Fundamentals of Pure and Applied Economics 18

Editors-in-Chief: Jacques Lesourne and Hugo Sonnenschein

Interprofile Conditions and Impossibility

Peter C. Fishburn

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This monograph reviews contributions to social change theory that have been stimulated by Kenneth Arrow's celebrated impossibility theorem, and addresses questions concerning the interactions between social choices and individuals' preferences and needs, and the ensuing possibility/impossibility results. The book examines Arrow's theorem in detail, elaborates on the types of conditions used for social choice, and surveys various multiprofile and singleprofile impossibility theorems. It then applies Arrow's approach to utility theory, and extends the theory for infinite sets of individuals. Concluding with an investigation of related results on probability aggregation, decision under certainty, and strategic voting, this book is a valuable reference for graduate students and researchers in social choice, economics, political science and applied mathematics.

About the author

Peter C. Fishburn received a B.S. in industrial engineering from Pennsylvania State University and a Ph.D. in operations research from the Case Institute of Technology. For the past twenty-five years he has engaged in research in areas of mathematical economics, political science, decision theory, and topics in combinatorial and discrete mathematics, and has held positions with the Research Analysis Corporation, the Institute for Advanced Study, and Pennsylvania State University. Dr. Fishburn is currently involved in research at the Mathematical Sciences Research Center, AT&T Bell Laboratories.

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Continued on inside back cover

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Interprofile Conditions and Impossibility

PETER C. FISHBURN

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1. INTRODUCTION

The modern era of social choice theory began with Kenneth Arrow's pathbreaking monograph [4] and his celebrated impossibility theorem. The purpose of the present monograph is to recount contributions to social choice that are based on Arrow's approach and succeeding developments. Its emphasis will be on the interactions among various conditions that relate social choices to individuals' values or preferences, and on the possibility/impossibility results that flow from these interactions. Special attention will be devoted to interprofile and intraprofile conditions [54] and their roles in generating impossibility theorems.

Although the nature of our subject requires a degree of mathematical analysis, it should be kept in mind that its conceptual core is eminently practical, and is discernible without the mathematical overlay. The basic question it addresses is: If a decision is required among competing alternatives, and if the decision is to depend on the values of the individuals in a society in certain specified ways, are there choice procedures that satisfy these specified dependencies? If the answer is "no" then we have an impossibility theorem. That is, the specifications relating the decision to the individuals' values are collectively incompatible; not all can be satisfied simultaneously, and we may wish to relax one or more of them to a point where the relaxed conditions are jointly compatible.

On the other hand, if the answer to the basic question is "yes", then we have a possibility theorem. But this is only part of the picture, for we still need to understand what kinds of choice procedures obey the conditions that say how the decision is to depend on the individuals' values. Once this is understood, we may discover that all such procedures are unsatisfactory in some unforeseen way or that even more conditions can be imposed without forcing the result into the realm of impossibility.

Thus, two questions emerge, an existence question and a characterization question. The existence question asks whether any choice procedure satisfies the stated conditions. And, when such procedures exist, the characterization question asks for their description. In many instances, the two are intertwined and inform one another. For example, one standard proof of Arrow's impossibility theorem is really a characterization proof since it shows that every social choice function that satisfies all but his nondictatorship condition lies within the class of dictatorial choice procedures.

The role of mathematics in all this is two-fold. First, it allows us to formulate precisely the structure of the social decision process at hand and the conditions that the choice procedure is to satisfy. Second, it facilitates the derivation of answers to the existential and/or characterization questions. This is especially helpful since these questions often involve combinatorial structures that are difficult to penetrate otherwise.

The next section of the monograph begins our inquiry into social choice impossibility by formulating the notion of a social choice function. Section 3 then examines Arrow's basic theorem in detail. Section 4 elaborates on the types of conditions for social choices used in Arrow's theorem since later theorems employ similar conditions. Section 5 considers a series of multiprofile impossibility theorems that are most closely related to Arrow's multiprofile theorem, and Section 6 follows suit for single-profile impossibility theorems.

In Section 7 we shall recast Arrow's approach in the language of utility theory and comment on allied results within this reformulation. Sections 8 and 9 then consider more rigidly specified utility structures for preference/utility profiles. The first of these sections looks at cardinal utilities for individuals with no interpersonal comparability. The second examines various degrees of interpersonal comparability of intrapersonal utilities.

Prior to Section 10, it is always assumed that the number of individuals is finite. This is relaxed in Section 10 where we note what can happen when infinite numbers of individuals are allowed.

The final section comments on possibility/impossibility theorems for four contexts that do not fall directly into the major theme of the monograph. These concern the aggregation of equivalence relations, probability distributions, and decisions under uncertainty, and the topic of strategic voting.

2. SOCIAL CHOICE AND IMPOSSIBILITY

The central object of our study is a social choice function. This describes how individuals' values are combined to select one or more alternatives from a specified set of feasible alternatives. Moreover, it does this for every one of a number of possible situations that might obtain. Each *situation* consists of two things, the set of feasible alternatives under consideration and a description of individuals' preferences or values on a set of alternatives that includes the feasible set. The set of possible situations will be referred to as the *domain* of the social choice function. The nonempty subset of feasible alternatives that the social choice function assigns to (chooses for) a situation will be referred to as the *choice set* for that situation.

We illustrate this with two examples. Suppose first that a professional society conducts a nomination process each year for the office of president. Each member is allowed to nominate up to three people by mail ballot, and each potential nominee who receives at least 15 percent of these votes is placed on the election ballot. If fewer than two nominees reach the 15 percent quota, only the two with the most nominations go on the election ballot. Thus the election ballot will have from two up to six names, and these names constitute one feasible subset of alternatives. The other part of a situtation is the members' preference over the names on the election ballot. Since these preferences might take many different forms, many situations could obtain for that election ballot. If the names on the election ballot are changed, then a different set of situations applies. All possibilities that could thus arise constitute the domain of the social choice function.

This function determines one or more names from the feasible set as the choice set for each possible situation in its domain. We do not necessarily require the choice set to contain only one candidate for every situation although a final unique choice is needed in any practical context. In a manner of speaking, the social choice function simply identifies the "best" candidates in a given situation according to its specifications or the conditions that are used to define how it makes choices on the basis of members' preferences.

We shall not be concerned here with particular ballot instructions or mechanisms by which individuals' preferences or values are elicited, or with voters' responses to such mechanisms. In other words, the implementation of social choice functions and the associated matters of ballot design and strategic voting lie beyond our present concerns. With the exception of the final section of the monograph, our sole focus is the specification of choice sets for different situations when individuals' preferences are presumed to be known.

For a second example, suppose each of three interest groups will nominate one policy for consideration by a committee or legislature, which will then adopt one of the three nominated policies. Assuming that different interest groups will not nominate the same policy, the potential feasible sets will be sets of three policies. If group i will nominate a_i or b_i ($i = 1, 2, 3$) and $\{a_1, b_1\}$, $\{a_2, b_2\}$, and $\{a_3, b_3\}$ are mutually disjoint, then there are eight potential feasible sets. The relevant preferences for the other part of the domain could be the preferences of the members of the committee or legislature over the six policies in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. The social choice function would then assign a nonempty subset of the three policies in a feasible set to every combination of a feasible set and a profile of the members' preferences over the policies.

An arbitrary social choice function will be described in the following way. First, we assume that there is a nonempty universal set X of *potential decision alternatives* and a nonempty set N of *individuals* whose preferences or values may be taken into account. We shall usually assume that N is finite, but will consider infinite sets of individuals in Section 10. The set X could be finite, as in the preceding examples, or infinite, as when it is the nonnegative orthant of a finite-dimensional Euclidean space.

Second, we suppose that there is a nonempty set $\mathscr{A}$ of nonempty subsets $A, B, \ldots$ of X, which are interpreted as the potential *feasible sets* that might arise. In the second example above,

$$\mathscr{A} = \left\{ \begin{matrix} \{a_1, a_2, a_3\}, \{a_1, a_2, b_3\}, \{a_1, b_2, a_3\}, \{a_1, b_2, b_3\}, \\ \{b_1, a_2, a_3\}, \{b_1, a_2, b_3\}, \{b_1, b_2, a_3\}, \{b_1, b_2, b_3\} \end{matrix} \right\}.$$

If X is the set of nonnegative m-dimensional real vectors $(x_1, x_2, \ldots, x_m)$ and $p_1, \ldots, p_m$, and b are positive real numbers, then sets in $\mathscr{A}$ might be described as subsets of X that satisfy a linear restriction such as

$$\sum_{i=1}^{m} p_i x_i \leqslant b.$$

Different sets in $\mathscr{A}$ are obtained by varying the p_i (prices) and b (budget).

Third, we suppose that there is a nonempty set $\mathscr{P}$, each element of which provides a description of the preferences or values of every individual in N. In most cases, the set on which each person's preferences are defined is either X or a set constructed from X by well-defined operations. Each element P in $\mathscr{P}$ will be referred to as a *preference profile*. Note that a preference profile, or *profile* for short, describes the preferences of *every* person in N. When individuals' preferences are represented by real valued utility functions, we shall often refer to a profile as a *utility profile*. Specific assumptions about preference relations or utility functions that are assumed to delineate admissible profiles in $\mathscr{P}$ will be introduced as they are needed.

The *domain* of a social choice function is a nonempty set $\mathscr{D}$ of ordered pairs in the Cartesian product of $\mathscr{A}$ and $\mathscr{P}$, i.e., $\mathscr{D} \neq \varnothing$ and

$$\mathscr{D} \subseteq \mathscr{A} \times \mathscr{P}.$$

In most cases, all (A, P) in $\mathscr{A} \times \mathscr{P}$ will be assumed to be in the domain, and, when this is true, we have $\mathscr{D} = \mathscr{A} \times \mathscr{P}$. Each member of $\mathscr{D}$ is a *situation* (A, P) composed of a feasible set A and a preference profile P.

Finally, a *social choice function* [4, 54, 56] is a mapping C from a domain $\mathscr{D}$ into the nonempty subsets of X such that, for every $(A, P) \in \mathscr{D}$,

$$C(A, P) \subseteq A.$$

We refer to $C(A, P)$ as the *choice set* for situation (A, P).

Under circumstances that will be described later, it is often convenient to replace $C(A, P)$ by a binary social preference relation $>_P$ or by a social utility function on X. When this is done, it is understood that, for every two-alternative subset $\{x, y\}$ in A, $x >_P y$ means the same thing as $C(\{x, y\}, P) = \{x\}$. When feasible sets with more than two alternatives are involved in $\mathcal{A}$, it is often assumed that $C(A, P) \subseteq \{x \in A : y >_P x \text{ for no } y \in A\}$, or $C(A, P) = \{x \in A : y >_P x \text{ for no } y \in A\}$, when the latter set is nonempty. However, various other connections between $C(A, P)$ and the $>_P$-maximal alternatives in A could be postulated.

Succeeding sections will consider various structures for $\mathcal{A}$ and $\mathcal{P}$ along with conditions or restrictions on C. A general classification of different types of conditions on social choice functions will be presented in Section 4 after we examine the structure of one version of Arrow's impossibility theorem in Section 3. Briefly stated, there are three main classes of conditions, namely structural conditions, existential conditions, and universal conditions. Structural conditions are concerned with the nature of $\mathcal{A}, \mathcal{P}$, and $\mathcal{D}$, and say nothing directly about C except by way of its domain $\mathcal{D}$. Existential conditions posit the existence of situations in $\mathcal{D}$ that satisfy certain conditions in regard to C. Universal conditions apply to all situations in $\mathcal{D}$. They may use the existential quantifier "there exists," but only in a secondary manner, and restrict the behavior of C in specified ways. The universal conditions subdivide into two main classes, which are referred to as intraprofile conditions and interprofile conditions. These will receive special attention although the others cannot be ignored since all are vital to the structure of impossibility.

Impossibility theorems arise when the conditions imposed on C cannot simultaneously hold for any social choice function. The interest in such theorems stems from two factors. First, their conclusions are often surprising or paradoxical, so they excite our intellectual curiosity and challenge us to understand their structures. Second, they provide very practical guidelines for the construction of viable social choice procedures by clarifying the boundary between the possible and the impossible. The role of possibility theorems, mentioned in the preceding section, is especially important in this regard.

Proofs of possibility/impossibility theorems follow the usual procedures of deductive mathematics. Ths most common form for impossibility theorems is proof by contradiction: we assume that C satisfies all of the imposed conditions and show that this leads to an absurdity. Alternatively, we may assume that C satisfies all but one of the conditions and then deduce the conclusion that it must violate the remaining condition. Proofs of possibility theorems may involve either a demonstration that there is a social choice function which satisfies the given conditions—without necessarily specifying the exact form of such a function, or a constructive definition of a class of social choice functions that are then shown to satisfy the conditions. When the possibility theorem provides an exact characterization of the class of social choice functions that satisfy specified conditions, it is also necessary to show that every C not in this class violates one or more of the conditions.

