, to obtain the proper proportion, q_{max}, of the portfolio to place in stock at the beginning of each of the forthcoming n years in order to maximize G, the geometric mean of the probability distribution of portfolio returns. Equation (4.1) may then be solved, using the given q_{max}, in order to find the maximum geometric average portfolio return (i.e., G_{max}) which can be obtained if the most likely combination of returns from stock occurs. For example, if $R_1 = 0$, $R_2 = 3.0$, $p_1 = .5$, $p_2 = .5$, and $C = 1.0$, equation (4.2) reduces to

$$.5[-1/(1 - q_{max})] + .5[2/(1 + 2q_{max})] = 0 ,$$

from which $q_{max} = .25$. This is the q_{max} used by the gambler faced with the payout matrix shown in Table 4.3.

G and q_{max} Estimated from the Arithmetic Mean and Variance of Returns from the Risk Asset

Equations (4.1) and (4.2) may be used to determine G, the geometric mean of the probability distribution of portfolio returns, and q_{max}, the proportion of the portfolio to allocate to risk assets in order to maximize G. However, these equations involve the whole probability distribution of returns from the risk assets and consequently may be difficult to apply. In this section a method is developed for approximating G and q_{max} based on the arithmetic mean and variance of returns from the risk asset. This method does not give good estimates of G and q_{max} in all cases. It does, however, fit a wide variety of distributions with reasonable accuracy, including typical distributions of returns from stocks.

Table 4.7 shows the payout matrix of returns from a portfolio allocated between a safe asset giving a return of C and a risk asset which will return R + s in a gain year and R - s in a loss year with gain years and loss years equally probable.

The first row of Table 4.7 shows the returns from a portfolio allocated entirely to the risk asset. The arithmetic mean of the probability distribution of returns from the risk asset is R and the variance of this distribution is s^2 where $n = 1$.[1] The geometric mean is a function

[1] When $n > 1$, the variance is not exactly s^2. However, when s/R is small so that $(s/R)^3$ and higher powers of s/R may be neglected, s^2 is a good estimate of the variance for all n's.

Table 4.7

Payout Matrix of Returns

Proportion in Risk Asset	Future Occurrence Gain Year	Loss Year	Criteria A	G
1.0	$R + s$	$R - s$	R	$(R^2 - s^2)^{.5}$
q	$q(R+s) + bC$	$q(R-s) + bC$	$qR + bC$	G_q
.0	C	C	C	C
Probability of Occurrence	.5	.5		

of the arithmetic mean and s^2. That is

$$G_1 = (R + s)^{.5}\ (R - s)^{.5}$$

$$= (R^2 - s^2)^{.5}.$$

The second row shows corresponding figures for a portfolio allocated q in stock and b in bonds with $q + b = 1$. In this case

$$G_q^2 = [q(R + s) + bC][q(R - s) + bC]$$

$$= (qR + bC + qs)(qR + bC - qs)$$

$$G_q^2 = (qR + bC)^2 - (qs)^2 \quad (4.3)$$

Here, too, the geometric mean is a simple function of the arithmetic mean and s^2.

G_q is maximized (i.e., $G_q = G_{max}$) when $q = q_{max}$. A condition for this is that the derivative of G with respect to q should equal zero. Differentiating equation (4.3) and collecting terms, it is apparent that this condition is satisfied when

$$(4.4) \qquad q_{max} = \frac{(R - C)C}{s^2 - (R - C)^2} .$$

The geometric mean of the probability distribution of annual returns for n years is equal to the corresponding geometric mean for one year[2] and the q_{max} which maximizes G when $n = 1$ will maximize G for any period of years. Consequently equations (4.3) and (4.4) give exact values for G and q_{max} for any period in terms of C, R, and s^2, when the risk asset returns $R + s$ in gain years and $R - s$ in loss years, with gain years and loss years equally probable. C is the return from the safe asset and R is the arithmetic mean of the probability distribution of the returns from the risk asset. When $n = 1$, s^2 is the variance of this distribution.

Equations corresponding in form to (4.3) and (4.4) also give good estimates for G and q_{max} for many other distributions of returns. This is true in all cases where the square of the geometric mean of a set of returns is approximately equal to the square of the arithmetic mean less the variance. That is, where

$$(4.5) \qquad G^2 \approx G'^2 = A^2 - s^2 ,$$

with G' being the estimated geometric mean.[3] Table 4.8, which shows the relationship of G' to G for three sets of returns, indicates that this approximation may be valid for a wide variety of distributions.

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2 See the paragraph following Table 2.5 on page 36.

3 This approximation holds if deviations (x) are small compared with the arithmetic mean of the distribution so that $(x/A)^3$ and higher powers of x/A may be neglected. See George U. Yule and M. G. Kendall, An Introduction to the Theory of Statistics (14th ed.; New York: Hafner Publishing Co., 1950), p. 150.

Table 4.8

Comparison of Estimated with Actual Geometric Means of Sets of Returns

	$\sqrt{R_i}$	R_i	Frequency	Log 10 R_i	f log 10 R_i	f R_i	f R_i^2
(a) Set of 252 Returns Distributed Normally into Seven Classes							
		.25	1	.398	.4	.2	.1
		.50	14	.699	9.8	7.0	3.5
		.75	61	.875	53.4	45.7	34.3
		1.00	100	1.000	100.0	100.0	100.0
		1.25	61	1.097	66.9	76.3	95.3
		1.50	14	1.176	16.5	21.0	31.5
		1.75	1	1.243	1.2	1.8	3.1
Sum			252		248.2	252.0	267.8
Average					L = 1.985	A = 1.0 A^2 = 1.0	X^2=1.063
(b) Set of 252 Returns with Logs Distributed Normally into Seven Classes							
		.35	1	.550	.6	.4	.0
		.50	14	.700	9.8	7.0	3.5
		.71	61	.850	51.9	43.4	30.8
		1.00	100	1.000	100.0	100.0	100.0
		1.41	61	1.150	70.1	86.0	121.2
		2.00	14	1.300	18.2	28.0	56.0
		1.82	1	1.450	1.4	2.8	7.9
Sum			252		252.0	267.6	319.4
Average					L = .00	A = 1.061 A^2 = 1.125	X^2 = 1.267
(c) Set of 252 Returns with Square Roots Distributed Normally into Seven Classes							
	.316	.100	1	.000	.0	.1	.0
	.544	.296	14	.471	6.6	4.1	1.2
	.772	.596	61	.775	47.3	36.3	21.6
	1.000	1.000	100	1.000	100.0	100.0	100.0
	1.228	1.502	61	1.177	70.8	91.6	137.5
	1.456	2.120	14	1.326	18.6	29.7	63.0
	1.684	2.840	1	1.453	1.5	2.8	7.8
Sum			252		244.8	264.6	331.1
Average					L = 1.971	A = 1.050 A^2 = 1.102	X^2 = 1.314

Statistics of Distribution of Returns

	A	s	A^2	s^2	G'^2	G'	G
(a)	1.000	.25	1.000	.063	.937	.968	.966
(b)	1.061	.38	1.125	.142	.983	.992	1.000
(c)	1.050	.46	1.102	.212	.890	.943	.935

Table 4.8 shows three sets of 252 returns distributed into seven classes in various ways. The returns , R_i, in set (a) range from .25 to 1.75 and are distributed approximately normally into seven classes with a mean of 1.00 and a standard deviation of .25 . The returns in set (b) range from .35 to 1.82 and are distributed so that the logs of $10\ R_i$[4] are distributed normally with a mean of 1.00 and a standard deviation of .15 . The returns for set (c) range from .10 to 2.84 and are distributed so that the square roots of the returns are distributed normally with a mean of 1.00 and a standard deviation of .288. The table shows the arithmetic mean of the logs of returns, L, the arithmetic mean return, A, and the arithmetic mean square return, X^2, for each set of returns. These data, in turn, were used to calculate the statistics of the distributions shown at the bottom of the table with G = antilog L , $s^2 = X^2 - A^2$, and $G'^2 = A^2 - s^2$.

