CRITERIA FOR CHOICE AMONG RISKY VENTURES

HENRY ALLEN LATANÉ
Chapel Hill, North Carolina

THE SUBGOAL

This paper is concerned with the problem of how to make rational choices among strategies in situations involving uncertainty. Such choices can be expressed through payout matrices stated in terms of some measure of value to be maximized. These matrices show the probabilities of all relevant future occurrences and the payouts resulting from the combined effects of each possible strategy, on the one hand, and each relevant future occurrence, on the other.[1] All this information is needed to choose the proper strategy rationally. It would be impossible for a decision-maker to choose rationally among strategies if he disregarded either the probability of the relevant future occurrences or any of the possible payouts.

The problem of rational decision-making can be broken down into three steps: (1) deciding upon an objective and criteria for choosing among strategies; (2) filling out a payout matrix; and (3) choosing among available strategies on the basis of this matrix and the criteria. In real life the second step—deciding upon the size of the payout matrix, measured by the number of columns representing relevant future occurrences and rows representing available strategies, and filling in the matrix with reasonable estimates of payouts and probabilities—is by far the most difficult part of the decision-maker's job. This paper has little to say about these problems. It deals largely with the first step: the problem of setting up criteria for choosing among strategies on the basis of a filled-in payout matrix.

A hierarchy of goals and guides for reaching these goals is involved in rational choices among strategies. This hierarchy consists of (1) a goal; (2) a subgoal; and (3) a criterion for choosing among strategies to reach the subgoal, that is, a measure that must be maximized to attain the subgoal. The goal in rational decision-making is the maximization of some measure of value. Each decision is made for the sake of the difference it will make in terms of this

[1] Payout matrices are shown in Tables 1, 2, and 3. The payouts represent the possible final outcomes of choices among strategies. The matrices have single-valued payouts and probabilities. Many, if not all, decision problems can be reduced to such form. Consider first the probabilities. A subjective probability distribution of an imperfectly known underlying probability can be reduced to a subjective probability of the event itself. For example, suppose a gambler believes that there is a 0.5 probability that a coin is biased so that it always comes up tails and a 0.5 probability that it is unbiased. He has a subjective probability of 0.25 for heads and 0.75 for tails, and these probabilities would be used in his payout matrix as long as his probability beliefs remain unchanged. Consider next the payouts. In much discussion of decision theory the payouts are taken as given, with only the probabilities subject to uncertainty. However, in real life the sizes of the payouts often are as subject to uncertainty as are the probabilities. But, even when the payouts are uncertain a matrix filled in with single values can be constructed. If we have probability distributions of payouts for all specified occurrences (such as heads and tails in the toss of a coin), a payout matrix can be constructed listing each possible payout and the subjective probability of its occurrence. These large matrices often can be reduced to simple two-valued distributions of payouts without much loss of information.

objective. The decision-maker is confronted with a payout matrix expressed in terms of either a subjective utility measure such as utiles or an objective measure such as money or bushels of wheat. He wishes to choose the strategy that will give him the maximum payout. This is his goal. When some one strategy gives a higher payout than any other strategy in all relevant future occurrences, the goal itself enables the decision-maker to choose among strategies. He merely chooses the dominant strategy.

When there is no strategy superior to all the rest in all possible future occurrences, the decision-maker needs some other guide for making decisions, since the goal itself does not enable him to make his choice. This guide is here called the "subgoal." The need for a subgoal exists because the outcome of specific strategies is subject to probabilistic uncertainty. In utility theory the payout matrix is expressed in terms of some measure of subjective utility, say, utiles. Choice of the strategy that will give the maximum payout in utiles is the goal, and choice of the strategy with the maximum expected utility[2] is taken as the subgoal. Given a completely filled-in matrix, this subgoal can surely be reached. Whether or not the goal is reached depends on future occurrences, but, in any event, the subgoal of maximization of the expected value of the payouts expressed in utiles is logically related to the goal of maximization of the forthcoming payout also expressed in utiles.