Two proofs of Arrow's impossibility theorem will be given in the next section to illustrate alternative proof techniques. Proofs of theorems in later sections will be provided only when they are relatively short and instructive.

3. ARROW'S THEOREM

Condorcet's phenomenon of cyclic majorities [34] occurs when the individuals in N have transitive binary preferences on the alternatives in X that lead to intransitive majority comparisons. The simplest example of this has $N = \{1, 2, 3\}$ and $X = \{a, b, c\}$ with the following preference rankings for the three individuals:

1. *abc* (1 prefers a to b to c)
2. *cab* (2 prefers c to a to b)
3. *bca* (3 prefers b to c to a).

Since two of the three individuals prefer a to b, another two prefer b to c, and yet another two prefer b to c, majority preferences are cyclic:

$a >_M b$ (a has a majority over b)
$b >_M c$ (b has a majority over c)
$c >_M a$ (c has a majority over a).

Because the simple-majority relation $>_M$ is cyclic, i.e., $a >_M b >_M c >_M a$, no alternative has a majority over each of the others and there is no clear way to specify a social choice on the basis of majority comparisons. We could take $C(\{a, b, c\}, P) = \{a, b, c\}$ for the given profile, but this resolves nothing.

There is an extensive literature on Condorcet's phenomenon and on social choice functions that select majority-dominant alternatives within feasible sets when they exist: see, for example, [4, 12, 54, 60, 63, 68, 136].

Preliminaries

Arrow's theorem [4] offers a striking generalization of Condorcet's phenomenon. In outline, Arrow's theorem says that if the majority relation $>_M$ is replaced by general profile-specific social preference relations $>_P$ that depend on individuals' preferences in certain appealing ways, and if a sufficient variety of preference profiles formed from transitive preference orders are included in $\mathscr{P}$, then there must be $P \in \mathscr{P}$ for which $>_P$ is not a weak order (defined below). If stronger conditions are imposed on how $>_P$ depends on individuals' preferences, then there must be profiles for which $>_P$ has cycles. We shall return to this case in Section 5.

The proofs at the end of the present section show that a key factor in Arrow's structure is the way that the $>_P$ relations for different profiles interact with one another. This is brought about by an interprofile condition that Arrow [4, p. 26] refers to as the independence of irrelevant alternatives. Its version used here will be called binary independence. Later, in Section 6, we shall note how the interactions facilitated by binary independence can be mimicked within a single profile, thus giving rise to what is referred to as a single-profile impossibility theorem.

A few definitions will be helpful in stating Arrow's theorem. Let $>_0$ denote an *asymmetric* binary relation on a set T, so that, for all s and t in T, if $s >_0 t$ then not $(t >_0 s)$. We shall say that $>_0$ on T is a *weak order* if it is *negatively transitive*, i.e., for all $r, s, t \in T$, not $(r >_0 s)$ and not $(s >_0 t)$ imply not $(r >_0 t)$, or, equivalently,

$$r >_0 t \Rightarrow [r >_0 s \text{ or } s >_0 t].$$

It is easily checked that $>_0$ is *transitive* ($r >_0 s$ and $s >_0 t$ imply $r >_0 t$) when $>_0$ is a weak order. Moreover, the *symmetric comple-*

ment $\sim_0$ of $>_0$, defined by

$$s \sim_0 t \text{ if not } (s >_0 t) \text{ and not } (t >_0 s),$$

is also transitive ($r \sim_0 s$ and $s \sim_0 t$ imply $r \sim_0 t$) when $>_0$ is a weak order. In this case,

$$[r >_0 s \text{ and } s \sim_0 t] \Rightarrow r >_0 t,$$

$$[r \sim_0 s \text{ and } s >_0 t] \Rightarrow r >_0 t,$$

as readers can readily verify. The derived relation $\sim_0$ partitions T into a number of classes such that $\sim_0$ holds between every pair of elements within each class and, wherever T_1 and T_2 are different classes, either $t_1 >_0 t_2$ for all $t_1 \in T_1$ and all $t_2 \in T_2$, or else $t_2 >_0 t_1$ for all $t_1 \in T_1$ and all $t_2 \in T_2$.

For convenience, we write the union of $>_0$ and $\sim_0$ as $\gtrsim_0$, so that $s \gtrsim_0 t$ if either $s >_0 t$ or $s \sim_0 t$. When $>_0$ is a weak order as defined above in the asymmetric sense, $\gtrsim_0$ is transitive and complete (for all s and t in T, $s \gtrsim_0 t$ or $t \gtrsim_0 s$), and such a relation is often referred to in the economics' literature as a weak order or complete preorder. In the present monograph, the asymmetric definition applies throughout. Moreover, *given any asymmetric binary relation* $>_*$, *its symmetric complement will always be denoted by* $\sim_*$, *and the union of* $>_*$ *and* $\sim_*$ *will always be denoted by* $\gtrsim_*$.

Figure 1 pictures three weak orders on a seven-element set. In each picture, $r >_0 t$ if r lies above t, and $r \sim_0 t$ if r and t are on the same level. Since $\sim_0$ never holds between distinct elements in the left picture, $>_0$ is a total order, or *linear order,* in that case. The middle picture shows a weak order with four $\sim_0$ classes. The right picture has only one $\sim_0$ class, in which case $>_0$ is empty, i.e., there are no r and t for which $r >_0 t$.

Individuals' preference relations in Arrow's theorem are assumed to be weak orders on X. When N consists of n individuals, indexed

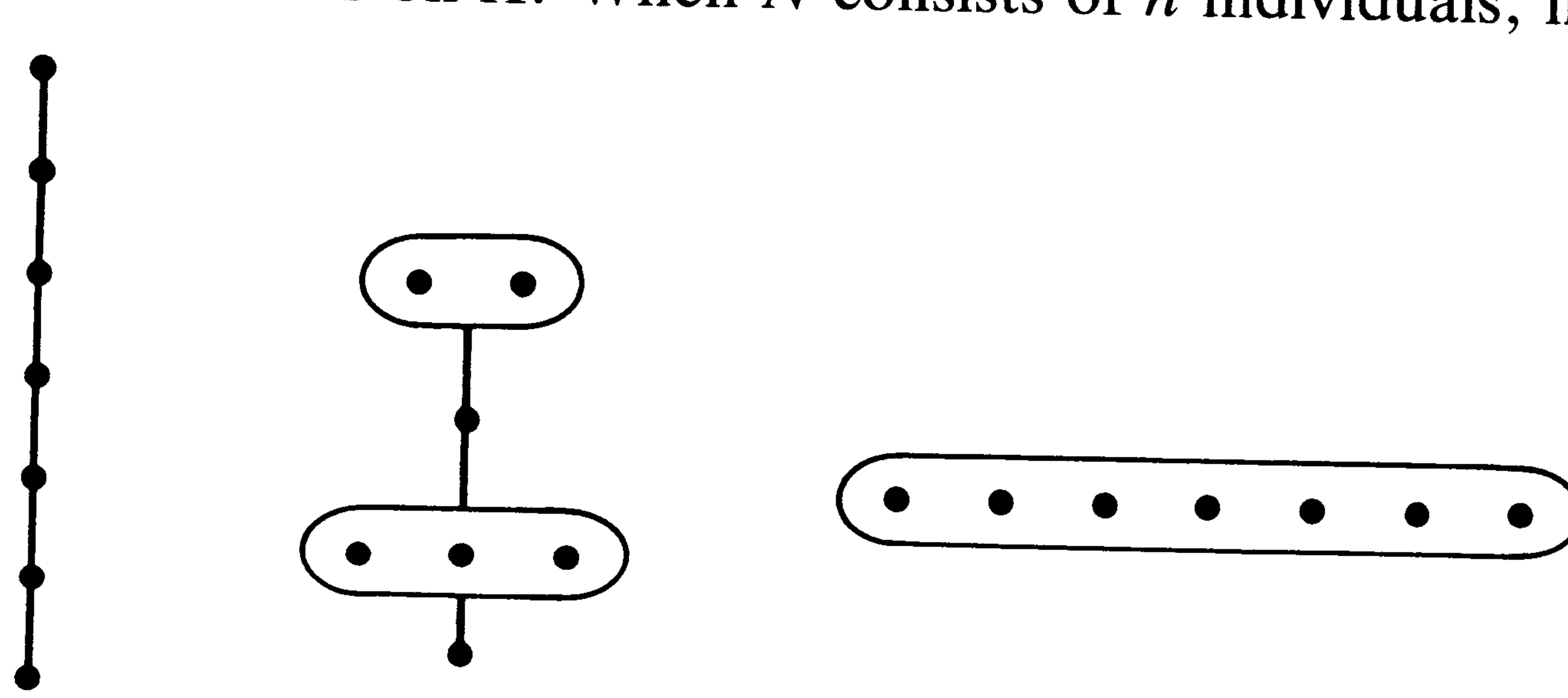

FIGURE 1

by i from 1 to n, a preference profile in $\mathscr{P}$ may be written as

$$P = (>_1, >_2, \ldots, >_n),$$

where $>_i$ denotes one possible preference relation on X for individual i. We read $x >_i y$ as "i prefers x to y," and $x \sim_i y$ (symmetric complement) as "i is indifferent between x and y." Superscripts, as in $P' = (>'_1, >'_2, \ldots, >'_n)$, are used to signify other profiles in $\mathscr{P}$.

The other half of Arrow's domain, the set $\mathscr{A}$ of feasible subsets of X, is assumed to contain every two-alternative subset $\{x, y\}$ with $x \neq y$. It may contain other subsets, but that is beside the point. The connection between the definition of a social choice function C in the preceding section and the social preference relations mentioned earlier in the present section is made explicit by the definition

$$x >_P y \text{ if } x \neq y \text{ and } C(\{x, y\}, P) = \{x\}.$$

That is, taking C as basic, we *define* x to be socially preferred to y under profile P if $x \neq y$ and $\{x\}$ is the choice set from $\{x, y\}$ when P obtains. For convenience, we shall use $>_P$ in the statement of Arrow's theorem in place of $C(\cdot, P)$ on the two-alternative sets in $\mathscr{A}$.

Arrow's theorem

THEOREM 1 *Suppose C on $\mathscr{D} = \mathscr{A} \times \mathscr{P}$ is a social function that satisfies*

A1. *N is a nonempty finite set,*
A2. *X has at least three elements, and $\mathscr{A}$ contains every two-element subset of X,*
A3. *$\mathscr{P}$ is the set of all functions from N into the set of weak orders on X.*

Then C cannot satisfy all of the following conditions:

A4. *For every i in N there exists a pair $\{x, y\} \in \mathscr{A}$ and a profile $P \in \mathscr{P}$ for which $x >_i y$ and $y \gtrsim_P 3x$,*
A5. *For all $P \in \mathscr{P}$, $>_P$ is a weak order on X,*
A6. *For all $x,y \in X$ and all $P \in \mathscr{P}$, if $x >_i y$ for all $i \in N$ then $x >_P y$,*
A7. *For all distinct x and y in X, and all $P,P' \in \mathscr{P}$, if $x >_i y \Leftrightarrow x >'_i y$ and $y >_i x \Leftrightarrow y >'_i x$ for all $i \in N$, then $x >_P y \Leftrightarrow x >_{P'} y$.*

Conditions A1, A2, and A3, along with $\mathscr{D} = \mathscr{A} \times \mathscr{P}$, are the structural conditions used in this version of Arrow's theorem. Up to the choice of the number of individuals in A1, and the possibility left open in A2 for other subsets of X in $\mathscr{A}$, they specify the domain of the social choice function.

Condition A4 is the one existential condition in the theorem. It prevents any individual from dictating social preferences. If A4 *fails,* then there exists an i in N such that, for *all* $x,y \in X$ and *all* $P \in \mathscr{P}$, $x >_P y$ whenever i prefers x to y. We refer to such an i as a *dictator.*

The final three conditions of Theorem 1 are universal conditions. The first two, A5 and A6, are *intraprofile* conditions since they apply to one profile P at a time. We refer to Arrow's social ordering condition A5 as a *passive* intraprofile condition since it does not place special restrictions on P as antecedents to its conclusion. On the other hand, the Pareto dominance condition A6 (if everyone prefers x to y, then x is socially preferred to y) is an *active* intraprofile condition since its conclusion, $x >_P y$, is based on specific aspects of P, namely $x >_i y$ for all i.