It is apparent that the arithmetic mean and the variance form a good basis for estimating the geometric mean for the sets of returns in Table 4.8. In set (a), the geometric mean differs from the arithmetic mean by .034, yet the estimated geometric mean, G', is only .002 greater than G. In set (b), the geometric mean is .061 less than the arithmetic mean, yet G' is only .008 less than G. In set (c) the difference between A and G is very large, being .116, yet G' is only .008 greater than G. This evidence supports the conclusion that equation (4.5) gives a good estimate of G for distributions of returns which fall within the tabled

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4 Logs of $10R_i$ are tabled rather than logs of R_i as a matter of convenience. Log R_i, of course, equals log $10R_i$ - 1.0 .

range of dispersion and regularity. It seems reasonable to believe that many probability distributions of returns from stock fall within this range. Ex post distributions of stock returns are used later in this paper to support this conclusion.[5]

Let a portfolio be divided q in stock and (1 - q) in bonds. The stock has a probability distribution of returns with an arithmetic mean of $\bar{R}$ and a variance of s^2, the bonds have a sure return of C. The arithmetic mean portfolio return, A, is equal to $q\bar{R} + (1 - q)C$. The variance is $(qs)^2$. Equation (4.5) becomes

$$(4.6) \qquad G_q^2 \approx G'^2_q = q\bar{R} + (1 - q)C - (qs)^2 .$$

G_q is approximately maximized when G'_q is maximized, that is, when

$$(4.7) \qquad q'_{max} \approx q_{max} = \frac{(\bar{R} - C)C}{s^2 - (\bar{R} - C)^2} .$$

Equation (4.7) corresponds to equation (4.4) except that s^2 now is the variance of the distribution of returns not only when n = 1 but for all values of n.

Obviously equations (4.6) and (4.7) do not apply when any one return is 0 proportion of $\bar{R} - s$. For example, consider the wealth- holder who is given the opportunity to bet on a gambling device which will return 1.40 in 5 out of 6 possible occurrences and 0 in the other occurrence. If equation (4.6) gave a good estimate under these conditions, a wealth-holder who bet all of his wealth on each roll of the dice would be estimated to receive

5 See page 110.

a long-run geometric average return of approximately 1.08 . In fact, however, he would lose all of his wealth at the first unfavorable occurrence so that the long-run geometric average return is zero. Equation (4.1) must be used to calculate G with this distribution of returns, and equation (4.2) must be used to calculate q_{max}.

A portfolio manager may form probability beliefs about the variance and arithmetic mean of the probability distribution of returns and wish to allocate his portfolio on the basis of these beliefs. He can use equation (4.7) to derive an estimate of the proportion of the portfolio to put into stock at the beginning of each of the next n years, n being large, to maximize portfolio returns if the mean and variance of the forthcoming n returns actually do agree with the mean and the variance believed most likely to occur. In this manner he will maximize his chance of obtaining a higher portfolio return than can be obtained by any other plan if his probability beliefs are correct and provided that the underlying distribution of returns from stock is not badly skewed.

Effects of Variance on Geometric Mean Return

The effects of variance on estimated geometric mean returns are shown in Table 4.9. This table is based entirely on hypothetical data. It shows the relationship between the arithmetic means and variances of sets of returns from stock on the one hand, and, on the other hand, the estimated geometric mean return, G', from portfolios allocated either in whole or in part to that stock. The table shows an array of estimated geometric mean portfolio returns based on sets of returns from stock with hypothetical combinations of sample means and variances. The sample

Table 4.9

Arrays of Estimated Geometric Mean Portfolio Returns

The return from bonds, C, is 1.04 per annum while the returns from stock have a sample arithmetic mean of $\bar{R}$, with $\bar{R}$ = 1.04, ..., 1.10, and a sample variance of s^2, with s^2 = .24, ..., .00 .

A. All Stock Portfolio

q = 1.0

$s^2 \backslash \bar{R}$	1.04	1.05	1.06	1.07	1.08	1.09	1.10
.24	.918	.928	.940	.951	.962	.974	.985
.20	.939	.950	.961	.972	.983	.994	1.005
.16	.960	.971	.982	.992	1.003	1.014	1.025
.12	.981	.991	1.002	1.012	1.023	1.033	1.044
.08	1.001	1.011	1.022	1.032	1.042	1.052	1.063
.04	1.021	1.031	1.041	1.051	1.061	1.071	1.082
.00	1.040	1.050	1.060	1.070	1.080	1.090	1.100

B. Diversified Portfolio

q = .5 at beginning of each year

C = 1.04

$q^2s^2 \backslash .5\bar{R}+.5C$	1.040	1.045	1.050	1.055	1.060	1.065	1.070
.06	1.011	1.016	1.021	1.026	1.031	1.036	1.042
.05	1.016	1.021	1.026	1.031	1.036	1.041	1.046
.04	1.021	1.026	1.031	1.036	1.041	1.046	1.051
.03	1.026	1.031	1.036	1.041	1.046	1.051	1.056
.02	1.031	1.036	1.040	1.046	1.051	1.056	1.061
.01	1.036	1.041	1.045	1.051	1.056	1.061	1.065
.00	1.040	1.045	1.050	1.055	1.060	1.065	1.070

B - A — Difference in Return Between B and A

$s^2 \backslash \bar{R}$	1.04	1.05	1.06	1.07	1.08	1.09	1.10
.24	.093	.088	.081	.074	.069	.062	.057
.20	.077	.071	.065	.059	.053	.047	.041
.16	.061	.055	.049	.044	.038	.032	.026
.12	.045	.040	.034	.029	.023	.018	.012
.08	.030	.025	.018	.014	.009	.004	-.002
.04	.015	.010	.004	.000	-.005	-.010	-.017
.00	.000	-.005	-.010	-.015	-.020	-.025	-.030

means range from 1.04 to 1.10 and the sample variances range from .24 to 0. Table 4.9A shows the estimated geometric mean returns for a portfolio allocated entirely to stock. For example, the upper left hand corner shows the G' for n years for a portfolio allocated entirely to stock which has a set of yearly returns during the n years with an arithmetic mean of 1.04 and a variance of .24 . Using equation (4.7) this geometric mean is estimated to be

$$G'^2 = (R^2 - s^2)^{.5} = (1.04^2 - .24)^{.5} = .918 .$$

Table 4.9B shows the geometric mean returns for a portfolio allocated at the beginning of each year half to bonds returning 1.04 and half to stock with the same returns as in A. For example, the upper left corner of Table B shows the G' from a portfolio divided half in bonds and half in stock having a set of returns with an arithmetic mean of 1.04 and a variance of .24 . The variance of a set of portfolio returns from a portfolio allocated .5 to a safe asset and .5 to stock with a variance of .24 is .06 and the estimated geometric mean return from the portfolio is 1.010. The estimated geometric mean portfolio return, when stock returns have a sample mean of 1.04 and a sample variance of .24, thus is .92 when the portfolio is allocated entirely to stock and 1.01 when the portfolio is allocated .5 to bonds and .5 to stock. This difference is .09 in favor of the diversified portfolio. Corresponding differences are tabulated for other sample means and variances in the last section of Table 4.9 .

Table 4.9 shows that, in spite of the fact that the bond return is no larger than the lowest arithmetic/stock return, the estimated geometric mean return over n years from the diversified portfolio is larger than the estimated mean return from the all-stock portfolio for many of the combinations of stock returns and variances. The table also indicates that, if the arithmetic mean return from stock is sufficiently large, geometric mean returns from an all stock portfolio are larger than the geometric mean returns from a half stock-half bond portfolio.[6] For example, when bonds return 1.04 and stock has a sample mean return of 1.08, the estimated geometric mean return from an all stock portfolio is larger than the G' from an equally apportioned portfolio even when the variance of stock returns is as large as .04 (that is, when the standard deviation is as large as .20). Table 4.9 indicates that the G' from the all stock portfolio is 1.061 under these conditions while the G' from the half stock portfolio is only 1.056.

It is indicated in Table 4.9 that the wealth-holder who feels confident that stock returns in the forthcoming n years will have an arithmetic average of between 1.04 and 1.09 and will have a variance of .06 or over can feel at least equally confident that a portfolio divided half in stock and half in bonds will give a better return than a portfolio

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6 The table shows only two portfolios: the all stock portfolio (i.e., $q = 1.0$) and the half stock-half bond portfolio (i.e., $q = .5$). The allocations to stock which actually would give the highest geometric mean return under the specified conditions (i.e., q_{max}) might fall between .5 and 1.0 or might be less than .5 or greater than 1.0 . In most cases where the all stock portfolio gives a higher geometric mean return than the half stock portfolio, returns on the equity in the portfolio would be maximized by borrowing on margin to buy stock (i.e., $q_{max} > 1$). See page 104.

placed entirely in stock.[7] When the wealth-holder knows the mean and variance of the distribution of all possible stock returns (as in honest games of chance) he also knows that the forthcoming set of returns may have a geometric mean higher than the most likely geometric mean and, hence, that the all-stock portfolio may give a higher return than the diversified portfolio. This may happen either because the arithmetic mean of the forthcoming set of returns is higher than the most likely arithmetic mean or the variance of the set of returns is lower than the most likely variance, or both. Such a wealth-holder can estimate the proportion of occasions on which either event may occur. When the wealth-holder is not certain of the mean and variance of the distribution of all possible risk asset returns, he will be unable to state after the unexpected occurrence whether the results came about by chance or because his probability beliefs were wrong.