In this paper a second subgoal is proposed for use when the choice is repetitious and has cumulative effects and when the goal is maximization of wealth at the end of a large number of choices. Under these conditions the choice of the strategy that has a greater probability (P') of leading to as much or more wealth than any other significantly different strategy at the end of a large number of choices also is a logical subgoal. The P' subgoal is not as general as the maximum expected utility subgoal. For example, it would not apply to unique choices. When a man is faced with a once-in-a-lifetime choice of risking his whole fortune and his life on a venture that will produce great rewards if successful, it does not help him to know that he is almost certain to be ruined if he takes such a risk often enough. The P' subgoal is not logically related to the goal in this case.[3] Such a man, however, could set up a payout matrix expressed in utiles and decide which course of action maximized his expected utility. Here the maximum expected utility subgoal is logically related to the goal even though the P' subgoal is not. The P' subgoal is less general but would seem to be more operational than the expected utility subgoal because of the difficulty of constructing a payout matrix expressed in terms of utiles, especially if the decision involves a firm or group of people.

The P' subgoal would seem to be particularly applicable to many business

[2] The expected utility of a strategy is computed by multiplying all possible payouts expressed in utiles by their respective probabilities and then summing the products.

[3] For certain utility functions and for certain repeated gambles, no amount of repetition justifies the rule that the gamble which is almost sure to bring the greatest wealth is the preferable one. For example, the P' subgoal is not appropriate for a decision-maker for whom the possibility of great gain, however small and diminishing, is more important than maximization of the probability of as much or more wealth than can be obtained by any other strategy in the long run. Such a decision-maker may adopt a course of action that is almost certain to result in less wealth in the long run. Whether or not his utility function is compatible with the specified goal of maximum long-run wealth is not at issue here.

decisions such as those involved in portfolio management. Wealth-holders have the option of holding their wealth in many different combinations of stocks, bonds, and cash. The allocation of wealth among these types of assets involves a series of choices extending over time. The fact that these choices are repetitive in nature with cumulative effects may be used as the key factor in defining a goal, a subgoal, and a criterion for choosing among portfolios.

The problem of choice among portfolios may be stated in terms of the payout matrix in Table 1. In this table p_j

TABLE 1

PAYOUT MATRIX FOR VARYING PORTFOLIOS

PORTFOLIO	RELEVANT FUTURE OCCURRENCES
	$1, \ldots, j, \ldots, k$
1	$a_{i1}, \ldots, a_{1j}, \ldots, a_{1k}$
.	
.	
.	
i	$a_{i1}, \ldots, a_{ij}, \ldots, a_{ik}$
.	
.	
.	
t	$a_{t1}, \ldots, a_{tj}, \ldots, a_{tk}$
Probability of occurrence	$p_1, \ldots, p_j, \ldots, p_k$

represents the probability of the jth occurrence, with $\Sigma p_j = 1$, and a_{ij} represents the return from the ith portfolio with $i = 1, \ldots, t$, if the jth occurrence takes place, with $j = 1, \ldots, k$. A return is the payout, including return of principal, per dollar of portfolio value per investment period (here called "year"). Returns cannot be negative, so that $a_i \geqq 0$. The portfolio manager is faced with such a payout matrix for n years and wants to choose in a rational manner one portfolio from all available portfolios in each of the n years.[4]

The goal of portfolio management is taken to be to select a portfolio so as to maximize wealth at the end of a period of years, assuming reinvestment of all returns.[5] Let W_i^n be the final value of \$1.00 placed in portfolio i if returns are reinvested n times. Then the goal of portfolio management is taken to be to select the optimum portfolio so that $W_{opt}^n \geqq W_i^n$, with $i = 1, \ldots, t$. This goal cannot be used as a basis for choice among portfolios, since which portfolio will have the maximum W^n depends on future occurrences.

The subgoal proposed here is the choice of the portfolio that has a greater probability (P') of being as valuable or more valuable than any other significantly different portfolio at the end of n years, n being large. It is shown below

[4] The idea of maximizing wealth at the end of a large number of separate decisions based on the same payout matrix may appear unrealistic, but portfolio managers are continually being faced with choices having cumulative effects and involving approximately the same payouts and probabilities time after time. For example, year after year a portfolio manager may have probability beliefs such as: "I look for conditions in the next ten years to be very similar to those prevailing in 1926 through 1935. Bonds will yield about 4 per cent per annum during the whole period. Some day we are going to have a boom and a bust in the stock market, but I do not know which is going to come first." Choosing one portfolio to hold in each of the n years is not the same as choosing one portfolio at the beginning of the period to hold throughout the n years. For example, if the probability beliefs at the beginning of one year are such that the maximum P' allocation of the portfolio is 40 per cent in bonds and 60 per cent in stock and if these beliefs remain the same at the beginning of the next year, then the maximum P' allocation again will be 40 per cent in bonds and 60 per cent in stock at the beginning of the second year. If the relative prices of stocks and bonds have changed between the two dates, it will be necessary for the portfolio manager to make some sales and purchases in his portfolio to bring it into line with the desired proportions even if these proportions themselves have not changed.