The third universal condition, A7, is an *interprofile* condition since it considers an interaction between two profiles. We referred to it earlier as a binary independence condition. This is because it is concerned with social choices from two-element subsets of X and stipulates that choices from $\{x, y\}$ under different profiles are to depend only on the individuals' preferences between x and y, independent of their preferences on all other pairs from X. In slightly different terms, A7 says that if $x \neq y$ and if the restriction of P to $\{x, y\}$ is identical to the restriction of P' to $\{x, y\}$, then $C(\{x, y\}, P) = C(\{x, y\}, P')$. If any confusion arises from the fact that the conclusion of A7 says only that $x >_P y \Leftrightarrow x >_{P'} y$, note that it also entails $y >_P x \Leftrightarrow y >_{P'} x$: simply interchange x and y throughout.

Proofs

Our two proofs of Theorem 1, which are essentially proofs by contradiction, approach the theorem in rather different ways. The first proof, which is similar to Arrow's proof, assumes that all conditions except A4 hold and then shows that some individual is a dictator. A variant of this proof appears in [8]. The second proof assumes that all conditions hold except for the finiteness of N in A1 and then uses an induction argument to show that N must be infinite.

Despite their differences, both proofs rely on preference profiles that have the following relations for some $i \in N$ and $a,b,c \in X$:

$$a >_i b >_i c,$$

$$a >_j b \text{ and } c >_j b \text{ for all } j \neq i \text{ in } N.$$

According to the Pareto condition A6, $a >_P b$ for any such profile. Hence, if $b \succsim_P c$, then the social preference condition A5 forces $a >_P c$; and if $c \succsim_P a$, then A5 forces $c >_P b$. This shows how new social preferences are generated by the Pareto condition and other social preferences or indifferences. It is a key step in both proofs.

For the first proof of Theorem 1, assume that all conditions except A4 hold. For all nonempty subsets I of N and all distinct $x,y \in X$, write

$$xIy \text{ if } x >_P y \text{ whenever } P \text{ has } x >_i y \text{ for all } i \in I \text{ and } y >_i x \text{ for all } i \in N \setminus I,$$

where in general $A \setminus B$ denotes the set of elements in A that are not also in B. Also write

$$xiy \text{ if } x >_P y \text{ whenever } P \text{ has } x >_i y.$$

The proof has two main steps. The first shows that $a\{i\}b$ for some $i \in N$ and some $a,b \in X$. Given this i, the second shows that xiy for *all* distinct $x,y \in X$. But then i is a dictator, so A4 must fail.

Step 1. By A6, xNy for all distinct $x,y \in X$. By A1 (N finite), there is a smallest I, say I^*, such that aI^*b for some $a,b \in X$. Fix $i \in I^*$. We claim that $I^* = \{i\}$. If not, consider any profile $P \in \mathscr{P}$ (justified by A2 and A3, with $x \in X \setminus \{a, b\}$) with

$$x >_i a >_i b,$$

$$a >_j b >_j x \text{ for all } j \in I^* \setminus \{i\},$$

$$b >_j x >_j a \text{ for all } j \in N \setminus I^*.$$

Then $a >_P b$ (by hypothesis and A7), $x \succsim_P a$ (else $a(I^* \setminus \{i\})x$, contradicting I^* as the smallest such I), and $b \succsim_P x$ (else $x\{i\}b$, again contradicting I^* as the smallest). But $\{b \gtrsim_P x, x \gtrsim_P a, a >_P b\}$ violates A5. Hence $I^* = \{i\}$.

Step 2. Given $a\{i\}b$, take $x \in X \setminus \{a, b\}$ by A2 and use A3 to construct P with

$$x >_i a >_i b,$$

$$x >_j a \text{ and } b >_j a \text{ for all } j \neq i \text{ in } N.$$

Then $a >_P b$ (by $a\{i\}b$) and $x >_P a$ (by A6), so $x >_P b$ according to A5. Since the relations between x and b for $j \neq i$ are arbitrary in P, it follows from A7 that xib. A similar argument shows that aix. These are true for all $x \notin \{a, b\}$. Moreover, since $x\{i\}b$ and $a\{i\}x$, the same arguments can be reapplied (start with x,b instead of a,b, for example) to get bix and xia, and then bia. Since this shows that xiy for all distinct x and y in X, the proof is complete.

For the second proof of Theorem 1, assume that all conditions hold except for the finiteness of N in A1. For all $i \in N$ and all distinct $x,y \in X$, write

$$xi^*y \text{ if } x >_P y \text{ whenever } P \text{ has } y >_i x \text{ and}$$

$$x >_j y \text{ for all } j \in N \setminus \{i\}.$$

Also, for each positive integer m, write

$$x\mathbf{m}y \text{ if } x >_P y \text{ whenever any } m \text{ individuals for } P$$

$$\text{have } y >_i x \text{ and all others have } x >_i y.$$

Again, there are two steps to the proof. The first shows that $x\mathbf{1}y$ for all distinct $x,y \in X$. The second then shows that if $x\mathbf{k}y$ for all $k \leq m$ and all distinct $x,y \in X$, then $x(\mathbf{m}+\mathbf{1})y$ for all $x \neq y$ in X. But then N must be infinite, so A1 must fail.

Step 1. Since $N \neq \varnothing$ by part of A1, choose any $i \in N$. Then A2, A3, and A4 (no dictator) imply that $b \gtrsim_P a$ for some $a,b \in X$ and some $P \in \mathscr{P}$ that has $a >_i b$. Take $x \in X \setminus \{a, b\}$ by A2, and by A3 construct P' so that

$$a >'_i x >'_i b,$$

$$a >'_j x \text{ and } b >'_j x \text{ for all } j \neq i \text{ in } N,$$

and such that, for all $i \in N$, $a >_i b \Leftrightarrow a >'_i b$ and $b >_i a \Leftrightarrow b >'_i a$. Then $b \gtrsim_{P'} a$ by A7 (binary independence) and $a >_{P'} x$ by A6, so $b >_{P'} x$

by A5. It follows from A7 that bi^*x. A similar demonstration gives xi^*a. With $y \in X\backslash\{x, b\}$ by A2, let $P^1 \in \mathscr{P}$ be the same as P' on $\{x, b\}$ and have $x >_i^1 y >_i^1 b$ along with $y >_j^1 b >_j^1 x$ for all $j \neq i$. Since $b >_{P^1} x$, and $y >_{P^1} b$ by A6, A5 implies that $y >_{P^1} x$, and therefore yi^*x by A7. This includes ai^*x. By similar proofs, xi^*b, bi^*a, and ai^*b, so we conclude that xi^*y for all distinct $x,y \in X$.

Since this is true for all $i \in N$, it follows that $x\mathbf{1}y$ for all distinct x and y in X.

Step 2. Conditions A4 and A6 require N to have at least two individuals. Let $|N|$ denote the cardinality of N, so $|N| \geq 2$. Suppose $|N| > m \geq 1$ and $x\mathbf{k}y$ for all $1 \leq k \leq m$ and all distinct $x,y \in X$. Choose $i \in N$ and $I \subset N$ such that $i \notin I$ and $|I| = m$, and let P be a profile in $\mathscr{P}$ for which

$$y >_i x >_i a,$$

$$a >_j y >_j x \text{ for all } j \in I,$$

$$x >_j a >_j y \text{ for all } j \in N\backslash(I \cup \{i\}).$$

Then $a\mathbf{1}y$ implies $a >_P y$, and $x\mathbf{m}a$ implies $x >_P a$, so $x >_P y$ by A5. Since this contradicts A6 if $|N| = m + 1$, we conclude that $|N| > m + 1$. Moreover, since i,x, and y are arbitrary, $x(\mathbf{m}+\mathbf{1})y$ for all distinct x and y. Since $x\mathbf{1}y$ by step 1, it follows by induction on m that $|N| > m$ for every positive integer m.

4. CONDITIONS ON SOCIAL CHOICE

The conditions on C in Theorem 1 are examples of the main types of conditions used in other possibility/impossibility theorems. The present section summarizes our classification of these types and then illustrates them further with conditions other than A1–A7. The initial illustrations weaken Arrow's conditions one by one to obtain a list of seven possibility theorems. We then introduce other conditions that play important roles in social choice theory.

Summary classification

To avoid confusion that might arise from negations, we assume that all conditions in the existential and universal categories of our classification use only the positive forms of the existential quantifier "there exists" and the universal quantifier "for all." For example, the version of the ordering condition A5 that applies to the classification is "for all $P \in \mathscr{P}$, $>_P$ is a weak order on X." This is a universal condition even though, with the use of negations, it is tantamount to "it is false that there exists a $P \in \mathscr{P}$ for which $>_P$ on X is not a weak order."

The three main types of conditions noted earlier are:

1. *Structural conditions* that describe restrictions on the domain $\mathscr{D}$ of C, including aspects of N (individuals), X (alternatives), $\mathscr{A}$ (feasible subsets of alternatives), and $\mathscr{P}$ (preference profiles);

2. *Existential conditions* that prescribe the existence of situations in $\mathscr{D}$ that have specified behaviors under C. Their formal statements must use the existential quantifier and may use the universal quantifier (all without prefatory negations);

3. *Universal conditions* that specify aspects of the behavior of C throughout $\mathscr{D}$ and whose formal statements do not use the existential quantifier, except perhaps in a secondary manner (see below). The universal conditions subdivide into two categories according to the number of profiles involved in their statements following the prefatory universal quantifiers:

3A. *Intraprofile* (single-profile) conditions consider one profile at a time;

3B. *Interprofile* conditions consider more than one profile at a time.

The only interprofile condition used thus far is A7, binary independence. It is a *two-profile* interprofile condition since it says that if P and P' in $\mathscr{P}$ relate to each other in a certain way, then $C(\cdot, P)$ and $C(\cdot, P')$ must relate to each other in certain ways. Interprofile conditions that may require simultaneous consideration of more than two profiles are sometimes referred to as *multiprofile* conditions.

Intraprofile conditions subdivide further into two categories according to whether they impose restrictions on $P \in \mathscr{P}$ beyond those used in the structural conditions that apply to $\mathscr{D}$:

3Ap. *Passive* intraprofile conditions impose no further restrictions on P;

3Aa. *Active* intraprofile conditions impose further restrictions on P.

For example, the passive intraprofile condition "for all $P \in \mathscr{P}$, $>_P$ is a weak order on X" applies equally to all feasible profiles regardless of their structures, whereas the active intraprofile condition "for all $x,y \in X$ and all $P \in \mathscr{P}$, if, for all $i \in N$, $x >_i y$, then $x >_P y$" applies only to profiles that contain dominance pairs ($x >_i y$ for all $i \in N$).

As suggested above, there are conditions that we will adopt as universal conditions even though they use the existential quantifier in a secondary way. The important aspect of such conditions is that they apply to all situations in $\mathscr{D}$ and do not posit the existence of situations with special features under C. A case in point is the following condition, which specifies the existence of a maximal alternative for every situation on the basis of binary comparisons:

For all $(A, P) \in \mathscr{D}$, *there exists an* $x \in A$ *such that* $x \in C(\{x, y\}, P)$ *for all* $y \in A$.

We regard this as a passive intraprofile condition, not as an existential condition.

Possibility theorems

We now consider seven kinds of possibility theorems that arise when the conditions of Theorem 1 are modified one at a time. There are three reasons for doing this beyond the mere exercise of exhibiting possibility theorems. First, it provides other examples for the preceding classification scheme. Second, it promotes an appreciation of the delicate interactions among the conditions that lead to Arrow's result. And third, it identifies points of departure for a number of other research topics in social choice theory.

In each of the seven succeeding paragraphs, we shall modify the condition that introduces the paragraph. All other conditions of Theorem 1, including $\mathscr{D} = \mathscr{A} \times \mathscr{P}$, are assumed to hold.

A1. Suppose N is allowed to be any nonempty set. Then, as first shown in [51] although it was independently discovered by Julian Blau about 1960, there are C that satisfy this modification of A1. By Theorem 1, they require N to be infinite. Section 10 gives more details.

A2. Change A2 by assuming that X has exactly two alternatives, say x and y. Then $>_P$ defined by the simple majority relation $>_M$ for each profile satisfies the conditions of Theorem 1 thus modified. An axiomatic characterization of $>_M$ for the two-alternative case was first given in [104]. A different change in A2 produces another possibility theorem as follows. Suppose X consists of the union of disjoint sets $X_1, X_2, \ldots, X_m$ $(m \geqslant 2)$, each X_j has at least three alternatives, and $A \in \mathscr{A}$ if and only if $|A| = 2$ and the two alternatives in A are in the same X_j. Then, by Theorem 1, there will be a dictator for each X_j set, but, when $|N| \geqslant 2$, different individuals can be dictators for different X_j, so A4 can be satisfied along with the other conditions. See [58] for additional details.