Chapter Summary

This chapter has dealt with the problem of discovering what proportion of a portfolio to allocate to a risk asset in order to maximize G, the geometric mean of the probability distribution of portfolio returns. The proportion of the portfolio to place in the risk asset at the beginning of each year in order to maximize G is called q_{max}. In terms of the whole probability distribution of returns from the risk asset, G and q_{max} are

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7 A wealth-holder with such beliefs would be less certain about the advantages of holding a half-stock portfolio as against the advantages of holding an all-bond portfolio. For a number of states of nature within his confidence range it would be better to hold only bonds yielding .04 .

determined by equations (4.1) and (4.2), that is:

$$(4.1)\quad \log G_q = p_1 \log[C + q(R_1 - C) + \dots + p_j \log[C + q(R_j - C) + \dots + p_k \log[C + q(R_k - C)]$$

$$(4.2)\quad p_1 \frac{R_1 - C}{C + (R_1 - C)q_{max}} + \dots + p_j \frac{R_j - C}{C + (R_j - C)q_{max}} + \dots + p_k \frac{R_k - C}{C + (R_k - C)q_{max}} = 0 \;.$$

These equations involve the whole probability distribution of returns and, consequently, are often difficult to solve except, possibly, by trial and error. It is possible, however, to develop equations which give good estimates of G and q_{max} for many probability distributions of returns from the risk asset.[8] These equations depend on the arithmetic mean, $\bar{R}$, and variance, s^2, of the distribution of returns from the risk asset and are as follows:

$$(4.6)\qquad G_q^2 \approx G'^2_q = q\bar{R} + (1 - q)C - (qs)^2$$

$$(4.7)\qquad q'_{max} \approx q_{max} = \frac{(\bar{R} - C)C}{s^2 - (R - C)^2} \;.$$

8 The problem of forming probability beliefs about returns from portfolios based on probability beliefs about individual stocks also may be simplified if it can be approached by means of standard returns in gain years and loss years. The standard return in a gain year is $(R + s)$ and the standard return in a loss year is $(R - s)$, with gain years and loss years equally probable. This problem is not part of the subject matter of this dissertation.

In the last section of the chapter a hypothetical numerical example is worked out to show the effects of variance on the estimated geometric mean returns from portfolios either entirely or half allocated to stock. This illustration is abstract but does demonstrate clearly that forming proper implicit or explicit beliefs as to variance is an essential part in rational portfolio allocation.

CHAPTER V

DEFINITIONS AND APPLICATIONS

In previous chapters the subgoals, criteria, and methods of rational portfolio management have been illustrated by the use of hypothetical payout matrices. In the first part of this chapter, an attempt will be made to explore some of the implicit and explicit assumptions underlying the payout matrices. Later in the chapter, <u>ex post</u> returns from stock and bonds in two reference periods will be used to show the hypothetical results of maximum chance portfolio allocation.

Both portfolio management and gambling often involve (1) returns which occur in series over time; (2) some reinvestment of returns so that a series of gains and losses is compounded; (3) some risk which cannot be eliminated by diversification; (4) repeated exposure to approximately the same risks time after time; (5) willingness and ability of the portfolio manager to take proper maximizing action; and (6) other restrictions. Each of these assumptions will be discussed in turn.

Returns from Stocks and Bonds Defined

Returns from stock and bonds occur in series over time. The exact determination of these returns for any one investment period is the subject of this section.

Both stocks and bonds are held in anticipation of a series of money payments to the owner over time. The series of payments for a bond held to maturity consists of the annual or semi-annual interest payments plus the principal amount at maturity. With high-grade bonds actual payment of the promised amounts is considered highly probable (hence the "high grade" rating) and the estimated yield to maturity closely approaches the promised yield to maturity. When the bond is sold before maturity, the series of payments is the interest receipts plus the estimated sales price. The estimated yield to planned sales date may differ widely from the indicated maturity yield. The estimated return (yield plus one) for the long-term bond held for one year is the interest receipts during the year plus estimated sales price at the end of the year per dollar of cost at the beginning of the year.

Stocks are held in anticipation of a series of dividend payments plus sales price at the end of the holding period. Both the future sales price and the future dividend receipts have to be estimated. The relative importance of the two estimates depends on the length of the planned holding period. Stocks may be thought of as long-term investments and be valued exclusively on the basis of anticipated dividend receipts,[1] or, on the other hand, anticipated price changes may receive

1 John Burr Williams, Theory of Investment Value (Cambridge: Harvard University Press, 1938). On page 6, Williams defines investment value as "the present worth of the future dividends in the case of a stock, or of the future coupons and principal in the case of a bond." He gives elaborate formulas for estimating the present value of expected dividends from "stocks with growth completed," "stocks with growth expected," as well as from stocks with other time-shapes of expected dividends.

major consideration.[2] Williams[3] divides stock buyers into investorsand speculators depending on whether they give primary attention to future dividends or future market price changes. However, the wealth-holder who looks for long-term growth in dividends, for example, also tends to look for market price appreciation, and the speculator who is concerned with price changes may well base his anticipations as to prices on anticipated earnings and dividends. Consequently, the rational "speculator," interested in capital growth, and the rational "investor" interested in dividends, may adopt the same investment plan.

It is agreed generally that the income of a firm for any given year consists of the gain in net worth for the year plus any withdrawals. The return for the year is the net worth at the end of the year plus withdrawals during the year per dollar of net worth at the beginning of the year. This holds true whether or not the firm holds assets with a productive life of greater than one year. The difficulty in measuring the rate of return for any particular firm-year arises largely because of the difficulty of determining the net worth at the beginning and at the end of the year. In like fashion, the <u>ex post</u> return from a stock for a specified year for both speculators and investors is the value at the end of the year plus dividend receipts per dollar value at the beginning of the year. The return from stock anticipated by a long-term investor, who thinks of stock as a permanent investment and disregards

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2 The "Dow Theory," for example, deals with price trends exclusively and disregards dividends. See <u>Barron's</u> for the current comment of "The Dow Analyst."

3 Williams, <u>op. cit.</u>, p. 4.

market prices, may differ widely from the return from the same stock based on anticipated market prices at the beginning and end of the year merely because of the different bases used in calculating the returns.

In this dissertation stock and bond holdings are not valued as permanent investments but are valued at going market prices at the beginning and end of each year. The ex post return from stock for a year is defined as the dividends received during the year plus market price at the end of the year, per dollar invested at the beginning of the year. This definition of returns combines market gains and losses and dividend receipts. It permits exact calculation of ex post returns for any year and permits comparison of these returns with other time series such as interest rates. In setting up a portfolio management model it is essential to have a clearly defined definition of returns. Returns are here considered a series of payouts occurring over time. The portfolio manager forms beliefs about the probability distribution of this series and wishes to allocate his portfolio on the basis of these beliefs.

Reinvestment of Returns

It is essential that there be some compounding of a series of gains and losses in order to justify the use of the geometric mean of the probability distribution of portfolio returns as a criterion. When there is no compounding, each gain and loss stands separately and total wealth after a series of such gains and losses is the sum, not the product, of the individual returns. Much gambling and practically all portfolio management involves some compounding of a series of gains and losses. In order to state the ultimate objective in the simplest terms, it has been postulated that the wealth-holder wishes to maximize his wealth at the end of a long series of gains and losses assuming reinvestment of all returns.[4]

This goal still is a valid objective even if all returns are not reinvested or if funds are added to the portfolio from time to time. Let a_i, with $i = 1, \ldots, n$, be the actual portfolio return in the i^{th} forthcoming year- let $(1 - x_i)$ be the proportion of the portfolio withdrawn so that x_i is the proportion retained;[5] and let W_i be the portfolio value at the end of the i^{th} year. Then the portfolio value at the end of one year is $W_1 = W_0 a_1 x_1$, that is, the wealth at the end of one year equals the initial wealth times the return times the proportion retained. Likewise, the wealth at the end of two years is

$$W_2 = W_1 a_2 x_2 = W_0 a_1 x_1 x_2 ,$$

and the wealth at the end of n years is

$$(5.1) \qquad W_n = W_0 \prod_{i=1}^{n} a_i \prod x_i \quad .$$

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4 The goal is stated in terms of total wealth at one date rather than in terms of a set of payments at various future dates in order to avoid the problem of time discounting. See Armen A. Alchian, "The Rate of Interest, Fisher's Rate of Return over Costs and Keynes' Internal Rate of Return," American Economic Review, 45 (December 1955), 938-943, for a discussion of bases for comparing various time-shapes of anticipated returns. The following quotation may apply: "If the time paths of the net receipts of the compared options are identical (except for a proportionality factor) the Keynesian internal rate of return ranking will agree with Fisher's maximum wealth criterion. In other words, in order to have the Keynesian ranking agree with Fisher's, either we must assume exactly similar time paths, or we must assume the net receipts from the two alternatives can be immediately and perpetually reinvested at their own internal rates of return."