[5] Few wealth-holders reinvest all returns, so the problem of maximizing wealth assuming no withdrawals is somethat unrealistic. However, this restriction can be modified. If withdrawals per unit of time are a fixed proportion of wealth (considered as interest, for example), they will not affect proper maximizing action. Whatever would maximize wealth, assuming no withdrawals, would maximize wealth, assuming proportionate withdrawals.

that the portfolio having a probability distribution of returns with the highest geometric mean, G, also has the greatest P'.

The central fact of this paper is a simple one: If the value of an asset, say, portfolio i, priced initially at \$1.00 is believed to change after a year to, alternatively, a_{i1} or $a_{i2}, \ldots,$ or a_{ik}, with respective probabilities $p_1, p_2, \ldots, p_k$, and if the proceeds are reinvested n times, then the final value of the investment, W_i^n, "converges in probability" to $G_i^n = a_{i1}^{p_1 n} \cdot a_{i2}^{p_2 n} \cdots a_{ik}^{p_k n}$. The probability that the absolute difference between W_i^n and G_i^n is smaller than any preassigned positive number will approach 1 as n increases indefinitely. In other words, the final return from \$1.00 invested in portfolio i, assuming reinvestment of all annual returns for n years will converge in probability on G_i^n, the nth power of the geometric mean of the probability distribution of annual returns from that portfolio. This relationship is intuitively obvious, since the a_{i1} return will "tend" to occur np_1 times, the a_{ii} return will tend to occur np_i times, and so forth, if n is large. It can be proved rigorously by use of the law of large numbers applied to the logarithms of the annual returns.

Let n_j be the number of occurrences of the jth relevant future occurrence, with $\Sigma n_j = n$, with $j = 1, \ldots, k$, then $n_j/n \underset{\lim}{\rightarrow} p_j$ and $n_j/n \log a_{ij} \underset{\lim}{\rightarrow} p_j \log a_{ij}$ as $n \rightarrow \infty$. But $\log W_i = \Sigma n_j/n \log a_{ij}$, with $j = 1, \ldots, k$ and $\log G_i = \Sigma_{pj} \log a_{ij}$, so $\log W_i \underset{\lim}{\rightarrow} \log G_i$ and $W_i \underset{\lim}{\rightarrow} G_i$ as $n \rightarrow \infty$.[6] It follows from this that, if $G_i > G_j$, then the probability, P', that $W_i^n > W_j^n$ at the end of n years approaches 1 as n increases indefinitely. The portfolio with the highest G is almost certain to be more valuable than any other significantly different portfolio in the long run. For this reason G is accepted here as a rational criterion for choice among portfolios.

SUBGOALS AND SUBJECTIVE UTILITY

Rational choice among strategies is the ancient problem of the gambler who has the option to choose among bets. Classical writers on probability theory recommended that problems of this kind be solved by first computing the expected winnings (possibly negative) for each available bet and then choosing the bet with the highest mathematical expectation of winning. Since there was no reason to assume that, of two persons encountering identical risks, either should expect to have his desires more closely fulfilled, the classical writers thought that no characteristic of the risk-takers themselves ought to be taken into consideration; only those matters should be weighed carefully that pertain to the terms of the risk.[7] In 1738 Daniel Bernoulli in four short paragraphs demonstrated that the use of the mathematical expectation of winnings did not always apply and proposed instead that gamblers should evaluate bets on the basis of the mathematical expectation of the utilities of winnings.[8]

In terms of subgoals as defined in this study, Bernoulli showed that use of the

[6] The asymptotic quality of G is used in information theory as developed by Dr. Claude Shannon and was applied to a gambling situation by John Kelly in "A New Interpretation of Information Rate," *Bell System Technical Journal*, August, 1956, pp. 917–26. See also R. Bellman and R. Kalaba, "Dynamic Programming and Statistical Communication Theory," *Proceedings of the National Academy of Science*, XLIII (1957), 749–51. I had no knowledge of this work when I first proposed the P' subgoal at a Cowles Foundation Seminar in February, 1956.