A3. Restrict $\mathscr{P}$ by requiring that all feasible preference profiles be single-peaked with linear orders (no indifference between distinct alternatives) for all individuals. Single-peakedness means that the alternatives in X can be linearly ordered in such a way that, for every $i \in N$, the preferences of i increase up to a unique most-preferred alternative and then decrease thereafter as we move through the underlying linear order on X. Then, so long as $|N|$ is odd and greater than or equal to three, $>_M$ will be a weak order on X for every such profile, and hence all conditions of Theorem 1 can hold under the single-peaked restriction on $\mathscr{P}$. Single-peaked profiles were first investigated extensively in [12]. Similar profile restrictions that guarantee the existence of an alternative that is beaten by no other alternative under simple-majority comparisons are discussed in [4, 31, 54, 116, 136, 140]. More generally, [55] shows how combinations of active intraprofile conditions and profile restrictions lead to transitivity and similar properties for binary social choices.

A4. Weaken A4 by assuming only that no $i \in N$ is an "absolute dictator" rather than that no $i \in N$ is a dictator. We say that i is an *absolute dictator* if, for all $x, y \in X$ and all $P \in \mathscr{P}$, $x >_i y \Rightarrow x >_P y$, and $x \sim_i y \Rightarrow x \sim_P y$. Stated as an existential condition in the assumed format, our weakening of A4 is: For every $i \in N$ there exists a pair $\{x, y\} \in \mathscr{A}$ and a $P \in \mathscr{P}$ for which either $x >_i y$ and $y \gtrsim_P x$, or $x \sim_i y$ and $x >_P y$." With $N = \{1, 2, \ldots, n\}$ and $n \geqslant 2$, all conditions in Theorem 1 thus modified will hold when we define $>_P$ lexicographically as $x >_P y$ if $x >_1 y$ or $(x \sim_1 y, x >_2 y)$ or $(x \sim_1 y, x \sim_2 y, x >_3 y)$ or ... or $(x \sim_1 y, \ldots, x \sim_{n-1} y, x >_n y)$. Then individual 1 is a dictator, but no individual is an absolute dictator. Note also that $>_P$ is a weak order on X for each profile.

A5. Suppose we replace the social ordering condition A5 by the less-demanding passive intraprofile condition "for all $P \in \mathscr{P}$, $>_P$ is transitive on X." Then the modified conditions of the theorem hold when $>_P$ is defined either by

$$x >_P y \text{ if } x >_i y \text{ for all } i \in N,$$

or by

$$x >_P y \text{ if } x \gtrsim_i y \text{ for all } i \in N, \text{ and } x >_i y \text{ for some } i \in N,$$

since $>_P$ is transitive for each type of Pareto ordering. We examine this and related weakenings of A5 in the next section.

A6. Replace the active intraprofile Pareto-dominance condition A6 by the existential condition "for all distinct $x,y \in X$ there exists a $P \in \mathcal{P}$ such that $x >_P y$." This condition is referred to as *citizens' sovereignty* by Arrow [4, p. 28]. To see that it can hold along with the other conditions of Theorem 1, just define $>_P$ by $x >_P y$ if $y >_1 x$. In a manner of speaking, this makes individual 1 an "absolute anti-dictator," but no person is a dictator, so A4 holds. In the original edition of his book, Arrow used citizens' sovereignty in concert with a condition involving positive association between individual and social preferences. These were later replaced by the single Pareto condition A6 [4, p. 97]. It is this later version that is stated as Theorem 1.

A7. Assume that X is finite and, for any weak order $>_*$ on X, define the distance d_* between x and y by $d_*(x, y) = 0$ if $x \sim_* y$, $d_*(x, y) = k > 0$ if $x >_* y$ and there are $(k-1) \sim_*$ classes between x and y, and $d_*(x, y) = k < 0$ if $y >_* x$ and $d_*(y, x) = -k$. Replace A7 by another interprofile condition, as follows: "For all distinct x and y in X, and all $P, P' \in \mathcal{P}$, if $d_i(x, y) = d_i'(x, y)$ for all $i \in N$, then $x >_P y \Leftrightarrow x >_{P'} y$." Then the modified conditions of Theorem 1 hold when $>_P$ is defined by

$$x >_P y \text{ if } \sum_{i=1}^{n} d_i(x, y) > 0.$$

Since $d_i(x, y) + d_i(y, z) = d_i(x, z)$ for any weak order $>_i$ and all $x, y, z \in X$, it follows readily that $>_P$ is a weak order for every profile. When every $>_i$ is a linear order on X, $>_P$ as just defined is referred to as the *Borda ordering* of X, named after Borda [21]. Borda's method is but one of a large number of ways to construct a social preference ordering on the basis of individuals' preferences.

An indication of the extensive research on positional scoring rules can be obtained from [49, 50, 54, 69, 141, 153, 154].

Universal conditions

Since the central topic of our study is the role of profile conditions in impossibility theorems, we shall say a bit more about universal conditions at this point. Other structural and existential conditions will be noted as they arise in settings considered in ensuing sections.

We consider passive intraprofile conditions first, then comment on interprofile conditions, and conclude with active intraprofile conditions. In all cases it is to be understood that the conditions apply to all situations in the domain of C.

Since passive intraprofile conditions apply to one P at a time and make no demands on P apart from what is already implied by $\mathscr{D}$, it is convenient to suppress P in $C(A, P)$ and simply write the choice set for situation (A, P) as $C(A)$. Thus, negative transitivity for $>_P$ with an arbitrary P understood could be written as $[y \in C(\{x, y\}), z \in C(\{y, z\})] \Rightarrow z \in C(\{x, z\})$, and transitivity for $>_P$ could be written as $[x \neq y \neq z \neq x, C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}] \Rightarrow C(\{x, z\}) = \{x\}$.

Passive intraprofile conditions, which are often referred to as conditions of consistency or collective rationality, interrelate choice sets $C(A)$ for different $A \in \mathscr{A}$ under the same preference profile. Three essentially different types of passive intraprofile conditions appear in the literature. We refer to these as uniform conditions, expansion conditions, and contraction conditions. Many of these have been classified by Sen [138], who notes a number of interrelationships among them. See also [5, 13, 54, 118].

Uniform conditions employ only $A \in \mathscr{A}$ that have the same size or cardinality. Examples for $|A| = 2$ include A5, the condition that every $>_P$ is transitive, and the condition which says that, for any $|A| \geq 2$, there is an $x \in A$ such that $x \in C(\{x, y\})$ for all $y \neq x$ in A. A generalization of the latter condition for $m > 2$ says that, for any $|A| \geq m$, there is an $x \in A$ such that $x \in C(B)$ for every subset B of A that contains exactly m alternatives, including x.

Expansion conditions restrict choices from larger sets on the basis of choices from their subsets. Four examples are:

1. $x \in C(A) \cap C(B) \Rightarrow x \in C(A \cup B)$,
2. $[x \in C(\{x, y\})$ for all $y \in A] \Rightarrow x \in C(A \cup \{x\})$,
3. $[A \subset B; x, y \in C(A); y \in C(B)] \Rightarrow x \in C(B)$,
4. $C(A \cup B) = C[C(A) \cup C(B)]$.

Each of these has a straightforward interpretation. For example, 1 says that if x is in the choice sets of both A and B, then it will be in the choice set of $A \cup B$, and 3 says that if x and y are in the choice set of A, y is in the choice set of B, and A is a proper subset of B, then x will be in the choice set of B. Condition 4 is Plott's "path independence" condition [118].

Contraction conditions restrict choices from smaller sets on the basis of choices from their supersets. Examples include:

1. $[A \subset B, A \cap C(B) \neq \varnothing] \Rightarrow C(A) = A \cap C(B)$,
2. $[x \in A \subset B, x \in C(B)] \Rightarrow x \in C(A)$,
3. $[x \in A \subset B, C(B) = \{x\}] \Rightarrow C(A) = \{x\}$,
4. There is an $x \in C(B)$ such that $x \in C(\{x, y\})$ for every $y \in B$.

Condition 3 is the specialization of condition 1 for unique choices. Condition 4 is another case of a passive intraprofile condition with an existential quantifier.

In contrast to the passivity of the preceding conditions with respect to P, interprofile and active intraprofile conditions base their conclusions on structures between or within profiles. Some of these conditions reflect notions of fairness or equity among individuals or alternatives, while others stress positive correlations between individuals' preferences and social choices.

We have already remarked on the interprofile condition of binary independence, A7. Later, we shall use its generalization which says that if $A \in \mathscr{A}$ and if the restrictions of profiles P and P' to A are identical, then $C(A, P) = C(A, P')$.

The most common interprofile conditions besides independence are monotonicity (positive association), anonymity, and neutrality conditions. There are several monotonicity conditions, but all are based on the idea that if $x \in C(A, P)$, and if P' is similar to P except that the standing of x is improved in one or more of the $>_i$ in going from P to P', then $x \in C(A, P')$.

To state general versions of the so-called symmetry conditions of anonymity (over individuals) and neutrality (among alternatives), we need definitions involving permutations on N and X. A *permutation* on a set T is a one-to-one mapping from T onto T. Given $P \in \mathscr{P}$ and a permutation σ on N, let P^σ be the profile that assigns the preference relation $>_{\sigma(i)}$ from P to individual i. For example, if $P = (>_1, >_2, >_3)$ and the relation in position k is for individual k, and if $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$, then $P^\sigma = (>_2, >_3, >_1)$. Hence individual 1 now has the relation originally held by individual 2, and so forth.

Anonymity. For all permutations σ on N, and all $(A, P), (A, P^\sigma) \in \mathscr{D}$, $C(A, P) = C(A, P^\sigma)$.

This is designed to treat individuals equally. Its companion, neutrality, is designed to treat alternatives equally. Given $P \in \mathscr{P}$ and a permutation λ on X, let $P^{(\lambda)}$ be obtained from P by λ acting on X for every $i \in N$: i.e., for all $i \in N$,

$$x >_i y \Leftrightarrow \lambda(x) >_i^{(\lambda)} \lambda(y), \text{ for all } x,y \in X.$$

In addition, for any nonempty subset B of X, let $\lambda(B) = \{\lambda(x) : x \in B\}$.

Neutrality. For all permutations λ on X, and all $(A, P), (\lambda(A), P^{(\lambda)}) \in \mathscr{D}$, $C(\lambda(A), P^{(\lambda)}) = \lambda(C(A, P))$.

This says, for example, that if $C(A, P) = \{x\}$, and if x and y are interchanged in every preference relation in P, all else unchanged, then y will be the unique choice from $\lambda(A)$ after the changes in the profile.

The Pareto condition A6, and its stronger version which says that $C(\{x, y\}, P) = \{x\}$ whenever $x \succsim_i y$ for all $i \in N$ and $x >_i y$ for some $i \in N$, represent the class of active intraprofile conditions which use the same $A \in \mathscr{A}$ throughout.

Other active intraprofile conditions vary A. Here are two examples:

1. For all $x,y \in X$, and all $(A, P) \in \mathscr{D}$, if $x,y \in A$, $x >_i y$ for all $i \in N$, and $(A \setminus \{y\}), P) \in \mathscr{D}$, then $C(A \setminus \{y\}, P) = C(A, P)$.
2. For all $x,y,a,b \in X$, and all $P \in \mathscr{P}$, if $x >_i y \Leftrightarrow a >_i b$ and $y >_i x \Leftrightarrow b >_i a$ for all $i \in N$, and if $(\{x, y\}, P), (\{a, b\}, P) \in \mathscr{D}$, then $a \in C(\{a, b\}, P) \Leftrightarrow x \in C(\{x, y\}, P)$.

The first condition says that a Pareto dominated alternative within A can be discarded from A without affecting the choice set. The second embraces aspects of neutrality and independence within the same profile. It says that if each $>_i$ behaves the same way on the ordered pair (a, b) as on (x, y), then the binary choices from $\{a, b\}$ and $\{x, y\}$ will be similar. It is this kind of mimicking of binary independence within a single profile that gives rise to some of the single-profile impossibility theorems that we shall encounter in Section 6.

5. MULTIPROFILE IMPOSSIBILITY THEOREMS

Arrow's theorem is a multiprofile impossibility theorem since it uses the interprofile condition of binary independence. We now describe other multiprofile results that were motivated by Arrow's work and which use structural assumptions that are similar to his. Other structural configurations will be considered in later sections.