5 x_i will be greater than 1 if funds are added to the portfolio.

In words, the portfolio value at the end of n years is a product of the initial wealth, the returns for each of the n years, and the proportion of the portfolio value retained in each year. For any given set of x_i, with $i = 1, \ldots, n$, the portfolio value at the end of n years is maximized when a_i is maximized. Rational spending policy—that is, the determination of x_i—is not the subject matter of this dissertation. It is assumed that the proportion of the portfolio value withdrawn during the i^{th} year, with $i = 1, \ldots, n$, is held constant or is otherwise specified. When x_i is specified, the portfolio manager who maximizes a_i (or the n^{th} root of a_i, which is the forthcoming geometric mean portfolio return) also will maximize the portfolio value at the end of n years.

In real-life portfolio management often the dividends and interest receipts are withdrawn and the remainder of the returns (i.e., the capital gains and losses) are reinvested. This corresponds to the situation above. The wealth-holder who plans to withdraw all cash dividends and interest, which he estimates to be a fixed proportion of his portfolio value so that $x_i = x_1$, with $i = 1, \ldots, n$, and who wishes to maximize the value of his portfolio at the end of a number of years is justified in adopting exactly the same investment plan as the wealth-holder who maximizes his wealth assuming reinvestment of all returns.

Risk Which Cannot Be Eliminated

In portfolio management the standard deviation of the probability distribution of portfolio returns often can be reduced, without lowering the mathematical expectation of the distribution, by proper diversification among the underlying securities. When returns from a group of stocks fluctuate together, however, it is impossible to eliminate all risks. This dissertation attempts to deal with risks which cannot be eliminated by diversification--that is, it deals with choices among whole portfolios.

Choice Among Efficient Portfolios

The problem of the allocation of portfolios among risk assets is not the subject matter here. Markowitz has outlined a basis for determining a set of efficient portfolios[6]--that is, portfolios with minimum variance for a specified expected return and with maximum kexpected return for a specified variance--on the basis of the expected returns, the variances, and the covariances of the available risk assets. Equation (4.4) may be used to choose the maximum chance portfolio from among these efficient portfolios. Figure 5.1(a) is a scatter diagram with the dots representing the expected returns, A_i, with $i = 1, \ldots, n$, and standard deviation, s_i, of all possible portfolios. The line with a positive slope marking the lower boundary of this set of dots represents the expected returns and standard deviations of returns from the efficient portfolios. The expected returns and standard deviations of returns of the efficient sets necessarily fall on a line (but not necessarily a straight line) because for any one specified expected return, A, there is only one minimum variance, s^2, and correspondingly, for any one specified variance there is only one maximum expected return. Figure 5.1(b) shows the estimated geometric mean returns, G', and expected returns from the

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6 Harry Markowitz, "Portfolio Selection," Journal of Finance, VII (March 1952), 77-91.

Figure 5.1(a)

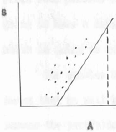
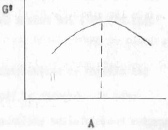


Figure 5.1(b)



efficient portfolios shown on Figure 5.1(a). G' is derived from equation (4.5) with $G_1'^2 = A_1^2 - s_1^2$. The maximum chance portfolio is the efficient portfolio with the highest G'.

Recurrent Risks

In order to illustrate the problem of recurrent risks the portfolio manager will be compared in this section with a professional gambler who is betting on a simple game such as dice. The gambler knows the odds and believes that he will be able to take the same type of risk time after time. In like fashion, it is postulated that the portfolio manager forms probability beliefs about forthcoming returns from risk assets. Such probability beliefs may be stated impbicitly or explicitly in the form of a payout matrix of returns. It is not necessary to separate probability as a measure of relative frequency from probability as a measure of degree of belief. The following is an illustration (used later in this chapter) of a portfolio manager's probability beliefs as to returns from stocks and bonds: "I look for conditions in the next ten years to be very similar to those prevailing in 1926 through 1935. I am certain that bonds will yield five percent per annum during the whole period. Some day we are going to have a boom and a bust in the stock market but I do not know which is going to come first."

The gambler knows the probability distribution of payoffs and knows that he will be told before this payoff is changed. In like manner the portfolio manager knows his probability beliefs about returns and will know when he changes his beliefs. Just as the gambler wants to find the best betting plan given the present odds, so does the portfolio manager want to find the best investment plan given his present probability beliefs about returns.

It is not necessary for the professional gambler to be convinced that he will have unlimited opportunity to play the same game on the same odds in order to justify his use of the maximum chance criterion for betting. He may recognize that he will have only one or few chances to bet at this time but thinks it likely that such opportunities will recur. He wants to maximize his chance of doing better than can be done with any other betting plan in a long series of such recurrent opportunities. In other words, the series of bets on game A may be interrupted by bets on games B, C, ..., N. In the long run he will maximize his chance of wealth at the end of a series of bets on such games if he maximizes his chance of wealth from each game separately. The situation in regard to portfolio management is exactly analogous. It is only necessary for the portfolio manager to recognize that there are recurrent opportunities to buy risk assets and that his portfolio will be exposed to the same general type of risk time after time.

Whether rational probability beliefs about returns from stock are apt to remain relatively stable over time or are apt to fluctuate when priceschange is not at issue here. Whether such beliefs fluctuate widely from period to period or whether they tend to be stable, the rational portfolio manager is interested in maximizing his chances of having a greater portfolio value than can be obtained by any other plan over a series of gains and losses and will adjust his investment plan accordingly. In similar fashion, the rational gambler would bet the same proportion of his wealth on the next toss whether he expected to play game A for a long period and then shift to game B or whether he mixes up the series of games.

Proper Maximizing Action

The rational portfolio manager like the rational gambler must actually carry out the plan which most nearly fulfills his objectives in the light of his probability beliefs. A proper series of maximizing actions does not include "letting the profits ride" either in the gambling situation or in portfolio management. This action probably will lead to poorer results than can be obtained by adopting the maximum chance plan. The gambler must bet the maximizing proportion, q_{max}, of his assets on each toss. If he won the previous toss this involves some holding of winnings in the form of cash; if he lost, some money must be added to the amount in play. In like fashion, the portfolio manager must adjust his portfolio so that he holds that proportion in stock at the beginning of each year which is correct in the light of his probability beliefs. If his beliefs change, the maximizing proportion also changes but otherwise it remains constant.

At the beginning of each year, the portfolio manager reviews his portfolio and readjusts it in the light of the probability beliefs about returns from securities which he holds at that time. Presumably his probability beliefs about returns will change to some extent from year to year and, consequently, it seems reasonable to anticipate some changes in the composition of the portfolio from year to year and some shift from safe assets to risk assets and vice versa. On the other hand, if the probability beliefs as to returns remain unchanged from year to year, the maximizing proportion to hold in the risk asset, q_{max}, will remain constant. For example, if the probability beliefs at the beginning of one year, year i, are such that the maximum chance allocation of the portfolio is 40 percent in bonds and 60 percent in stock and these beliefs remain the same at the beginning of year i + 1, then the maximum chance allocation again will be 40 percent in bonds and 60 percent in stock at the beginning of the year i + 1. If the relative prices of stocks and bonds have changed between the two dates, it will be necessary for the portfolio manager to make some sales and purchases in his portfolio to bring it into line with the desired proportions even if the proportions themselves have not changed.[7] Under this plan of action, the

7 Allocating a fixed proportion of the portfolio to risk assets at the beginning of each year is somewhat similar in effect to the constant ratio, or equalizing, formula investment plan. In this type of investment plan, the total fund is divided initially into determined percentages of aggressive and defensive securities (e.g., 50 percent

portfolio manager whose probability beliefs about returns from stocks and bonds remain unchanged exposes the same proportion of his portfolio to the same risk time after time.