[7] See Daniel Bernoulli, "Exposition of a New Theory on the Measurement of Risk," trans. Louise Sommer, *Econometrica*, XXII (January, 1954), 23.

[8] *Ibid.*, p. 24.

expected-value subgoal did not always lead to choices that seemed rational to him and proposed instead the use of the expected-utility subgoal. He used the following example:

> Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chances of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble.[9]

Bernoulli's example is somewhat aside from the daily business of living, but, when stripped of its gambling wrappings and expressed in terms of payouts and returns, it is seen to represent a major segment of economic decision-making. The hypothetical market price of the ticket, which has an equal probability of paying 20,000 ducats or nothing, is 9,000 ducats. Both the poor man and the rich man have the option either to hold the lottery ticket or to hold 9,000 ducats. Possible payouts range from 2.22 per ducat risked to 0. Payouts with ranges such as this—indeed, much greater ranges—are ordinary economic occurrences. The magnitude of the choice faced by the rich man is well within the range of ordinary business decisions, and the "poor man" today is continually faced with implicit or explicit decisions as serious as that faced by Bernoulli's lottery-ticket owner. He must decide whether to move to a new job, buy a new home, sign a second mortgage. He is continually offered the opportunity to undertake such risky ventures as purchasing his own truck, opening a restaurant, buying some uranium stock, some oil stock, or some investment shares. Some of these options may be highly advantageous, and he must choose some one course of action in each case. The effects of these choices are cumulative; that is, the decision-maker never comes back to exactly the same position he occupied before making his choice. The major difference between Bernoulli's problem and other choices among courses of action is that the ticket-holder's choice is clearly defined, while the other opportunities are usually ignored, or the choices are muddled.

TABLE 2

PAYOUT MATRIX OF GAINS AND LOSSES

Strategy	Future Occurrence: Ticket Wins	Future Occurrence: Ticket Loses	Criterion A
a) Poor man:			
Hold ticket...	20	0	10
Sell ticket....	9	9	9
b) Rich man:			
Buy ticket...	11	−9	1
Not buy ticket	0	0	0
Probability of occurrence........	0.5	0.5	

Thus Bernoulli's example is representative of a wide class of choices. The decision-maker is being faced continually with such choices, and the outcome of each decision affects his entire future. In the following discussion this example is stated in payout matrices constructed to illustrate choices based on (*a*) classical mathematical expectation (the expected value subgoal); (*b*) Bernoulli's subjective utility (the expected-utility subgoal); and (*c*) the maximum chance (P') subgoal.

Table 2 shows the classical approach to choosing among risky ventures. The payout matrix, expressed in terms of thousands of ducats, shows the probability of the lottery ticket paying off or not and the net payout to the poor man and to the rich man for each of two courses of

[9] *Ibid.*

action. The classical writers would calculate the mathematical expectation, A, of the net payouts and choose that strategy which maximizes A. In this case they would recommend that the rich man buy the ticket for 9,000 ducats and that the poor man refuse to sell it at this price.

The mathematical expectation (that is, the arithmetic mean) of the probability distribution of payouts is, indeed, a good criterion when there are large numbers of independent trials. Even decision-makers who make repeated choices with cumulative effects (for example, the operators of roulette wheels and insurance companies) are rightly interested in this average when each risk is small in relation to total wealth. There is little or no conflict under these conditions between the use of the arithmetic mean as a criterion and the use of the geometric mean of the probability distribution of payouts per dollar of wealth as a criterion.[10] When a decision-maker can surely bet the same small amount on a large number of independent trials, he can maximize the expected value of his gain, and also the likelihood of having more gain than can be obtained by any other plan, by choosing that set of bets which gives him the greatest mathematically expected payout. For example, if Bernoulli's poor man had found 10,000 tickets involving 10,000 independent drawings, each with a payout equally likely to be 2 ducats or nothing, he clearly would be unwise to sell his block of tickets for 9,000 ducats. His winnings on 10,000 different trials would be almost certainly very close to 10,000 ducats, the mathematical expectation of the value of the set of tickets, and the advice of the classical writers would be sound.