No attempt will be made to list all of the Arrow-type impossibility theorems. We shall, however, note their main lines of development. Slightly different perspectives on these lines are given by Sen [138] and Kelly [95].

The theorems presented in this section are divided into eight categories according to their changes in Arrow's conditions. The first six categories use A2 and A7 (two-alternative feasible subsets and binary independence), though moves away from A2 and A7 are discussed for the sixth category. The final two categories consider choices from larger feasible sets and significantly modify A7. A brief outline of the categories follows.

1. Tighten A3 by using fewer profiles.
2. Relax A5 to partial orders. Either tighten A4 to no vetoer or add a strong monotonicity condition plus $|N| \geq 3$.
3. Relax A5 in other ways, e.g., to semiorders or interval orders, and require $|X| \geq 4$.
4. Relax A5 to acyclic relations, strengthen A4 to no vetoer, tighten A1 and/or A2 by requiring more individuals and/or alternatives, and perhaps add a monotonicity condition.
5. Replace the Pareto condition A6 by others.
6. Apply independence to larger sets, or retain A7 and expand $\mathscr{A}$ to include all nonempty subsets of X in conjunction with other modifications.

7. Change A2 by assuming that $\mathscr{A}$ need only contain all A for which $|A| \geqslant m$ or for which $|A| = m$, for some $m \geqslant 3$. Modify A7 and other conditions accordingly.
8. Change A2 drastically by assuming that $\mathscr{A}$ can have only one feasible set. Modify other conditions.

Profiles

Several writers [4, 17, 54, 90, 95, 150], including Arrow, note that certain subsets of the profiles described in A3 suffice for impossibility when no other changes are made in Theorem 1. For example, the proofs in Section 3 are not affected if $\mathscr{P}$ is the set of all functions from N into the set of *linear orders* on X. It also suffices to use linear orders for individuals such that each triple of alternatives in X can have any configuration in a profile. For example, suppose $X = \{a, b, c, d\}$ and $\mathscr{P}$ consists of all profiles that can be formed from the following six linear orders on X:

$$\begin{array}{cccc} a\,b\,c\,d & & c\,b\,a\,d \\ a\,d\,c\,b & & c\,d\,a\,b \\ b\,d\,a\,c & & d\,b\,c\,a. \end{array}$$

Then, if $\{x, y, z\}$ is any three-alternative subset of X, all six of xyz, xzy, yxz, yzx, zxy, and zyx appear in the preceding orders, so each assignment of these six to the $i \in N$ corresponds to a profile in $\mathscr{P}$.

Partial orders and vetoers

One of the earliest variations on Arrow's theme [71, 77, 110, 131, 134] changed only A4 and A5, to:

A4*. *For every i in N there exists a pair $\{x, y\} \in \mathscr{A}$ and a $P \in \mathscr{P}$ such that $x >_i y$ and $y >_P x$;*

A5*. *For all $P \in \mathscr{P}, >_P$ is transitive on X.*

An $i \in N$ who violates A4* ($x \gtrsim_P y$ whenever $x >_i y$) will be referred to as a *vetoer*; some writers prefer the term "weak dictator." For A5* we presume, by the definition of $>_P$ in terms of C, that $>_P$ is asymmetric, and will refer to any asymmetric and transitive binary relation $>_0$ as a *partial order*. When $>_0$ is a partial order, the relation $\gtrsim_0$ is sometimes referred to as a "quasi-transitive" order since only $>_0$ (and not $\sim_0$) is assumed to be transitive.

THEOREM 2A *Suppose* C *on* $\mathcal{D} = \mathcal{A} \times \mathcal{P}$ *is a social choice function that satisfies* A1, A2, A3, A6, *and* A7. *Then it must violate either* A4* *or* A5*.

Thus, when A5 is relaxed to partial orders, some individual must be a vetoer, or have "veto power." A more definitive result was noted by Gibbard [71]. Call $I \subseteq N$ an *oligarchy* if I is not empty, every $i \in I$ is a vetoer, and I collectively dictates social preferences, i.e., $x >_P y$ whenever $x >_i y$ for all $i \in I$. Given A1, A2, A3, A5*, A6, and A7, Gibbard proved that N includes exactly one oligarchy. Proofs appear in [54, 77, 95]; the last two of these have further observations on oligarchies.

A proof of Theorem 2A follows readily from the second proof of Theorem 1. In step 1, where A5 is first used, we use A5* instead to get $b >_{P'} x$ from $b >_{P'} a$ and $a >_{P'} x$, where $b >_{P'} a$ follows from the change of A4 to A4*. All other uses of A5 in that proof require only transitivity for $>_P$, so again we conclude that N must be infinite.

Mas-Colell and Sonnenschein [102] show that the no-dictator condition A4 can be used in Theorem 2A instead of A4* if we add the following interprofile condition of strong monotonicity along with $|N| \geq 3$.

A8. *For all* $i \in N$, *all* $x,y \in X$, *and all* $P,P' \in \mathcal{P}$, *if* $>_j = >'_j$ *for all* $j \in N \setminus \{i\}$, *either* $(y >_i x, x \sim'_i y)$ *or* $(x \sim_i y, x >'_i y)$, *and* $x \gtrsim_P y$, *then* $x >_{P'} y$.

THEOREM 2B *Suppose* C *on* $\mathcal{D} = \mathcal{A} \times \mathcal{P}$ *is a social choice function that satisfies* A1 *with* $|N| \geq 3$, A2, A3, A6, A7, *and* A8. *Then it must violate* A4 *or* A5*.

The reason that at least three individuals are used is that $>_M$ satisfies the other conditions when $|N| = 2$. This is reflected in a proof of Theorem 2B that modifies step 1 of the second proof of Theorem 1. If $b >_P a$ in the first few sentences of that proof, then the rest of the proof carries through with A5* in place of A5. To avoid $b >_P a$ for any i in the presence of A8 and A5*, it is easily seen that there is an $a_i \in A$ for $i = 1,2$ such that, for all $x \in X$,

$(a_1 >_1 x, x >_j a_1$ for all $j \neq 1) \Rightarrow a_1 \sim_P x$, and $(a_2 >_2 x, x >_j a_2$ for all $j \neq 2) \Rightarrow a_2 \sim_P x$. In particular, *if* $a_1 \neq a_2$, then we have $a_1 \sim_P a_2$ when $(a_1 >_1 a_2, a_2 >_2 a_1, a_2 >_j a_1$ for all $j \geq 3)$; hence, by A8, $a_1 >_P a_2$ when $(a_1 >_1 a_2, a_2 >_2 a_1, a_1 >_j a_2$ for all $j \geq 3)$. But this contradicts the derived property of a_2 versus x for individual 2. On the other hand, if $a_1 = a_2 = a$, then the use of $x,y \in X$ with $|\{a, x, y\}| = 3$ and $(y >_1 a >_1 x, x >_2 a >_2 y, x >_j a$ and $y >_j a$ for all $j \geq 3)$ leads to a contradiction.

Specialized social preference relations

It is impossible to categorically weaken A5 without changing some other condition and still have impossibility since the full force of A5 is needed when $|X| = 3$. However, if X has more than three alternatives, then A5 can be weakened without changing other conditions or adding new conditions. Some of these weakenings are partial orders, and lie between partial orders and weak orders in the hierarchy of ordering relations; other weakenings are not partial orders, and need not even be acyclic. Some of these will be mentioned after we look at orders intermediate between partial and weak orders. The next subsection then takes a closer look at acyclicity.

Two conditions on an asymmetric binary relation $>_0$ on T that separately imply that $>_0$ is transitive but collectively do not imply that $>_0$ is a weak order are:

(1) for all $a,b,x,y \in T$, $[a >_0 x, b >_0 y] \Rightarrow [a >_0 y$ or $b >_0 x]$;
(2) for all $a,b,c,x \in T$, $[a >_0 b >_0 c] \Rightarrow [a >_0 x$ or $x >_0 c]$.

We refer to $>_0$ as an *interval order* if it satisfies (1), a *semitransitive order* if it satisfies (2), and a *semiorder* if it satisfies both (1) and (2): see [64] for a general discussion of these relations.

A5(1). *For all* $P \in \mathcal{P}$, $>_P$ *is an interval order on* X.

A5(2). *For all* $P \in \mathcal{P}$, $>_P$ *is a semitransitive order on* X.

THEOREM 3 *Suppose* C *on* $\mathcal{D} = \mathcal{A} \times \mathcal{P}$ *is a social choice function that satisfies* A1, A2 *with* $|X| \geq 4$, A3, A6, *and* A7. *Then it violates either* A4 *or* A5(1), *and it violates either* A4 *or* A5(2).

This was proved independently by Blau [19] and Blair and Pollak [14]. In fact, they proved much more, as we now note.

Let us say that an asymmetric relation $>_0$ on T is *α-transitive* if, whenever the x_i are in T,

$$[x_1 >_0 x_2 >_0 \cdots >_0 x_{\alpha+1}] \Rightarrow x_1 >_0 x_{\alpha+1},$$

and is *(α, β)-transitive* if, whenever the x_i and y_j are in T,

$$[x_1 >_0 \cdots >_0 x_{\alpha+1} \sim_0 y_1 >_0 \cdots >_0 y_{\beta+1}] \Rightarrow x_1 >_0 y_{\beta+1}.$$

Then, if A1, A2, A3, A6, and A7 hold, and if $\alpha, \beta \geq 0$, $\alpha + \beta \geq 2$, and $|X| \geq \alpha + \beta + 2$, either A4 fails or it is false that every $>_P$ is (α, β)-transitive.

Blau [19] also shows that even weaken conditions can be used for $>_P$ when X is infinite, and Blair and Pollak [14] prove that an oligarchy exists when A1, A2, A3, A6, and A7 hold and there is an $\alpha \geq 2$ such that $|X| \geq \alpha + 1$ and every $>_P$ is α-transitive. When $\alpha = 2$, the latter result reduces to Gibbard's oligarchy theorem.

The interesting technical aspect of these other results is that, when $\alpha + \beta \geq 3$, α-transitive and (α, β)-transitive relations need not be partial orders and may even have $>_P$ cycles. However, as we shall now explain, social choice theory seems somewhat more concerned with acyclic social preferences.

Acyclicity

An asymmetric binary relation $>_0$ on a set T is *acyclic* if there is no integer $m \geq 3$ and t_i in T such that $t_1 >_0 t_2 >_0 \cdots >_0 t_m >_0 t_1$. It is easily seen that $>_0$ is acyclic if and only if $\{s \in S : s \gtrsim_0 t \text{ for all } t \in S\}$ is nonempty for every nonempty finite subset S of T. We shall say that $>_0$ is *3-acyclic* if it is never true that $x >_0 y$, $y >_0 z$, and $z >_0 x$ for three elements in T. Relations that are 3-acyclic need not be acyclic in general.

Acyclic social preference relations $>_P$ have attracted wide attention since they guarantee that every nonempty finite $A \subseteq X$ has a *$>_P$-maximal* alternative, i.e., an $a \in A$ such that no $b \in A$ has $b >_P a$. Moreover, acyclicity is the weakest property for $>_P$ that offers this guarantee. A social choice function that satisfies A1–A3 and the following condition is often referred to as a "social decision function" (SDF) [134].

A5**. *For all $P \in \mathcal{P}$, $>_P$ is acyclic on X.*

proved that A1, A2, A3, A6, and his version of m-ary independence imply A7. Hence, in each of the preceding theorems that use A1, A2, A3, A6, and A7, we can replace A7 by m-ary independence for any fixed finite $m \geqslant 2$ that is smaller than $|X|$.

A different move, made by a number of people [7, 13, 46, 54, 95, 113, 118, 136, 138], is to retain A7 along with A1 and A3 but also to expand $\mathscr{A}$ to contain every nonempty subset of X. This is done to allow the use of passive intraprofile (consistency, rationality) conditions of the expansion or contraction types in place of direct ordering conditions like A5 and A5**. In many cases, the expansion/contraction conditions imply ordering properties for $>_P$ and may therefore be used in preceding impossibility theorems in place of uniformity conditions so long as $\mathscr{A}$ is suitably expanded. This may appeal to some people since the expansion-contraction conditions often have a direct intuitive flavor that is not obvious for conditions like A5* and A5**.

Two examples will illustrate this: see [54, 95, 138] for additional discussion. First, Plott's path independence condition $C(A \cup B) = C(C(A) \cup C(B))$ implies that each $>_P$ is transitive, hence that A5* holds. Second, the contraction condition $A \subset B \Rightarrow A \cap C(B) \subseteq C(A)$ implies that each $>_P$ is acyclic, hence that A5** holds. Therefore, with $\mathscr{A}$ suitably expanded, A5* can be replaced by path independence, and A5** can be replaced by the noted contraction condition.