Reference Period Returns

The years 1926-1935 and 1890-1899 may be used as two reference periods to illustrate the results of portfolio diversification. Actual stock returns for these periods are tabulated in Tables 5.1 and 5.2. These returns are based on a very high proportion of the total value of all stocks listed on the New York Stock Exchange. They are derived from a series representing stock prices including cash dividends developed by Alfred Cowles and published by the Cowles Commission.[8] This series includes reinvested dividends so the return per dollar invested in the year t is equivalent to the ratio of the index for the year t + 1 to the index for the year t.

Returns from stock for the 10 individual years in the 1926-1935 period varied from 1.47 to .55 per annum per dollar invested. The

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stock-50 percent bonds) and then the aggressive and defensive funds are adjusted periodically to restore these percentages. However, the adjustment schedule usually depends on changes in relative values rather than time. For example, a 50 percent stock fund may be restored to the desired ratio by appropriate purchases and sales whenever the stock proportion falls to 45 percent or rises to 55 percent. This dissertation is concerned with the problem of determining a proper allocation between risk assets and safe assets based on beliefs about forthcoming returns. It is not an attempt to find a formula which will assure satisfactory investment performance by arbitrary adjustments. See J. Fred Weston, "Some Theoretical Aspects of Formula Timing," Journal of Business of the University of Chicago, XXII (1949), 250-263, for a discussion of some of the assumptions underlying formula investment plans.

8 Alfred Cowles and Associates, Common Stock Indexes (Bloomington, Indiana: Principia Press, 1939), pp. 168, 169.

Table 5.1

<u>Ex Post</u> Stock Returns and Hypothetical Portfolio Returns

1926-1935 Reference Period

	Stock Price Index	Stock Return q=1.0	Hypothetical Portfolio Returns (a_q)			Hypothetical Cumulative Portfolio Value (W) Jan. 1, 1926 = 1.0			
	S	R	q=.4	q=.6	q=.8	q=.4	q=.6	q=.8	q=1.0
1926	459	1.25	1.13	1.17	1.21	1.13	1.17	1.21	1.25
1927	572	1.32	1.16	1.21	1.27	1.31	1.42	1.54	1.65
1928	756	1.31	1.15	1.21	1.26	1.51	1.71	1.94	2.16
1929	993	.82	.96	.91	.87	1.45	1.56	1.69	1.77
1930	813	.66	.89	.82	.74	1.29	1.28	1.25	1.17
1931	540	.55	.85	.75	.65	1.10	.96	.81	.64
1932	297	1.37	1.18	1.24	1.31	1.29	1.19	1.06	.88
1933	408	1.19	1.11	1.13	1.16	1.44	1.34	1.23	1.05
1934	486	1.12	1.08	1.09	1.11	1.55	1.46	1.37	1.17
1935	546	1.47	1.22	1.30	1.39	1.89	1.91	1.89	1.73
1936	804								
Arithmetic Mean (A)		1.106	1.073	1.083	1.097				
Geometric Mean (G)		1.057	1.065	1.067	1.065				
Variance $(qs)^2$		.093	.015	.033	.060				
$G' = \sqrt{A^2 - (qs)^2}$		1.063	1.064	1.069	1.068				

S : Stock Prices Including Cash Dividends, Alfred Cowles and Associates, <u>Common Stock Indexes</u> (Bloomington, Indiana: Principia Press, 1939), <u>Table C-1</u>, pp. 168-169.

R_t : S_{t+1}/S_t

C : 1.05

a_q : $qR + (1 - q)C$

W_t : $(W_{t-1})a_{qt}$

arithmetic mean of these returns was 1.06 and the geometric mean was 1.057. The latter figure indicates the average annual long-term return from one cross-section share of stock held over the whole period with all dividends reinvested. That is, an average annual yield of 5.7 percent would have been obtained if one cross-section share of stock had been held for the whole ten years. This continuous holding results in a higher dollar investment in stocks when prices prove to be high on an ex post basis and yields are low or negative and a lower dollar investment when ex post prices are low and yields are high. The long-term stock return, that is yield plus one, is equivalent to the geometric average of the one year returns. The arithmetic average of the yields for one year investment periods is higher than the long-term yields. It shows what the average results would have been if exactly the same amount of money had been invested in stocks in each of the years and held in stocks for one year.

Hypothetical returns and cumulative values for portfolios with specified proportions put in stock are also shown in Tables 5.1 and 5.2. For example, the first row of the table shows that a portfolio allocated 40 percent to stock (i.e., $q = .4$) and 60 percent to bonds yielding 5 percent per annum would have produced a portfolio return of 1.13 in 1926 and would have had a value of 1.13 at the end of that year if all returns are reinvested (the value on January 1, 1926, being taken as 1). The table shows also the arithmetic means, the geometric means, and the variances of the portfolio returns, as well as G' which is an estimate of G based on the arithmetic mean and variance of the returns.

Table 5.1 shows that a portfolio adjusted at the beginning of each year to 40 percent in bonds yielding 5 percent per annum and 60 percent in stock at the actual yields prevailing in 1926-1935 would have yielded approximately 6.7 percent per annum for the period as a whole, rather than the 5.7 percent which would have been obtained if all of the portfolio had been concentrated in stocks or the 5 percent which would have been obtained if all of the portfolio had been held in bonds. In other words, diversification would have increased yields from 5.7 percent for the all stock portfolio and 5 percent for the all bond portfolio to 6.7 percent for the diversified portfolio in this period.

The hypothetical portfolio returns shown in Tables 5.1 and 5.2 are based on the assumption that the wealth-holder kept the same probability beliefs as to returns from stock throughout the period and so adjusted his portfolio to hold the same proportion, q, with $q = .4, \ldots, 1.0$, in stock and $(1 - q)$ in bonds at the beginning of each of the ten reference years. It is an attempt to give a numerical example of the results of maximizing behavior based on correctly knowing the returns for a set of ten forthcoming years but not the order of occurrence. In real life, there is little doubt that the wealth-holder would have changed his probability beliefs at some point. These changes could have been for the better, so that he would have held a bigger proportion of stock when stock proved to be low on an <u>ex post</u> basis; or for the worse, so that the wealth-holder increased his stock holdings at the peak prices and got more conservative at the bottom of the depression. A wealth-holder who changed his beliefs in a correct direction so that he could recognize the opportunity to buy more stock when they afterwards proved to be low, and recognized the opportunity to sell more when they proved to be high could get a bigger portfolio return than that obtained from a constant proportion placed in stock. Even such a wealth-holder would need to know how to reallocate his portfolio after he changed his beliefs about forthcoming returns.

Negative Bond Holding

So far it has been assumed that the proportions of the portfolio placed in bonds and stock at the beginning of each year are positive, though there are no restrictions of this nature in the various equations. Henceforth, it is assumed that a wealth-holder owns the equity in a group of securities and this equity will be called his portfolio. A wealth-holder's portfolio includes not only his securities but also the associated debt, if any. The proportion of the portfolio, as thus defined, held in bonds can be either positive or negative. Negative bond holding corresponds to borrowing to buy stock. In all cases the proportion of the net value of the portfolio in bonds, plus the proportion in stock add to one (i.e., $p + b = 1$). The proportion, q, of the net portfolio held in stock can be adjusted over a very wide range. A negative value for q corresponds to selling stock short and holding the proceeds in cash or in bonds. A value of q greater than 1 implies negative bond holding, that is, holding stock on margin. In practice, some portfolio managers may not be able to sell stock short or hold stock on margin, but either option often is available.