Bernoulli used the lottery-ticket example to show that the expected values of the payouts are not good guides in making choices involving large risks. He proposed instead that the expected value of the utilities of the payouts be used as a criterion. He would fill in the payout matrix in Table 2 not with the money value of the gains and losses but with their utilities and then would use the mathematical expectation of these utilities as his criterion.

Whether or not particular payout matrices expressed in terms of subjective utility are realistic is not a problem here. But Bernoulli's procedure is very much at issue. He defines the "mean utility" of a course of action as the mathematical expectation of the probability distribution of the possible utilities from that course of action. He then states, with no discussion, that this mean utility can be used as a basis for valuing risks, that is, as a basis for choosing among courses of action. In other words, he explains why he expresses his profits (or losses) in terms of subjective utility, but he does not give any justification for maximizing the mathematical expectation of these utilities. Bernoulli's use of subjective utility has had wide recognition, and his use of mathematical expectation also has been widely adopted with little or no discussion.[11]

[10] When a gambler who has the choice of betting or not betting bets all his wealth on the toss of a fair coin with a payout of \$3.00 per \$1.00 bet if heads occur and nothing per \$1.00 bet if tails occur, he is maximizing the expected value of the payout but not G. When he can bet only 1 per cent of his wealth, however, he will maximize both A and G by betting.

[11] Mathematical expectation now is used as a basis for defining utility. The present emphasis on the axiomatic approach to utility is largely derived from John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior* (rev. ed.; Princeton, N.J.: Princeton University Press, 1935). They say: "We have practically defined numerical utility as being that thing for which the calculus of mathematical expectations is legitimate" (p. 28).

Bernoulli's problem can also be solved by the use of the maximum-chance (P') subgoal. Table 3 shows the payout matrix of returns (that is, payouts, including return of principal, per dollar of wealth) for the poor man, who is assumed to have a wealth of 1,000 ducats aside from his lottery ticket, and for the rich man, who is assumed to have a wealth of 100,000 ducats. The arithmetic mean, A, of the probability distribution of returns is higher for the poor man when he holds the ticket and for the rich man when he buys the ticket. The geometric mean, G, of returns for the poor man is higher when the ticket is sold, however, and G for the rich man is higher when he buys the ticket.

TABLE 3

PAYOUT MATRIX OF RETURNS

Strategy	Future Occurrence: Ticket Wins	Future Occurrence: Ticket Loses	Criterion: A	Criterion: G
a) Poor man:				
Hold ticket.	2.1	0.1	1.1	0.46
Sell ticket..	1.0	1.0	1.0	1.0
b) Rich man:				
Buy ticket.	1.11	0.91	1.01	1.005
Not buy ticket....	1.00	1.00	1.00	1.00
Probability of occurrence......	0.5	0.5		

Over a long enough period of time many economic choices involving returns of the same order of magnitude repeat themselves. Bernoulli's poor man may never find another lottery ticket, but he probably will have many options among courses of action with as wide, or wider, a range of returns. It is assumed here that both the rich man and the poor man will have many opportunities to risk the same proportions of their respective fortunes on approximately the same terms and that both men prefer more wealth to less wealth, everything else being equal. If these assumptions are valid, the maximization of P', the probability of having more wealth at the end of a long series of such choices than can be obtained by any other specified course of action, is a rational subgoal, and G is a rational criterion. The use of the maximum-chance subgoal results in courses of action for both the rich man and the poor man which seemed rational to Bernoulli.

The decision-maker who is interested in maximizing his wealth at the end of a long series of choices should ask himself how he would come out in the long run if he made the same choice on the same terms over and over again. It is not necessary for him to ask himself what his individual subjective utility of winning is. This is not to say that other goals, rather than the goal of maximum wealth at the end of a long series of choices, are irrational. Indeed, the use of subgoals based on the goal of maximum wealth often may be irrational. For example, the man who desperately needs \$10.00 to escape a jail sentence and who has only \$1.00 may well be justified in taking a gamble to get his money, even though this gamble would not stand the maximum-chance subgoal test. Even under these conditions, however, it would be useful for the man to know that he should not often act in such a manner, if he wants to build up his fortune so as to avoid similar predicaments in the future.

