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Discrete Sequential Search

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INTRODUCTION AND STATEMENT OF PROBLEM

In this search problem, a single target is in one of several regions and cannot move. The search consists of a sequence of discrete "looks" terminating when the target is found. For each region i, we specify:

c_i the cost per look in the ith region
p_i the prior probability that the target is in the ith region
m_i the conditional miss probability, the probability that a target in the ith region will *not* be detected on a single look there

In what sequence should regions be inspected to minimize the expected cost of the search?

This problem has been extensively studied by dynamic programming techniques.[1] Below we show a simpler, more transparent approach, which is more suitable to the problem.

SOLUTION

Before the search it is impossible to predict how many looks will be necessary to locate the target. A policy is a plan of search that tells where the nth look will be made—should it be necessary. For any policy, if $P(k)$ is the probability that the target is found on or before the kth look, $C(k)$ is the total cost of the first k looks, and C is the random total cost, then the expectation of C is[2]:

$$E(C) = \sum_{k=1}^{\infty} (C(k) - C(k-1))(1 - P(k-1)) \qquad (1)$$

[1] An early example is Pollock (1960). A more recent, but similar paper, is that of Matula (1964).

[2] Since $P(k)$ is the probability that the target is found on or before the kth look, the probability that it is found on the kth look is $P(k) - P(k-1)$. There

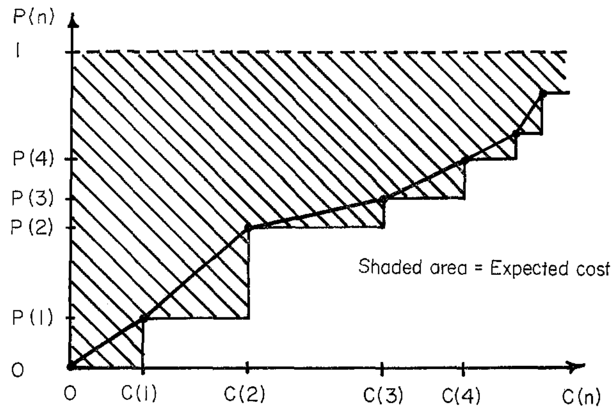

FIG. 1. A cumulative probability-cost plot. The coordinates of the point corresponding to the nth look are $(C(n), P(n))$. That the shaded area equals the expected cost is evident if the area is divided into vertical strips, each strip having an area identified with one of the terms in Eq. (1).

Avoiding trivial cases we assume that no c_i, p_i, or m_i, is zero. In this case any policy providing for only a finite number of looks in some region cannot be optimal, since with nonzero probability it allows the search to last forever. Thus $E(c)$ can be finite only for policies that provide for an infinite number of visits to every region.

For any policy we can construct a cumulative probability—cost plot as shown in Fig. 1. Equation (1) shows that the expected cost of a search using this policy is equal to the shaded area. This area consists

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is a temptation to write:

$$E(C) = \sum_k C(k)(P(k) - P(k-1))$$

This expression is incorrect, since it ignores the possibility of never finding the object. For example, if $C(k) = k$ and $P(k) = 0$ for all k, the above sum is zero, but the search would continue forever and the expectation of k is infinite. A better form is:

$$E(C) = \operatorname*{Lim}_N \sum_{k=1}^{N} C(k)(P(k) - P(k-1)) + C(N)(1 - P(N))$$

which by elementary manipulation gives equation one. It is well known that the erroenous expression gives the correct value for $E(C)$ when $C(k)$ is monotone and $\mathrm{Lim}_{N\to\infty} P(N) = 1$.

of two portions, that contained in the triangles and that above the triangles.

An important point is that *all policies with finite expected cost have the same triangles in their probability-cost plot, with only their order changed.* More precisely, every such policy must eventually provide[3] for an nth look in region i and, when it does, a triangle of height $p_i(1 - m_i)m_i^{n-1}$ and base c_i occurs on the plot.[4] Thus all policies with finite expected cost contain the same area in the triangles of their plots, and our problem reduces to arranging these triangles to minimize the area above them.

Clearly the policy that places the triangles in order of decreasing steepness is optimal if it is feasible.[5] We can imagine all the numbers of form:

$$\frac{p_i(1 - m_i)m_i^{n-1}}{c_i}$$

arranged in a two dimensional array (by n and i).

Then the optimal policy makes the first look in the region corresponding to the largest entry in the array, the second look in the region corresponding to the next largest, etc. There is, however, one point to clear up—this policy must be feasible. That is, it must not make the third look in region one before it makes the second, etc. That the optimal policy is an honest policy is evident from the fact that the:

$$\frac{p_i(1 - m_i)m_i^{n-1}}{c_i}$$

are monotone decreasing[6] in n.

Thus we have solved the problem and found the policy minimizing the expected search cost.

[3] This is a consequence of our earlier remark that the expected cost of a policy can be finite only if it provides for an infinite number of visits to every region.

[4] The height of a triangle corresponding to a given look is just the probability that the target is found on that look—i.e. the probability that the target is in the region examined, was overlooked on all previous visits to the region, and is found on the given look.

[5] Although every policy is a rearrangement of the same triangles, not every rearrangement corresponds to a policy. (This point is discussed later.) Moreover, since the number of triangles is infinite it is not clear that there is a steepest, etc.

[6] The monotone decreasing nature of the numbers in the array also guarantees that they can be arranged in order of steepness—i.e. there is a steepest, a next steepest etc.

SOME PROPERTIES OF THE OPTIMAL POLICY

A simple application of Bayes rule shows that the policy with minimum expected cost, as derived above, is identical to that generated by the rule:

> Always look in the region for which the *posterior* probability (given the failure of earlier looks) of finding the object divided by the cost is maximum.

This is in accord with the intuitive principle of shopping where you expect to get the most per dollar.

Since the logarithm is monotone increasing in its argument, we can construct the optimal policy by arranging the numbers:

$$\log\left[\frac{p_i(1 - m_i)}{c_i}\right] + (n - 1)\log m_i$$

in decreasing order. Viewing these numbers as points along a line, the points corresponding to any particular region will be equally spaced. The nth look in the optimal policy will be in the region whose point is nth from the right. This viewpoint displays clearly certain interesting structural properties of the optimal policy. For example, if the $\log m_i$ are commensurate the optimal policy is eventually periodic.

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