The stock returns for 1890-1899, shown in Table 5.2, may be used to give a further example of the use of both stock and bonds (that is, borrowing) to obtain a larger yield on the net value of the portfolio

Table 5.2

Ex Post Stock Returns and Hypothetical Portfolio Returns

1890-1899 Reference Period

	Stock Price Index S	Stock Return q=1.0 R	Hypothetical Portfolio Returns (a_q) q=2.0	q=3.0	q=4.0	Hypothetical Cumulative Portfolio Value (W) Jan. 1, 1890 = 1.0 q=2.0	q=3.0	q=4.0
1890	320	1.00	.96	.93	.90	.96	.93	.90
1891	319	1.15	1.27	1.39	1.51	1.22	1.28	1.36
1892	367	.90	.77	.64	.51	.94	.82	.69
1893	330	.96	.90	.83	.77	.84	.69	.52
1894	318	1.08	1.13	1.18	1.23	.95	.81	.64
1895	343	.97	.92	.86	.81	.87	.70	.52
1896	334	1.10	1.17	1.24	1.31	1.02	.86	.68
1897	367	1.18	1.33	1.48	1.63	1.36	1.27	1.10
1898	433	1.29	1.55	1.81	2.07	2.11	2.31	2.29
1899	559	1.02	1.00	.99	.97	2.11	2.28	2.23
Arithmetic Mean (A)		1.065	1.100	1.135	1.171			
Geometric Mean (G)		1.059	1.077	1.086	1.083			
Variance $(qs)^2$		.014	.055	.123	.220			
$S' = \sqrt{A^2 - (qs)^2}$		1.058	1.074	1.080	1.072			

S : Stock Prices Including Cash Dividends, Alfred Cowles and Associates, Common Stock Indexes (Bloomington, Indiana: Principia Press, 1939), Table C-1, pp. 168-169.

R_t : S_{t+1}/S_t

C : 1.03

a_q : $qR + (1 - q)C$

W_t : $(W_{t-1})a_{qt}$

than can be obtained either from an all stock or from an all bond portfolio. In those reference years, a portfolio adjusted at the beginning of each year so that the investor borrows \$2.00 at 3 percent for each of his own dollars and invests all in stock, would yield 8.6 percent on the equity as against an arithmetic stock yield of 6.5 percent and a geometric average return from stocks equivalent to a yield of 5.9 percent. It is to be noticed that the small variance in the 1890-1899 set of stock returns, as reflected in the small difference between the arithmetic and the geometric average returns, makes such large scale borrowing possible and profitable. In the 1926-1935 period, borrowing on the same scale would have resulted in a return of zero in one or more years and the consequent elimination of the portfolio. A lot of people jumped out of the window in 1929 and 1930 because they underestimated the variance of stock returns for that period.

These results, in themselves, indicate that it is not necessary to appeal to individual risk preference or other utility considerations to justify portfolio diversification, that is, the use of both stocks and bonds or borrowing. Allocation of a portfolio between a risky security giving a high expected yield and a safe security giving a lower certain yield usually is justified by a gain in portfolio value at the end of n years provided that the specified proportions are correctly chosen. In some instances the maximizing proportion to be allocated to stock (i.e., q_{max}) may be 1 or 0. This does not affect the general conclusion.

Empirical Tests of Formulas

In Chapter IV two methods were described for calculating the geometric means, G_q, of the probability distributions of portfolio returns and for selecting the proportion, q_{max}, to allocate to stock in order to maximize G_q. One method (equations 4.1 and 4.2) involved using the full probability distribution of returns from stock as a basis for calculating G_q and q_{max}. The second method (equations 4.3 and 4.4) is much simpler and is based solely on the arithmetic mean and variance of the probability distribution of returns from stock. In this section the reference period returns are used to compare results from using these two methods.

A wealth-holder may have probability beliefs such as: "I look for conditions in the next ten years to be very similar to those prevailing in 1926 through 1935. I am certain that bonds will yield five percent per annum during the whole period. Some day we are going to have a boom and a bust in the stock market, but I do not know which is going to come first." These beliefs may be stated in terms of the payout matrix of returns shown in Table 5.3. It should be noted that the usual content of rows and columns is reversed in this matrix. The first column lists the possible future occurrences, that is, the occurrence of a year such as 1926, ..., 1935, and the last column lists the probability of each such occurrence (i.e., .1). The middle five columns show the matrix of returns for each possible future occurrence when .0, ..., 1.0 of the portfolio is allocated to stock.

In Table 5.3, the column headed .4, for example, shows returns from a portfolio divided .4 in stock and .6 in bonds if a series of years

Table 5.3

Payout Matrix of Returns

Occurrence of Year Such As:	Proportion Allocated to Stock .0	.4	.6	.8	1.0	Probability of Occurrence
1926	1.05	1.13	1.17	1.21	1.25	.1
1927	1.05	1.16	1.21	1.27	1.32	.1
1928	1.05	1.15	1.21	1.26	1.31	.1
1929	1.05	.96	.91	.87	.82	.1
1930	1.05	.89	.82	.74	.66	.1
1931	1.05	.85	.75	.65	.55	.1
1932	1.05	1.18	1.24	1.31	1.37	.1
1933	1.05	1.11	1.13	1.16	1.19	.1
1934	1.05	1.08	1.09	1.11	1.12	.1
1935	1.05	1.22	1.30	1.39	1.47	.1
Arithmetic Mean (A_q)	1.05	1.073	1.083	1.097	1.106	
Geometric Mean (G_q)	1.05	1.065	1.067	1.065	1.057	

such as 1926-1935 occurs. The arithmetic mean of the probability distribution of returns, A_q, with $q = .0, \ldots, 1.0$, increases as q increases. When $q = 1$, that is, when the portfolio consists of stock, $A_q = \bar{R}$. The geometric mean, G_q, apparently reaches a peak when $q = .60$. The wealth-holder, with probability beliefs such as those reflected in the matrix, would maximize A by holding an all stock portfolio. He would maximize G by allocating approximately .6 of his portfolio to stock and .4 to bonds at the beginning of each year.

Table 5.4 shows a payout matrix of returns for the portfolio manager who bases his probability beliefs on the actual returns from stock during the 1890-1899 reference period. Such a portfolio manager assumes that the forthcoming return is equally likely to be like that of any one of these ten years. He believes, for example, that there is a .1 probability of the occurrence of such a year as 1896 in which stocks returned 1.10. This also is the portfolio return when the proportion of the stock allocated entirely to stock is 1.0. The return on the net portfolio when a year such as 1896 occurs and when the proportion allocated to stock is 2.0 (shown in the second column of the matrix) is 1.17.[9]

Table 5.4

Payout Matrix of Returns

Occurrence of Year Such As:	Proportion Allocated to Stock 1.0	2.0	3.0	4.0	Probability of Occurrence
1890	1.00	.96	.93	.90	.1
1891	1.15	1.27	1.39	1.51	.1
1892	.90	.77	.64	.51	.1
1893	.96	.90	.83	.77	.1
1894	1.08	1.13	1.18	1.23	.1
1895	.97	.92	.86	.81	.1
1896	1.10	1.17	1.24	1.31	.1
1897	1.18	1.33	1.48	1.63	.1
1898	1.29	1.55	1.81	2.07	.1
1899	1.02	1.00	.99	.97	.1
Arithmetic Mean (A_q)	1.065	1.100	1.135	1.171	
Geometric Mean (G_q)	1.059	1.077	1.086	1.083	

9 Under these conditions the wealth-holder borrows $100 at an interest cost of $3 and holds $200 in stock for each $100 of net portfolio

This matrix indicates that G_q is approximately maximized when $q_{max} = 3.0$, that is, when \$3.00 is held in stock and \$2.00 is borrowed for each \$1.00 of net portfolio value. The geometric mean return from the portfolio with $q = 3.0$ is greater than the corresponding return from the portfolio with $q = 2.0$ or with $q = 4.0$.

As was shown in Chapter IV, probability beliefs about returns from stock often can be stated, with little loss of information, in terms of the arithmetic mean and variance of the probability distribution of returns. The wealth-holder who has the probability beliefs stated in Table 5.3, that is, who uses the 1926-1935 reference period as a basis, believes that the probability distribution of returns from stock has an arithmetic mean, $\bar{R}$, of 1.106 and a variance, s^2 , of .093 (i.e., the standard deviation $s = .305$). G_q and q_{max} can be estimated directly from these statistics by using equations (4.6) and (4.7). That is,

$$(4.6) \qquad G_q^2 \approx G'^2_q = A_q^2 - (qs)^2$$

and

$$(4.7) \qquad q_{max} \approx q'_{max} = \frac{C(\bar{R} - C)}{s^2 - (\bar{R} - C)^2} \ .$$

G'_q is compared directly with G_q at the foot of Tables 5.1 and 5.2 . $q'_{max} = .65$ when probability beliefs are based on the 1926-1935 reference period and $q'_{max} = 2.85$ when beliefs are based on 1890-1899. It is apparent that, with probability beliefs such as these, the G' and q'_{max} based on the mean and variance of returns from stocks differ little from those based on the whole probability distribution of returns.

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value. The gain from the \$200 in stock would be \$20 so the net gain per \$100 of portfolio value is \$17. Thus the return is 1.17.