In his paper Bernoulli uses the expected utilities of the payouts as his criterion. He then reaches the conclusion that the utility resulting from any small increase in wealth usually is inversely proportional to the quantity of goods previously possessed.[12] Under these conditions the utilities of the returns vary as their logarithms, and the geometric mean, G, of the probability distributions

[12] This is generally credited with being the first use of a utility function.

of returns can be used as a criterion instead of expected utility. The arithmetic mean of the logarithms (utilities) of returns is maximized when G is maximized.[13]

Bernoulli gives a number of applications of his formula to gambling and to insurance. In each instance he is able to give a specific answer. He says that everyone who bets any part of his fortune on a mathematically fair game of chance is acting irrationally, and he then determines what odds a gambler with a specified fortune must obtain to break even in the long run. Most of his problems still are interesting in their own right, and many have a bearing on proper portfolio management. For instance, he demonstrates with numerical examples the advantages of diversification among equally risky ventures and between risky and safe assets.

Bernoulli's approach to the valuation of risky ventures is not contradictory to the maximum-chance (P') approach. Not only do the two approaches lead to the same conclusion when they both can be applied but they tend to support each other. Wealth-holders may be divided into two groups. The first group contains those to whom each risk is a unique event either because they do not expect it to recur or because they keep its effects entirely separate from the results of other risks. For example, the man who each year sets aside a small sum to bet on the races during his vacation, with the intention of "living it up" if he wins and writing it off to experience if he loses, presumably is not actuated by long-run profit-maximizing motives. The effects of each risk are kept separate. Analysis based on maximum chance has nothing to offer this first class of wealth-holders. The choice between profit and safety or expected return and variance is a matter of subjective utility. Bernoulli's assumption that the satisfaction derived from a small gain tends to vary in inverse proportion to the initial wealth may or may not be a shrewd guess.

The second class of wealth-holders includes those who expect to be faced repeatedly with risks of the same general type and magnitude. This group includes those making most business and portfolio decisions and hence is of great importance. It includes, specifically, all those who want to maximize the value of their portfolio at the end of n years, assuming reinvestment of all returns. Here there is a definite rule for choosing between risk and return, the P' subgoal, based on maximum-chance principles. This class may be subdivided further into (*a*) those who undertake only one risky venture at a time and (*b*) those who are able to diversify their risky ventures. Because so many economic phenomena, including yields on stocks, tend to fluctuate together over time, diversification among risky ventures cannot go as far toward eliminating risk as otherwise would be the case. Final choice among efficient portfolios for both groups, (*a*) and (*b*), is based on maximization of G, not because this maximizes subjective utility, but because it maximizes P'.

Bernoulli states that the wealth-holder should ask himself whether the added satisfaction associated with the expected gain justifies undertaking the risky venture. He bases an exact rule of behavior on his assumption as to how the added satisfaction varies with the size of the potential gain or loss in relation to the size of the portfolio. The rule may or

[13] As pointed out to me by Professor L. J. Savage (in correspondence), not only is the maximization of G the rule for maximum expected utility in connection with Bernoulli's function but (insofar as certain approximations are permissible) this same rule is approximately valid for all utility functions.

may not be empirically useful, but it is grounded on rather shaky evidence about the exact shape of the utility function. According to maximum-chance analysis, the wealth-holder or portfolio manager should ask himself how he can maximize his chances of getting as good or better return than can be obtained with any other specified plan, assuming that he risks the same proportion of his portfolio on the same terms over and over again. It turns out that the formula which enables the portfolio manager to answer the maximum-chance question is the same as that developed by Bernoulli on grounds of subjective utility.