The approach of retaining A7, expanding $\mathscr{A}$, and using other passive intraprofile conditions in place of the direct ordering conditions has perhaps received its deepest expression in Bandyopadhyay [7]. He retains A7 along with A1 with $|N| \geqslant 2$ and A3, and assumes that $\mathscr{A}$ contains every nonempty subset of X. For reasons discussed in [7], he considers only expansion conditions for the passive intraprofile category.

Five expansion conditions figure in Bandyopadhyay's impossibility theorems. One of these says that, for all nonempty index sets J and all $A_j \in \mathscr{A}$:

A5$_1$. *If* $|A_j| \geqslant 2$ *for all* $j \in J$, *then* $C(\bigcup_{j \in J} C(A_j)) \subseteq C(\bigcup_{j \in J} A_j)$.

The other four are, for all $x,y \in X$ and all $A \in \mathscr{A}$ such that

$\{x, y\} \subset A$:

A5$_2$. *If* $x \in C(\{x, y\})$ *and* $y \in C(A)$, *then* $x \in C(A)$,
A5$_3$. *If* $C(\{x, y\}) = \{x, y\}$ *and* $y \in C(A)$, *then* $x \in C(A)$,
A5$_4$. *If* $C(\{x, y\}) = \{x\}$ *and* $y \in C(A)$, *then* $x \in C(A)$,
A5$_5$. *If* $x \neq y$ *and* $C(\{x, y\}) = \{x\}$, *then* $C(A) \subset A$.

As before, we suppress P in $C(A, P)$ throughout. Each condition applies to every $P \in \mathscr{P}$. The weakness of A5$_5$ is especially noteworthy. It simply says that if $|A| \geq 3$ and if $x >_P y$ for at least one pair $\{x, y\} \subset A$, then something in A (not necessarily y) will not appear in its choice set $C(A)$.

Bandyopadhyay considers a variety of existential, active intraprofile, and interprofile conditions in conjunction with the preceding passive intraprofile conditions to illustrate the boundary between possibility and impossibility. Because his results lie at the edge of recent research in this area, I shall summarize them here.

His existential conditions are A4 (no dictator), A4* (no vetoer), and three others:

A4** (no oligarchy). *For every nonempty* $I \subseteq N$ *there exist* $x,y \in X$ *and* $P \in \mathscr{P}$ *such that either* $x >_i y$ *for all* $i \in I$ *and* $y \gtrsim_P x$, *or* $x >_i y$ *for some* $i \in I$ *and* $y >_P x$.

A4° (no "strict" dictator). *For every* $i \in N$ *there exist* $x,y \in X$ *and* $P \in \mathscr{P}$ *such that either* $x >_i y$ *and* $y >_P x$, *or* $x >_i y$, $x \gtrsim_j y$ *for some* $j \in N\backslash\{i\}$, *and* $y \gtrsim_P x$.

A4′ (non-weak resoluteness). *For all distinct* $x,y \in X$ *and all* $A \in \mathscr{A}$, *if* $\{x, y\} \subseteq A$ *then there exists a* $P \in \mathscr{P}$ *such that* $x \sim_P y$ *and either* $x >_i y$ *or* $y >_i x$ *for some* $i \in N$.

Two active intraprofile conditions are used, namely A6 and its strong companion A6* ($x >_P y$ whenever $x \gtrsim_i y$ for all i and $x >_i y$ for some i).

Finally, Bandyopadhyay adopts three interprofile conditions in addition to A7. The following apply to all $x,y,a,b \in X$ and all $P,P' \in \mathscr{P}$.

A8* (monotonicity). *If* $(x >_i y \Rightarrow x >_i' y,\ x \sim_i y \Rightarrow x \gtrsim_i' y)$ *for all* $i \in N$, *then* $x >_P y \Rightarrow x >_{P'} y$ *and* $x \gtrsim_P y \Rightarrow x \gtrsim_{P'} y$.

A8** (strict monotonicity). *If the hypotheses of* A8* *hold and either* $(y \gtrsim_i x,\ x >_i' y)$ *or* $(y >_i x,\ x \gtrsim_i' y)$ *for some* $i \in N$, *then* $x \gtrsim_P y \Rightarrow x >_{P'} y$.

A9 (binary independence-neutrality). *If* $x >_i y \Leftrightarrow a >'_i b$ *and* $y >_i x \Leftrightarrow b >'_i a$ *for all* $i \in N$, *then* $x >_P y \Leftrightarrow a >_{P'} b$.

Note that each of A8* and A9 implies A7.

The following theorem presents a summary statement of the possibility/impossibility theorems in [7]. There are seven sub-theorems, (a) through (g). Each emphasizes the critical nature of the very weak extension condition $A5_5$.

THEOREM 6 *Suppose* C *on* $\mathcal{D} = \mathcal{A} \times \mathcal{P}$ *is a social choice function with* $\mathcal{A} = \{A \subseteq X : A \neq \emptyset\}$ *that satisfies* A1, A3, A7, *and* $|N| \geq 2$. *If, in addition, C satisfies*

(a) $A5_2$ *and* A6, *then* A4 *holds* $\Leftrightarrow$ $A5_5$ *fails*;
(b) $A5_1$, $A5_4$, *and* A6, *then* A4** *holds* $\Leftrightarrow$ $A5_5$ *fails*;
(c) $|N| \geq 4$, $A5_1$, A6, *and* A8**, *then* $A4^0$ *holds* $\Leftrightarrow$ $A5_5$ *fails*;
(d) $|N| \geq 4$, $A5_4$, A6, *and* A8**, *then* $A4^0$ *holds* $\Leftrightarrow$ $A5_5$ *fails*;
(e) $|X| \geq |N|$, $A5_1$, A8* *and* A9, *then* A4* *holds* $\Leftrightarrow$ $A5_5$ *fails*;
(f) $|X| \geq |N|$, $A5_4$, A8*, *and* A9, *then* A4* *holds* $\Leftrightarrow$ $A5_5$ *fails*;
(g) $A5_3$, A4′, *and* A9, *then* A6* *holds* $\Leftrightarrow$ $A5_5$ *fails*.

Thus, in each case, when all other conditions hold, $A5_5$ must fail, i.e., there is an A with $|A| \geq 3$ and a $P \in \mathcal{P}$ such that $x >_P y$ for some $x, y \in A$, and $C(A, P) = A$. Conversely, if $A5_5$ holds along with the antecedent conditions in any given case, then the condition named immediately before "$\Leftrightarrow$" must fail. For additional discussion, see [7] and its references. A related contribution in [48].

Nonbinary theorems

Our move away from A2 and A7 is completed by considering families of feasible sets that need not contain any subset of X with fewer than m alternatives for a specified $m \geq 3$. To highlight this change, we prefix our conditions with "B" instead of "A" but maintain the previous numbering arrangement. Thus, B1 refers to N, B2 to X and $\mathcal{A}$, B3 to $\mathcal{P}$, B4 to existential conditions, and so forth.

We begin with a theorem of Grether and Plott [75], then consider two theorems of Fishburn [57, 61]. The conditions used by Grether and Plott compare fairly directly with those in Theorem 1.

THEOREM 7A *Suppose C on $\mathscr{D} = \mathscr{A} \times \mathscr{P}$ is a social choice function such that, for a fixed m with $3 \leq m < |X|$:*

B1. *N is a nonempty finite set with $|N| \geq 2$,*
B2. *X is a finite set with at least four elements, and $\mathscr{A}$ contains every subset of X with at least m elements,*
B3. *$\mathscr{P}$ is the set of all functions from N into the set of weak orders on X.*

Then C cannot satisfy all of the following conditions:

B4. *For every $i \in N$ there exists an $A \in \mathscr{A}$ and a $P \in \mathscr{P}$ such that $C(A, P) \subseteq \{x \in A : x \succsim_i y \text{ for all } y \in A\}$,*
B5. *For all $P \in \mathscr{P}$ and all $A, B \in \mathscr{A}$, if $A \subset B$ and $A \cap C(B, P) \neq \varnothing$, then $C(A, P) = A \cap C(B, P)$,*
B6. *For all $x \in A \in \mathscr{A}$ and all $P \in \mathscr{P}$, if $x >_i y$ for all $i \in N$, then $y \notin C(A, P)$,*
B7. *For all $A \in \mathscr{A}$ and all $P, P' \in \mathscr{P}$, if for all $x, y \in A$ and all $i \in N$ $(x >_i y \Leftrightarrow x >'_i y,\ y >_i x \Leftrightarrow y >'_i x)$, then $C(A, P) = C(A, P')$.*

Condition B4 is a no-dictator condition, B5 is the passive intraprofile condition often referred to in the context of individual choice as the "weak axiom of revealed preference," B6 is an obvious Pareto condition, and B7 is a traditional independence axiom [4].

Grether and Plott prove Theorem 7A by showing that its conditions imply the existence of a social choice function that satisfies the same conditions after m is replaced by $m-1$. This reduction in m can be continued down to $m = 2$, at which point Theorem 1 establishes impossibility.

For convenience, let $\mathscr{A}_m = \{A \subseteq X : |A| = m\}$. Theorem 7A assumes that $(\mathscr{A}_m \cup \mathscr{A}_{m+1} \cup \cdots) \subseteq \mathscr{A}$. Fishburn [57, 61] restricts $\mathscr{A}$ further by assuming only that $\mathscr{A}_m \subseteq \mathscr{A}$ for a fixed m with $3 \leq m < |X|$. Because of this, B5 in inappropriate and the following will be used instead:

B5*(m). *For all $A, B \in \mathscr{A}$ with $|A| = |B| = m$, and for all $P \in \mathscr{P}$, if $A \cap C(B, P) \neq \varnothing$, then $C(A, P) \cap [B \setminus C(B, P)] = \varnothing$;*

B5**(m). *For all $B \subseteq X$ with $|B| > m$, and for all $P \in \mathscr{P}$, some $x \in B$ is in $C(A, P)$ for every $A \subset B$ for which $x \in A$ and $|A| = m$.*

The first of these, which is stronger than the second, says that if something in the choice set for $B \in \mathscr{A}_m$ is also in $A \in \mathscr{A}_m$, then no "loser" for B will be in A's choice set. The second requires the presence of an alternative in any B with $|B| > m$ that is chosen from every m-element subset of B which contains that alternative. It is a straightforward extension of the idea behind acyclicity, A5**.

The following theorem [57] is an m-ary version of Theorem 4A. Like the earlier theorem, it uses a no-vetoer condition and a strong monotonicity assumption.

THEOREM 7B *Suppose C on $\mathcal{D} = \mathcal{A} \times \mathcal{P}$ is a social choice function such that, for a fixed m with $3 \leq m < |X|$:*

B1′. *N is a nonempty finite set with $|N| \geq 3$.*
B2′. *X is finite, $|X| \geq 4$, and $\mathcal{A}_m \subseteq \mathcal{A}$,*
B3′. *$\mathcal{P}$ is the set of all functions from N into the set of linear orders on X.*

Then C cannot satisfy all of B5**(m), B6, B7,

B4*. *For every $i \in N$ there is an $A \in \mathcal{A}_m$, an $x \in A$, and a $P \in \mathcal{P}$ such that $x >_i y$ for all $y \in A \setminus \{x\}$ and $x \notin C(A, P)$,*

and

B8. *For all $A \in \mathcal{A}_m$, all $x \in A$, all $P, P' \in \mathcal{P}$, and all $i \in N$, if for all $j \in N \setminus \{i\}$ the restrictions of $>_j$ and $>'_j$ to A are identical, and if $x \in C(A, P')$ and $(x >_i y,\ y >'_i x)$ for some $y \in A \setminus \{x\}$ and $>_i$ and $>'_i$ are the same on $A \setminus \{x\}$, then $C(A, P) = \{x\}$.*

Condition B8 says that if x is in $C(A, P')$ and if P on A is exactly the same as P' on A except that some individual moves x up one notch, then x will be the *unique* choice in $C(A, P)$. This condition, like A8, is felt by some people to be unduly strong. It is therefore omitted in the next theorem, which uses the symmetry properties of anonymity and neutrality that were defined near the end of Section 4.

THEOREM 7C *Suppose C on $\mathcal{D} = \mathcal{A} \times \mathcal{P}$ is an anonymous and neutral social choice function that satisfies* B1, B2′, B3, B5*(m), *and* B7 *for a fixed m with $3 \leq m < |X|$. Then it violates*

B4′. *There exists an $A \in \mathcal{A}_m$ and a $P \in \mathcal{P}$ such that $C(A, P) \neq A$.*

This theorem, from [61], is a natural generalization of a similar binary theorem in Hansson [81]. Both results say that if C is required to satisfy fairly appealing symmetry conditions along with an independence axiom and an apparently innocuous passive intraprofile condition, then C is completely indecisive, i.e., $C(A, P) = A$, for all A in a substantial part of $\mathscr{A}$.