Standard Returns in Gain Years and Loss Years

In the absence of the unusual, probability beliefs about returns from common stocks and from portfolios consisting of stocks and bonds often can be represented by a payout matrix such as Table 5.5.[10] In matrices such as this, common stocks are looked upon as risk assets which equally probably will return R + s, which may be called the standard return in a gain year, and R - s, which may be called the standard return in a loss year.

Table 5.5

Payout Matrix of Returns
C=1.05 R=1.106 s=.305

Proportion of Portfolio in Stock	Occurrence: Gain Year	Occurrence: Loss Year	Criteria: A	Criteria: G
1.0	1.41	.80	1.106	1.063
.65	1.28	.89	1.086	1.069
.0	1.05	1.05	1.05	1.05
Probability of Occurrence	.5	.5		

The payout matrix in Table 5.5 is based on the 1926-1935 reference period. R + s = 1.106 + .305 = 1.41 and R - s = 1.106 - .305 = .80. These values for R and s (and C = 1.05) result in a q_{max} of .65 and a G_{max} of 1.069.

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10 The matrix of returns in Table 5.5 has already been used for illustrative purposes as Table 4.5. It is repeated here for convenience in reference.

Table 5.6, below, shows a corresponding payout matrix of returns when the wealth-holder has probability beliefs based on returns in the 1890-1899 reference period.

The matrix of portfolio returns in Table 5.6 is based on the assumption that the wealth-holder can borrow to buy stocks at a net interest cost of 3 percent. The portfolio is the equity in a group of securities. To say that 4.0 proportion of the portfolio is placed in stock, for example, is to say that the wealth-holder borrows $300 out of each $400 invested in stock. G_q is maximized when $q = 2.85$ under these conditions.

Table 5.6

Payout Matrix of Returns

Proportion in Stock	Occurrence: Gain Year	Occurrence: Loss Year	Criteria: A	Criteria: G
4.0	1.642	.698	1.171	1.072
2.85	1.466	.794	1.130	1.080
2.0	1.336	.864	1.100	1.074
1.0	1.183	.947	1.065	1.058
.0	1.030	1.030	1.030	1.030
Probability of Occurrence	.5	.5		

Payout matrices such as those in Tables 5.5 and 5.6 often may be good representations of probability beliefs about returns from stock. When this is so, equations (4.6) and (4.7) are useful tools for allocating portfolios between stocks and bonds and for estimating the geometric mean of the probability distribution of portfolio returns.

Application of Formulas

A wealth-holder who adopts the maximum chance subgoal will be guided by equation (4.2) or (4.7) in determining the portion of his wealth he should risk repeatedly on the same terms. He also would use equation (4.1) or (4.6) to determine how much a risk asset or the avoidance of a risk would be worth to him. When the underlying probability distribution of returns is highly skewed, as in disaster insurance or lottery tickets, the wealth-holder would have to determine his q_{max} and G by using equations (4.1) and (4.2). When the chances of gain and loss are more evenly distributed about the mean, as is usually the situation in allocating a portfolio between stocks and bonds, equations (4.6) and (4.7)--i.e., the standard returns method--may give satisfactory results.

Whether the long formulas or the short formulas are used, the proper allocation of a wealth-holder's resources between risk and safety depends on the return from the safe asset and the level and dispersion of returns from the risk asset. The interrelations of these three factors have a considerable bearing on proper maximizing behavior. For example, the theoretical effects of risk on the demand for funds to buy stock by a wealth-holder who adopts the maximum chance subgoal is shown dramatically by comparing the effects of various assumptions as to interest rates on proper maximizing behavior by a wealth-holder faced with distributions of stock returns similar to those which occurred in the two reference periods shown in Tables 5.1 and 5.2. These effects are shown in Table 5.7 .

Table 5.7

Maximum Chance Proportion of Portfolio to be Placed in Stock with Specified $\bar{R}$ and s^2 for Various Anticipated Bond Yields

Anticipated Bond Yield	Maximum Chance Proportion (q'_{max})	
	1926–1935 Reference Period	1890–1899 Reference Period
C–1	R=1.106 s^2=.093	R=1.065 s^2=.014
.00	1.29	6.50
.03	.90	2.80
.05	.65	1.10
.07	.42	.00

Source: Equation (4.7) $\quad q'_{max} = \dfrac{(R - C)C}{s^2 - (R - C)^2}$

Table 5.7 shows the interrelations between specified C, R, and s^2, on the one hand, and q'_{max} on the other. For example, the third line in the table shows that, when bonds yield 5 percent and R and s^2 equal those of the 1926–1935 reference period, q'_{max} = .65 but when R and s^2 equal those of the 1890–1899 reference period, q'_{max} = 1.10. When the "maximum chance" portfolio manager looks for highly varying stock returns, such as those which occurred in 1926–1935, and anticipates that bonds will yield 5 percent, the table indicates that he should hold .65 of his portfolio in stock in order to maximize G—but if he anticipates that bonds will yield only 3 percent, he should hold .90 of his portfolio in stock. That is, he will hold .35 of his portfolio in bonds if he looks for bonds to yield 5 percent and .10 in bonds if he looks for bonds to yield 3 percent. Under these conditions a two point difference in anticipated interest rates would account for a difference in bond holdings equivalent to 25 percent of the net portfolio of the maximizing wealth-holder. On the other hand, when such a wealth-holder expects stability in stock returns such as occurred in 1890-1899,[11] the same difference in anticipated interest rates would account for a difference in borrowing or negative bond holding equivalent to 170 percent of his net portfolio.

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11 Table 5.6 shows the results of various interest rate assumptions on proper maximizing action under these conditions. If the "maximum chance" portfolio manager could borrow at 3 percent he would borrow 180 percent of his net portfolio and use the proceeds to hold stock (i.e., $q'_{max} = 2.80$). If he had to pay 5 percent interest he would borrow only a small amount, .1 of his net portfolio, and would hold 1.10 of his net portfolio in stock.

CHAPTER VI

SUMMARY AND CONCLUSIONS

This dissertation is concerned with rational decision making in portfolio management. Every wealth-holder who has a portfolio consisting of stocks, bonds, and cash, with a given market price can choose to continue to hold this combination of assets or can choose to hold any other combination (including holding stocks on margin) available to him at this price. Rational choice among portfolios involves forming probability beliefs about returns from portfolios and choosing among portfolios on the basis of these beliefs. Probability beliefs as to portfolio returns may be expressed in terms of payout matrices which show the probability of all relevant future occurrences and the payouts resulting from the combined effects of the holding of each available portfolio on the one hand and each relevant future occurrence on the other. The relevant future occurrences may include such possibilities as "prosperity" and "depression" or "gain year" and "loss year."

The first three chapters of the study are devoted to an analysis of rational choices among portfolios on the basis of given probability beliefs. The fact that these choices are repetitive in nature with cumulative effects is used as the key factor in developing a goal, a subgoal, and a criterion for choosing among portfolios.

The goal of portfolio management is taken to be maximization of portfolio value at the end of a period of time broken down into a large number of investment periods called years. Since it is impossible to specify which portfolio actually will be the most valuable at the end of a number of years, it is necessary to use another basis—the subgoal—for choosing among portfolios.

The subgoal is an objective which can be reached at the time of making the choice by the decision maker who has a filled-in payout matrix. For example, choice of that portfolio which has the highest mathematically-expected value at the end of a number of years (i.e., the expected-value subgoal) may be the subgoal of a portfolio manager who takes as his goal the maximization of portfolio value at the end of the same period of time. The expected-value subgoal is not accepted as a rational subgoal because, in many instances, another portfolio is almost certain to be more valuable at the end of a long period of years than the portfolio with the highest mathematical expectation of returns.

To be rational, a subgoal must be based on balanced consideration of the probabilities and payouts from all relevant future occurrences. The minimax subgoal (i.e., maximization of returns if the most unfavorable event occurs) is rejected for portfolio management for this reason. It gives weight only to unfavorable occurrences and disregards the probability of favorable occurrences. The subgoal also must be related logically to the goal. Selection of the portfolio having the probability distribution of returns with the smallest variance, for example, is rejected because such portfolios often are certain to be less valuable at the end of a number of years than other portfolios whatever the relevant future occurrences.

The maximum chance subgoal proposed in this dissertation is based on balanced evaluation of the whole payout matrix of returns and is coupled logically with the goal. It is the choice of that portfolio which has the greatest probability (P') of being more valuable than any other specified portfolio at the end of n years, n being large. It is proved that P' for that one portfolio approaches 1 as n approaches infinity. Consequently, the portfolio with highest P' when n is large is almost certain to be more valuable than any other specified portfolio in the long run. Selection of the portfolio with the maximum P' when n is large is accepted as a rational way to reach the goal of maximum long run portfolio value.