In conclusion Bernoulli says:

> Though a person who is fairly judicious by natural instinct might have realized and spontaneously applied much of what I have here explained, hardly anyone believed it possible to define these problems with the precision we have employed in our examples. Since all of our propositions harmonize perfectly with experience it would be wrong to neglect them as abstractions resting upon precarious hypotheses.[14]

Professor Stigler, in a review article,[15] gives considerable space to Bernoulli's hypothesis about the slope of the wealth-holder's utility function, even though the major emphasis of the article is on utility not affected by probability. He mentions that Laplace and Marshall, among others, have accepted the law as a realistic guide. He also points out the similarity of Bernoulli's law to the Weber-Fechner psychological hypothesis that the just noticeable increment to any stimulus is proportional to the stimulus. Stigler says: "Bernoulli was right in seeking the explanation[16] in utility and he was wrong only in making a special assumption with respect to the slope of the utility curve for which there was no evidence and which he submitted to no tests."[17]

More recently Savage in a section on "Historical and Critical Comments on Utility" had this to say:

> Bernoulli went further than the law of diminishing marginal utility and suggested that the slope of utility as a function of wealth might, at least as a rule of thumb, be supposed, not only to decrease with, but to be inversely proportional to, the cash value of wealth. To this day, no other function has been suggested as a better prototype for Everyman's utility function. . . . Though it might be a reasonable approximation to a person's utility in a moderate range of wealth, it cannot be taken seriously over extreme ranges.[18]

INDIVIDUAL RISK PREFERENCE

As indicated in the previous section, Bernoulli took the following steps to develop his utility function and to justify diversification among risky ventures and between risk assets and safe assets. (1) He showed—subject to the implicit assumption about subgoals previously discussed—that the value of a risky venture to the individual wealth-holder is not the arithmetic mean of the probability distribution of returns (the mathematical expectation of returns) but may be taken to be the arithmetic mean of the probability distribution of the utilities of the returns. (2) He stated that, in the absence of the unusual, the gain in utility

[14] *Op. cit.*, p. 31.

[15] George J. Stigler, "The Development of Utility Theory," *Journal of Political Economy*, LVIII (1950), 373–77.

[16] Bernoulli is explaining the reason for the limited value of the game involved in the St. Petersburg paradox. This game is a type of risky venture with an infinitely large mathematically expected value but with an extremely small probability of winning.

[17] Stigler, *op. cit.*, p. 375.

[18] Leonard J. Savage, *The Foundations of Statistics* (New York: John Wiley & Sons, 1954), p. 94.

resulting from any small increase in wealth may be assumed to be inversely proportional to the quantity of goods previously possessed. (3) He developed a formula for calculating the utility of a risk asset to the individual wealth-holder using as a criterion the utility function developed in step 2. According to Bernoulli, the subjective utility of the wealth-holder's assets, including the risky venture, is measured by the geometric mean, G, of the probability distribution of payouts from such assets. (4) Using this formula, he was able to calculate exactly the utility of the wealth-holder's assets, including the risky venture, and to show that diversification among risky ventures increases the utility.[19]

Bernoulli's step 2 may be a reasonable assumption about utility,[20] but it is subject to so many qualifications and exceptions (it does not explain gambling, for example) that it has not been accepted as a suitable basis for erecting the superstructure of steps 3 and 4. The valuation of risky ventures has been left to individual risk preference without any criterion for deciding what this preference is likely to be. For example, Makower and Marschak present a hypothetical table in which an asset's marginal contribution is determined by adding together its contribution to "lucrativity" and safety measured in "lucrativity units" determined by the safety preference rate for a single individual.[21] These individual safety preference rates, in turn, are a matter of taste and must be accepted as given. Friedman and Savage build on Bernoulli's step 1 but modify step 2 by developing a doubly inflected curve comparing utility with income.[22]

Markowitz begins his analysis of portfolio selection by pointing out that "the portfolio with the maximum expected return is not necessarily the one with the minimum variance. There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return."[23] He assumes that the investor considers, or should consider, expected return a desirable thing and variance of return an undesirable thing, and he defines an efficient portfolio as a portfolio with minimum variance for a given expected return or more and a maximum expected return for a given variance or less. He develops a method for selecting efficient portfolios from the set of all possible portfolios but does not give any basis for choice among the efficient portfolios except the individual's safety preference rate.

THE NEED FOR AN OBJECTIVE CRITERION

The difficulty of evaluating subjective risk preference and the need of an objective criterion are well indicated in the

[19] Bernoulli, *op. cit.*, pp. 24, 25, 28, 30.

[20] Cf. Alfred Marshall, *Principles of Economics* (8th ed.; New York: Macmillan Co., 1950), p. 135. Marshall says: "In accordance with a suggestion made by Daniel Bernoulli, we may regard the satisfaction which a person derives from his income as commencing when he has enough to support life, and afterwards as increasing by equal amounts with every equal successive percentage that is added to his income; and vice versa for loss of income."