Single-set theorems

We conclude our discussion of Arrow-type multiprofile theorems with structures similar to Theorem 1 by remarking on two theorems of Hansson [80, 81] that allow $\mathscr{A}$ to contain only one set. For convenience, we assume that $X \in \mathscr{A}$.

THEOREM 8A *Suppose C on $\mathscr{D} = \mathscr{A} \times \mathscr{P}$ is a social choice function that satisfies* B1 *with* $|N| \geqslant 1$, B3, B6, *and*

B2^o. $|X| \geqslant 3$ *and* $X \in \mathscr{A}$.

Then C cannot satisfy both of the following conditions:

B4**. *For every $i \in N$ there are $x, y \in X$ and $P \in \mathscr{P}$ such that $x >_i y$ and $y \in C(X, P)$,*

B7*. *For all nonempty $A \subseteq X$ and all $P, P' \in \mathscr{P}$, if for all $x, y \in A$ and all $i \in N$ $(x >_i y \Leftrightarrow x >'_i y,\ y >_i x \Leftrightarrow y >'_i x)$, then either $A \cap C(X, P) = A \cap C(X, P')$ or else one of these two intersections is empty.*

Condition B4** is another form of no-dictator condition, and B7* is a type of independence condition that restricts $C(X, \cdot)$ over $\mathscr{P}$. If there are $A \in \mathscr{A}$ other than X, B7* has nothing to do with these other feasible sets. Note that there is no direct passive intraprofile condition in the theorem apart from the requirement that $C(X, P)$ be nonempty for every P. However, B7* carries within itself aspects of such a condition.

This is shown by the proof of the theorem, which demonstrates that if all the conditions hold, then it is possible to define another choice function C' on $\{A \subseteq X : A \neq \varnothing\} \times \mathscr{P}$ that satisfies the conditions of Theorem 1. Since we know that the latter conditions are inconsistent, the conditions of Theorem 8A must also be inconsistent. To define $C'(A, P)$ when $\varnothing \subset A \subset X$ [we take $C'(X, P) = C(X, P)$], let $P^o \in \mathscr{P}$ be such that P^o is the same as P on A and, for all $x \in A$ and all $y \in X \backslash A$, $x >^o_i y$ for every $i \in N$. Then let

$$C'(A, P) = A \cap C(X, P^o) \text{ for all such } (A, P).$$

Condition B6 insures that $C'(A, P) \neq \varnothing$, so C' is a social choice function on the enriched domain. We then use B7* to show that C' satisfies the passive intraprofile condition $[A \subseteq B, A \cap C'(B, P) \neq \varnothing] \Rightarrow C'(A, P) = A \cap C'(B, P)$, and this in turn implies that every $>_P$, defined on the basis of C' on pairs of alternatives, is a weak order. It is then straightforward to show that C' satisfies Arrow's other conditions.

As might be expected from Theorem 8A and its proof sketch, B7* is a very strong condition. But then so are A5 and A7 taken together. The message of Theorem 8A is that the essence of A5 and A7 can be captured with a single interprofile condition that applies to the simplest possible structure for $\mathscr{A}$.

Hansson's other theorem noted here follows the theme of Theorem 7C. It uses the following modification of the Pareto condition B6:

B6*. *For all* $x, y \in X$ *and all* $P \in \mathscr{P}$, *if* $x >_i y$ *for all* $i \in N$ *and* $y \in C(X, P)$, *then* $x \in C(X, P)$.

Rather than excluding y from the choice set when it is dominated by x, this allows a dominated y to be in $C(X, P)$ but, when this happens, all dominators of y must also be in $C(X, P)$. It is easily checked that $C(X, P) = X$ for all $P \in \mathscr{P}$ satisfies the conditions of Theorem 8A when $\mathscr{A} = \{X\}$ and B6 is replaced by B6*. However, there is no other way to achieve consistency under this change.

THEOREM 8B *Suppose* C *on* $\mathscr{D} = \mathscr{A} \times \mathscr{P}$ *is a social choice function that satisfies* B1 *with* $|N| \geqslant 1$, B2°, B3, B4**, B6*, *and* B7*. *Then it violates*

B4″. *There exists a* $P \in \mathscr{P}$ *such that* $C(X, P) \neq X$.

Complete proofs of Theorems 8A and 8B are also given in [54].

6. SINGLE-PROFILE THEOREMS

In contrast to the single-set theorems that conclude the preceding section, we now consider impossibility theorems based on single profiles. The crux of these theorems lies in the structure of the profile used to demonstrate inconsistency with various intraprofile conditions. Since interprofile conditions are, by definition, inapplicable to the single-profile context unless their intraprofile specializations are nontrivial, single-profile theorems that emulate Arrow's multiprofile theorem and its companions must use conditions that facilitate manipulations within a profile that mimic interprofile comparisons made possible by independence conditions like A7. As we shall see, some structure is needed to do this.

Part of the motivation behind single-profile theorems lies in the observation that any specific realization of a social choice problem gives rise to only one preference profile. Consequently, it may be possible to circumvent the difficulties raised by multiprofile impossibility theorems by using only intraprofile conditions and avoiding interprofile conditions that apply to different profiles that cannot possibly obtain simultaneously. The message behind the single-profile theorems is that there is no simple escape from impossibility by this route.

Examples

We shall begin our discussion of single-profile impossibility theorems with a few examples designed to clarify the perimeter of these theorems. Later subsections expand on the ideas introduced in the examples. To add variety, we shall consider Sen's Paretian-liberal paradox [135, 136] alongside of Arrow's theme.

Assume initially that A1, A2, and A6 hold with $N = \{1, 2\}$ and $X = \{a, b, c\}$. For definiteness, suppose that the two individuals have the following linear preference orders on X:

1. abc $(a >_1 b >_1 c)$
2. bca $(b >_2 c >_2 a)$.

This specifies a single profile that we shall use instead of condition A3. Let P denote the profile. Then, by A6, $b >_P c$. Given $b >_P c$, we note two ways an impossibility might occur.

First, as in Sen's Paretian-liberal paradox, suppose each individual is a "dictator" over a single pair of alternatives, and assume that $>_P$ is to be acyclic, as in A5**. Let individual 1 dictate the social preference on $\{a, b\}$, and let individual 2 dictate the social preference on $\{a, c\}$. This might be appropriate if the welfare of 1 is affected by the choice from $\{a, b\}$ but the welfare of 2 is not affected by this choice, and similarly for $\{a, c\}$ with the roles of 1 and 2 reversed. We then have $a >_P b$ and $c >_P a$, which in conjunction with $b >_P c$ violate A5**.

Second, following Arrow's theme, suppose we adopt A4, A5, and the following intraprofile specialization of the interprofile independence-neutrality condition A9 used in Theorem 6:

A9*. *For all* $x,y,a,b \in X$ *and all* $P \in \mathscr{P}$, *if* $(x >_i y \Leftrightarrow a >_i b,\ y >_i x \Leftrightarrow b >_i a)$ *for all* $i \in N$, *then* $x >_P y \Leftrightarrow a >_P b$.

Condition A9* is identical, under A2, to the final condition in Section 4. When $|\mathscr{P}| = 1$, it is identical to A9, but we list it separately to emphasize its intraprofile intention.

Given the particular profile of our example, namely $P = (abc, bca)$, A9* requires $a >_P b \Leftrightarrow a >_P c$ and $b >_P a \Leftrightarrow c >_P a$. By A6, $b >_P c$. Hence, by A5 (negative transitivity) either $b >_P a$ or $a >_P c$. If $b >_P a$, then A5, A6, and A9* require $b >_P c >_P a$, and in this case individual 2 is a dictator. On the other hand, if $a >_P c$, then A5, A6, and A9* require $a >_P b >_P c$, and therefore 1 is a dictator. Consequently $>_P$ must violate at least one of A4, A5, A6, and A9* for the single profile $P = (abc, bca)$.

The simple two-individual profile of our initial example was chosen to illustrate the inconsistencies in nontrivial ways that show the interactions involved within a single profile. It should be evident that the cited conditions could be consistent for other profiles. For example, if $P = (abc, cab)$, then $c >_P a >_P b$ (with $c >_P b$) is consistent with A5** and the dictatorial powers of 1 and 2 over their two pairs. And, if we choose a profile P in the Arrow context for which the simple majority relation $>_M$ is a weak order, then all of A4, A5, A6 and A9* will hold so long as some $>_i$ is not included in $>_M$.

There are also many profiles for which $>_M$ is not a weak order but A4, A5, A6, and A9* are consistent. One example is the cyclic-majorities profile

1. *abc*
2. *cab*
3. *bca*,

where $a >_M b >_M c >_M a$. If we take $>_P$ to be the linear order $a >_P c >_P b$, then no individual is a dictator (or vetoer), and A6 and A9* hold trivially.

Hence, when A9* is used along with conditions like A5 and A6, it is necessary to supply the single profile with structure that facilitates the use of A9* when A5 and A6 by themselves do not force a dictator or vetoer. Our initial example with two individuals specified a minimal structure of this kind. Following a brief discussion of the Paretian-liberal case, we shall describe a more varied structure for use with A9* that leads to an immediate proof of impossibility via the first proof of Theorem 1.

Paretian-liberal theorems

Impossibility theorems that stem from Sen's work [135, 136] that are based on specialized powers for different individuals are usually presented in a multiprofile format. However, since their proofs are essentially single-profile proofs, they are classified here as single-profile theorems. The proofs use the Pareto condition and conditions for the powers of individuals to construct a profile that violates a passive intraprofile condition like acyclicity. We could formulate the special profile's structure as part of the theorem to yield a single-profile statement, but since this would virtually include the proof within the statement of the theorem we shall follow the usual format.

Three Paretian-liberal theorems will be presented. The first is Sen's. The others are due to Gibbard [73], who provided a new way of formulating the problem. Many other contributions to the topic are noted in [6, 73, 92, 95, 137, 145] and in references therein. The central theme of these theorems is the clash between individual powers and rights and "collective rationality" conditions such as A5**.

THEOREM 9A *Suppose C on* $\mathscr{D} = \mathscr{A} \times \mathscr{P}$ *is a social choice function that satisfies* A1, A2, A3, *and* A6. *Then C must violate either* A5** *or*

A4^2. *There exist distinct* $i,j \in N$ *and pairs* $\{x_i, y_i\}$ *and* $\{x_j, y_j\}$ *in* $\mathscr{A}$ *such that, for all* $P \in \mathscr{P}$, $x_i >_i y_i \Rightarrow x_i >_P y_i$ and $x_j >_j y_j \Rightarrow x_j >_P y_j$.

The existential condition A4^2 grants each of two individuals dictatorial power over a pair of alternatives. The proof of the theorem uses A4^2 and A6 to construct a profile that violates A5**. For example, if $\{x_i, y_i\}$ and $\{x_j, y_j\}$ are disjoint, then a profile P

with

$$y_j >_i x_i >_i y_i >_i x_j$$

$$y_i >_j x_j >_j y_j >_j x_i$$

$$y_i >_k y_j >_k x_i >_k x_j \quad \text{for each } k \in N \setminus \{i, j\}$$

yields $x_i >_P y_i$ (A4[2]), $y_i >_P x_j$ (A6), $x_j >_P y_j$ (A4[2]), and $y_j >_P x_i$ (A6) for a $>_P$ cycle.

Gibbard observed that libertarian claims seem more reasonable if individuals are associated with specific factors in a multidimensional alternative space. For example, a specific factor might pertain to aspects of individual 1's life that have nothing to do with other individuals' lives. To make the multidimensional structure explicit we shall use

A2*. *$X = X_1 \times X_2 \times \cdots \times X_m$ for a positive integer m, $|X_t| \geq 2$ for $t = 1, \ldots, m$, and $\mathscr{A}$ includes every two-alternative subset of X.*

The next two theorems are usually presented in terms of a general choice function C, but nothing essential is lost by focussing on the two-alternative subsets of X. We shall write xE_ty when $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ are in X and have $x_s = y_s$ for all $s \in \{1, \ldots, m\} \setminus \{t\}$. Thus xE_ty means that x and y differ at most only in their t-th components.