A criterion, or measure, to be maximized is needed in order to enable the decision maker to reach his subgoal. It is shown that the portfolio with the probability distribution of returns with the highest geometric mean (G) also has the greatest probability of being more valuable than any other specified portfolio at the end of n years, n being large. For this reason G is accepted as a rational criterion for choice among portfolios.

The classical writers used the mathematical expectation of the probability distribution of payouts as the basis for choice among risky ventures. Daniel Bernoulli showed that this approach sometimes gave results which seemed irrational to him. He proposed, instead, that the mathematical expectation of the utilities of the payouts be used as a basis for choice. He suggested that often the utility of a small gain or loss varies inversely with the amount of wealth possessed. When this is so, the mathematical expectation of the utilities of the payouts is maximized when the geometric mean of the probability distribution of original wealth plus or minus gains or losses (i.e., G) is maximized. For this reason he advocated the use of G as a basis for choice among risky ventures. Bernoulli's criticism of the expected value subgoal is widely accepted today but his utility function is not generally used. What portfolio to hold or what risky venture to undertake is left to individual risk preference.

The wealth-holder who adopts the maximum chance subgoal can reach this subgoal by using the geometric mean, G, of the probability distribution of returns as his criterion and by choosing that portfolio which has the probability distribution of returns with the highest G. Bernoulli also has shown that choice of that portfolio with the highest G is a rational choice if: (1) maximization of the mathematical expectation of the utilities of the payouts is a rational subgoal; and (2) if the utility of a small gain or loss varies inversely with the amount of wealth already possessed.

Most economists recognize that the mathematical expectation and the variance of the probability distribution of returns, and the chance of ruin, are important to the wealth-holder—but they leave it to individual risk preference to balance one factor against the others. The geometric mean of the probability distribution of returns (G) depends on both the mathematical expectation and the variance of the distribution. Further, G would equal zero if there were any possibility of a return of zero (i.e., ruin). When the portfolio with the highest G is chosen, with G greater than zero, there is no chance of ruin if the wealth-holder's probability beliefs are correct. Consequently, maximization of G falls within the generally accepted range of rational behavior. This is not to say that G is the only rational criterion for portfolio management; it is to say, however, that it is a useful criterion when dealing with a broad range of problems. When the portfolio with maximum G is not chosen, there must be justification for choosing to hold a portfolio which has little chance of being the most valuable portfolio in the long run.

Chapter IV deals with the problem of determining what proportion of a portfolio to allocate to a risk asset in order to maximize G, the geometric mean of the probability distribution of portfolio returns. Both G and this proportion (i.e., q_{max}) are functions of the whole probability distribution of returns from the risk asset and may be determined by means of equations (4.1) and (4.2). These equations involve the whole probability distribution of returns and, consequently, are often difficult to solve except, possibly, by trial and error. However, equations involving only the mean and variance of the probability distribution of returns from the risk asset often give good estimates of G and q_{max}. These equations, that is, equations (4.6) and (4.7), also may be used when probability beliefs take the form of estimated standard returns in gain years and standard returns in loss years. A standard return for the risk asset in a gain year is R + s and in a loss year is R - s, with gain years and loss years equally probable.

The analysis in this study is based on a number of concepts underlying the payout matrices and the equations which determine proper maximizing action. Two of the most important of these concepts have to do with returns and with proper maximizing action. Returns occur in series over time. In order to determine the return for a specified year it is necessary to know the value at the beginning and at the end of the year as well as the cash dividend and interest receipts during the year. Market values are used as a basis for this determination. If the probability beliefs as to returns remain unchanged from year to year, the maximizing proportion to hold in the risk asset (q_{max}) will remain constant. For example, if the probability beliefs at the beginning of the year, year i, are such that the maximum chance allocation of the portfolio is 40 percent in bonds and 60 percent in stock and these beliefs remain the same at the beginning of year i + 1, then the maximum chance allocation again will be 40 percent in bonds and 60 percent in stock at the beginning of the year i + 1. If the relative prices of stocks and bonds have changed between the two dates, it will be necessary for the portfolio manager to make some sales and purchases in his portfolio to bring it into line with the desired proportions even if the proportions themselves have not changed. Under this plan of action, the portfolio manager whose probability beliefs about returns from stocks and bonds remain unchanged exposes the same proportion of his portfolio to the same risk time after time.

Reference period returns were used in Chapter V to illustrate the hypothetical results of proper allocation of a portfolio between bonds and stocks. Stock returns in the 1926-1935 period ranged between .55 and 1.47. The arithmetic mean return was 1.106 and the geometric mean return was 1.057. If 40 percent of an hypothetical portfolio had been placed in bonds yielding 5 percent at the beginning of each year and 60 percent had been placed in stock, the geometric mean portfolio return over the whole period would have been 1.067. Such diversification would have increased yields from 5.7 percent for the all stock portfolio and 5 percent from the all bond portfolio to 6.7 percent for the diversified portfolio in this period. In the 1890-1899 period, on the other hand, returns from stocks ranged only from .90 to 1.29 with an arithmetic mean return of 1.065 and a geometric mean return of 1.059. In those reference years, a portfolio adjusted at the beginning of each year so that the investor borrowed $2.00 at 3 percent for each of his own dollars and invested all in stock would have yielded 8.6 percent on the equity. The small variance in the 1890-1899 set of stock returns made such hypothetical borrowing profitable. In the 1926-1935 period borrowing on the same scale would have resulted in a return of zero in one or more years and the consequent elimination of the portfolio.

The proper allocation of a wealth-holder's resources between stock and bonds (or borrowing) depends on returns from bonds and the cost of borrowing on the one hand, and the probability distribution of returns from stock on the other. The effects of different levels of interest rates on q_{max} depends on the probability distribution of returns from stock. A given difference in interest rates will have less effect on q_{max} when the probability distribution is dispersed than when the variance is small. These relationships may be quantified in the case of the wealth-holder who uses the maximum chance subgoal as the basis for his decisions. If the wealth-holder believes that a standard gain year and a standard loss year are equally probable and that stocks will return

1.41 in the gain year and .80 in the loss year, a two point difference in anticipated interest rates (5 percent vs. 3 percent) would account for a difference in bond holdings equivalent to 25 percent of the net value of the portfolio. If he believes that stocks will return 1.183 in the gain year and .947 in the loss year, the same two point difference in anticipated interest rates would account for a difference in borrowing or negative bond holding equivalent to 170 percent of his net portfolio. This analysis indicates the effect of uncertainty on the interest elasticity of demand for funds to buy stocks.

GLOSSARY OF MATHEMATICAL SYMBOLS

A : the arithmetic mean (i.e., mathematical expectation) of the probability distribution of portfolio returns.

A_i : the arithmetic mean (i.e., mathematical expectation) of the probability distribution from portfolio i.

A_i' : an estimate of A_i.

a_{ij} : the return from portfolio i if the j^{th} event occurs.

b : the proportion of the portfolio allocated to bonds (or other safe assets) at the beginning of each year.

C : the return from bonds (or other safe asset).

G : the geometric mean of the probability distribution of portfolio returns.

G_i : the geometric mean of the probability distribution of portfolio returns from the i^{th} portfolio.

G_{max}: the geometric mean of the probability distribution of returns from the portfolio having the highest G.

G_q : the geometric mean of the probability distribution of returns from the portfolio allocated q to stock at the beginning of each year.

G' : an estimate of G.

g_{ij} : the geometric mean portfolio return from the i^{th} portfolio if the j^{th} combination of events occurs.

n : number of investment periods called years.

P' : the probability of having a larger return than any other specified portfolio.

P'_{in} : the probability of portfolio i being more valuable than any other specified portfolio at the end of n years.

p_j : the probability of the j^{th} occurrence.

q : the proportion of the portfolio allocated to stock (or other risk asset)at the beginning of each year.

q_{max}: the proportion of the portfolio to be allocated to stock at the beginning of each year in order to maximize G.

q'_{max}: an estimate of q_{max} .

$\bar{R}$ or R: the arithmetic mean of the probability distribution of returns from stock (i.e., the risk asset).

R_j : the return from stock if the j^{th} event occurs.

s : the standard deviation of the probability distribution of returns from the risk asset (stock).

s : the deviation from R in a standard gain year and a standard loss year.

S_t : Stock price index including cash dividends in year t.

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