See also Savage's comment quoted previously.

[21] Helen Makower and Jacob Marschak, "Assets, Prices and Marketing Theory," *Economica*, V (1938), 261–88. Reprinted in American Economic Association, *Readings in Price Theory* (Chicago: Richard D. Irwin, Inc., 1952), pp. 301–2.

[22] Milton Friedman and L. J. Savage, "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy*, LVI (1948), 279–304.

[23] Harry Markowitz, "Portfolio Selection," *Journal of Finance*, VII (March, 1952), 79.

following quotation from a recent journal article dealing with selection of an optimum combination of crops for a farmer:

> The introduction of risk into an economic model of a firm and consequently into a linear programming model of a firm has been accomplished by describing risky outcomes as probability distributions and choosing from among alternate possible distributions by the expected utility hypothesis.
>
> Two basic weaknesses have appeared in applying this method of incorporating risk. One difficulty arises in choosing a value for the constant α, which in this case is some sort of risk aversion indicator, and is, to some degree, governed by the personal characteristics of the entrepreneur. A large value for α indicates that the entrepreneur places a great weight on the variance as a deciding factor and is consequently highly averse to risk, and vice versa. The estimation of such a constant to be used in a model is thus quite important; the wrong choice will invalidate any results obtained. The derivation of this constant is a delicate task beyond the scope of this paper.[24]

A major advantage of the criterion for choice among risky ventures developed in this paper is that it avoids the necessity for direct subjective determination of such factors as Marschak's "lucrativity units" or Freund's "risk aversion indicator." As Roy remarks, "A man who seeks advice about his actions will not be grateful for the suggestion that he maximize expected utility."[25]

The criteria for choice between risk and safety in portfolio management can be illustrated by assuming that a gambler has the choice of holding his money in cash or of betting on a gambling device which, with equal probability, will return $R-s$ on loss occasions and $R+s$ on gain occasions with an expected return of R per dollar played. The gambler's portfolio at any time consists of the proportion of his wealth held in cash plus the proportion bet on the gambling device. When the gambler bets none of his wealth, the expected return from his portfolio is 1, and the standard deviation of returns is 0. As the proportion bet increases, both the expected portfolio return and the standard deviation of returns increase. When he bets all his wealth, the expected portfolio return is R, and the expected standard deviation of returns is s. As long as R is greater than 1, and $R-s$ is less than 1, all possible combinations of the two assets in this range are efficient portfolios in that any one of the combinations gives the maximum possible expected return for some standard deviation or variance and the minimum standard deviation or variance for some expected return. Neither Marschak nor Friedman and Savage nor Markowitz would be able to help the gambler in choosing among these efficient portfolios beyond telling him that he should gamble heavily if he has a high preference for risk and should be very conservative in his betting if he has a high risk-aversion factor. In this paper an attempt is made to give the gambler (and wealth-holders, in general) an objective criterion for making this choice.

The wealth-holder who adopts the maximum-chance (P') subgoal can reach this subgoal by using the geometric mean, G, of the probability distribution of returns as his criterion and choose the strategy that has the probability distribution of returns with the highest G. Bernoulli also has shown that choice of that risky venture with the highest G is a rational choice (1) if maximization of the mathematical expectation of the utilities

[24] Rudolph J. Freund, "The Introduction of Risk into a Programming Model," *Econometrica*, XXIV (July, 1956), 253–63.

[25] A. D. Roy, "Safety First and the Holding of Assets," *Econometrica* XX (1952), 433.

of the payouts is a rational subgoal and (2) if the utility of a small gain or loss varies inversely with the amount of wealth already possessed.

Most economists recognize that the mathematical expectation and the variance of the probability distribution of returns and the chance of ruin are important to the wealth-holder—but they leave it to individual risk preference to balance one factor against the others. Since G depends on both the mathematical expectation and the variance of the probability distribution of returns, when G is maximized, there is no chance of ruin if the wealth-holder's probability beliefs are correct. Consequently, maximization of G falls within the generally accepted range of rational behavior. This is not to say that G is the only rational criterion for choice among strategies; it is to say, however, that it is a useful criterion in dealing with a broad range of problems.