THEOREM 9B *Suppose C on $\mathscr{D} = \mathscr{A} \times \mathscr{P}$ is a social choice function that satisfies* A1, A2*, *and* A3. *Then C violates either* A5** *or*

> A4[3]. *There exist distinct $i, j \in N$ and $t_i, t_j \in \{1, \ldots, m\}$ such that, for each $k \in \{i, j\}$ and all $x, y \in X$, if $xE_{t_k}y$ and $x >_k y$, then $x >_P y$.*

The structure added by A2* coupled with the more sweeping nature of A4[3] compared to A4[2] allows the omission of A6 from Theorem 9B. With $(i, j) = (1, 2)$ for A4[3] and x' any fixed element in $X_3 \times \cdots \times X_m$ if $m \geq 3$, a profile P with

$$(x_1, x_2, x') >_1 (y_1, y_2, x') >_1 (x_1, y_2, x') >_1 (y_1, x_2, x'),$$

$$(y_1, x_2, x') >_k (x_1, y_2, x') >_k (y_1, y_2, x') >_k (x_1, x_2, x')$$

for all $k \in N \setminus \{1\}$ yields $(x_1, x_2, x') >_P (y_1, x_2, x') >_P (y_1, y_2, x') >_P (x_1, y_2, x') >_P (x_1, x_2, x')$, for a $>_P$ cycle.

The preceding proof relies on the feature that individual 1 prefers x_1 to y_1 when all else is fixed at (x_2, x'), but prefers y_1 to x_1 when all else is fixed at (y_2, x'). The next theorem tightens A4^3 by granting dictatorial power only when such conditional preference reversals are not present. To reflect this we write $a \gg_i b$ for $a,b \in X_t$ if and only if $x >_i y$ for all $x,y \in X$ such that $xE_t y$, $x_t = a$, and $y_t = b$. Condition A6 is reinstated here since the other conditions are consistent without it.

THEOREM 9C *Suppose C on* $\mathscr{D} = \mathscr{A} \times \mathscr{P}$ *is a social choice function that satisfies* A1, A2*, A3, *and* A6. *Then C violates either* A5** *or*

A4^4. *There exist distinct* $i,j \in N$ *and* $t_i, t_j \in \{1, \ldots, m\}$ *such that, for each* $k \in \{i, j\}$ *and all* $x,y \in X$, *if* $xE_{t_k}y$ *and* $x_{t_k} \gg_k y_{t_k}$, *then* $x >_P y$.

The proof is similar to the proof of Theorem 9A. In [73], Gibbard shows that a further tightening of the existential libertarian condition allows consistency with A6 and the other conditions of Theorem 9C. Additional analyses appear in the references cited earlier.

An existential profile condition

We now return to Arrow's theme under the intraprofile independence-neutrality condition A9*. Although this condition is not used in Theorem 1, it is essentially implied by conditions in that theorem. In particular, given A1, A2, A5, and A7, A9* is implied by the combination of A6 and A3 modified to include only linear orders for individuals, or by the original A3 and the stronger Pareto condition A6*.

The other component needed for a direct single-profile analogue of Theorem 1 is a structural condition for the profile used in the theorem. The condition that we adopt is similar to profile conditions used by Parks [114], Kemp and Ng [96], and Pollak [119]. Its statement here is essentially that in [119].

A3*. $\mathscr{P}$ *consists of a single function* P^* *from N into the set of weak orders on X such that, for every function* P' *from N into the set of weak orders on* $\{a_1, a_2, a_3\}$ *with the* a_j *distinct, there exist*

$x_1, x_2, x_3 \in X$ *such that, for all* $i \in N$ *and all* $j,k \in \{1, 2, 3\}$, $x_j >_i^* x_k \Leftrightarrow a_j >_i' a_k$.

This says that every possible 3-alternative profile of weak orders can be found within P^* by a suitable relabeling of alternatives. To illustrate A3*, we simplify slightly by considering only profiles with linear orders. (This substitution can be made in the ensuing theorem without altering its conclusion.)

When $N = \{1, 2\}$, the simplest profile of two linear orders that satisfies the linear version of A3* is

$$P^* = (x_1x_2x_3x_4x_5,\ x_2x_5x_3x_1x_4).$$

Six subprofiles that correspond to the six possible combinations of two 3-element linear orders up to relabeling are:

$$\begin{array}{lcl}
(x_2x_3x_4,\ x_2x_3x_4) & \text{like} & (a_1a_2a_3,\ a_1a_2a_3) \\
(x_2x_3x_5,\ x_2x_5x_3) & & (a_1a_2a_3,\ a_1a_3a_2) \\
(x_1x_3x_4,\ x_3x_1x_4) & & (a_1a_2a_3,\ a_2a_1a_3) \\
(x_1x_2x_5,\ x_2x_5x_1) & & (a_1a_2a_3,\ a_2a_3a_1) \\
(x_1x_4x_5,\ x_5x_1x_4) & & (a_1a_2a_3,\ a_3a_1a_2) \\
(x_1x_3x_5,\ x_5x_3x_1) & & (a_1a_2a_3,\ a_3a_2a_1).
\end{array}$$

When $N = \{1, 2, 3\}$, we need a P^* with three linear orders that contains each of the 36 distinct combinations of three 3-element linear orders. This appears to require at least nine elements in X. The original weak-order version of A3* requires considerably more alternatives for a given $|N|$ than does its linear-order counterpart.

The first single-profile impossibility theorems like our next theorem were proved independently by Parks [114] and Kemp and Ng [96]. The following theorem is similar to those in [114, 119]. See also [78].

THEOREM 10 *Suppose C on* $\mathcal{D} = \mathcal{A} \times \mathcal{P}$ *is a social choice function that satisfies* A1, A2, *and* A3*. *Then it cannot satisfy all of* A4, A5, A6, *and* A9*.

The proof is almost identical to the first proof of Theorem 1. The only change needed is to use A9* in place of A7 and in other places where specialized 3-alternative subprofiles are employed. Condition A3* guarantees the existence of suitable subprofiles.

Parks [114] also notes modifications of Theorem 10 that are closely related to Theorem 5A and to the version of Arrow's theorem that uses monotonicity, and Pollak [119] shows that if A6 is strengthened to A6*, then the conditions of Theorem 10 apart from A4 imply a lexicographic dictatorship scheme (see paragraph *A4* in Section 4). In addition, Muller [109] presents a generalization of Theorem 10 that considers restrictions on P' in condition A3*.

Correspondences for other multiprofile theorems

Additional correspondences between multiprofile and single-profile impossibility theorems are developed by Roberts [124]. At this point we note only the part of [124] that fits in with our preceding discussion.

Roberts shows how A3* and A9* can be used with a single profile P^* to define a binary relation $>_P$ for every $P \in \mathscr{P}$ under A3 so that each $>_P$ satisfies A9 (defined immediately before Theorem 6) and also satisfies a passive intraprofile condition based on conditions like transitivity that use no more than three alternatives in their characterizations, provided that the parent relation $>_{P^*}$ satisfies the same condition. The definition for each $>_P$ that accomplishes this is as follows for all $x,y \in X$:

$$x >_P y \text{ if, for all } a,b \in X,\ a >_{P^*} b \text{ whenever}$$
$$(x >_i y \Leftrightarrow a >_i^* b,\ y >_i x \Leftrightarrow b >_i^* a) \text{ for all } i \in N.$$

As usual, the $>_i$ are for P, and the $>_i^*$ for P^*.

The following result is a three-part theorem. One part is for A5, the second applies to A5*, and the third refers to 3-acyclicity (for $>_{P^*}$ and the $>_P$) as defined prior to Theorem 4A. For convenience we shall denote 3-acyclicity by $A5^3$.

THEOREM 11 *Suppose* N,X *and* $\mathscr{A}$ *satisfy* A1 *and* A2, *and* P^* *is a profile that satisfies* A3*. *Suppose further that* C^* *on* $\mathscr{A} \times \{P^*\}$ *is a social choice function that satisfies* A9* *along with* A5 [A5*, $A5^3$, *respectively*]. *Then there exists a social choice function* C *on* $\mathscr{A} \times \mathscr{P}$ *that satisfies* A3 *and* A9 *along with* A5 [A5*, $A5^3$, *respectively*].

Since A9 implies A7, the hypotheses of Theorem 11 for the single-profile case generate the kinds of structure and conditions used in many multiprofile theorems. It is also easily seen that conditions similar to A4, A4*, A6, and A8 imposed on $>_{P^*}$ imply the corresponding conditions for all $>_P$ under A3* and A9*. As a consequence, Theorem 11 generates single-profile impossibility theorems that are natural correspondents of Theorems 2A, 2B, 4A (use $A5^3$ instead of general acyclicity), 5A, and 5B.

Further single-profile analogues of theorems in the preceding section can be obtained by strengthening A3* to include more subprofile configurations. For example, if it is assumed that every possible 4-alternative profile of weak orders can be found in P^*, then we can obtain the single-profile version of Theorem 3, for interval orders, semitransitive orders, and semiorders. And if every finite-alternative subprofile can be found in P^*—which of course forces X to be infinite—then single-profile versions of multiprofile theorems based on A5** can be generated.

An analysis of the relationship between the single-profile and multiprofile approaches that is based on formal logic appears in [125].

7. ORDINAL UTILITY AND IMPOSSIBILITY

This section is intended as a bridge between the preceding sections and the two that follow. Its purpose is to recast some of the earlier formulation in terms of utility functions and to mention other results within the ordinalist approach of Arrow [4] that are facilitated by the use of utilities.

Heretofore, profiles have been characterized as mappings from N into a set of weak orders or linear orders on X. An alternative used in much of the social choice literature is to represent individuals' preferences by real valued utility functions, or by classes of such functions, and to speak of profiles in terms of these functions.

Given a weak order $>_0$ on X, we shall say that u is a *utility function* for $>_0$ if u is a real valued function on X, and, for all $x,y \in X$,

$$x >_0 y \Leftrightarrow u(x) > u(y).$$

Since this numerical representation of $>_0$ implies that $>_0$ must be a weak order, it cannot serve for more flexible types of binary relations such as those that allow $\sim_0$ to be nontransitive. Other numerical representations for such cases are discussed in [52], but we shall not go into them here since the assumption of weak orders will be maintained for individual preferences.

On the other hand, the existence of a utility function for $>_0$ is not assured by the assumption that $>_0$ is a weak order. It *is* assured if X is finite or denumerable, but not otherwise. The simplest example of a weak order not representable by a real valued function is the lexicographic order of the Euclidean plane defined by $(x_1, x_2) >_0 (y_1, y_2) \Leftrightarrow [x_1 > y_1 \text{ or } (x_1 = y_1, x_2 > y_2)]$, for it can be easily seen [39, 52] that the existence of a utility function for this case would imply that the set of all rational numbers is uncountable, which is false.

Sufficient or necessary and sufficient conditions for a weak order $>_0$ on X to have a utility function are presented in [39, 41, 42, 52, 99]. These references also discuss conditions for $>_0$ on X that ensure the existence of a *continuous* utility function with respect to a topology defined on X. An example familiar in economics is the case in which X is the nonnegative orthant of a finite-dimensional Euclidean space, i.e., the space of commodity bundles $(x_1, \ldots, x_m)$ with $x_j \geq 0$ for all j, with $u(x_1^k, \ldots, x_m^k)$ converging to $u(x_1, \ldots, x_m)$ for any sequence of bundles $\{(x_1^k, \ldots, x_m^k) : k = 1, 2, \ldots\}$ that approaches $(x_1, \ldots, x_m)$.

A utility function for $>_0$ as defined above is often referred to as an order-preserving or *ordinal* utility function since it places no demands on u other than $x >_0 y \Leftrightarrow u(x) > u(y)$. Any other real valued function u on X which satisfies $x >_0 y \Leftrightarrow u(x) > u(y)$, for all $x, y \in X$, is also an ordinal utility function for $>_0$. Consequently, it is sometimes useful to think of $>_0$ in terms of the set of all real valued functions that represent it in the manner indicated. Such a set is an equivalence class in the intuitive sense that all functions in the set are utility functions for $>_0$, and no function not in the set can be a utility function for $>_0$. More formally, suppose F is the set of all real valued functions on X, and let $F(>_0)$ denote the functions in F that represent $>_0$. So long as each weak order $>_0$ has a utility function, the weak orders on X partition F into nonempty and mutually disjoint classes $F(>_0)$. This partition defines an equivalence relation (reflexive, symmetric, transitive) on F, with u equivalent to v if and only if u and v are members of the same class in the partition.

If we assume for preceding theorems that the individual weak orders used in A3, B3, and so forth have utility functions, then a profile $P = (>_1, >_2, \ldots, >_n)$ can be thought of as an n-tuple of utility functions $(u_1, u_2, \ldots, u_n)$ where u_i is a utility function for $>_i$. More precisely, the n-tuple of weak orders $(>_1, >_2, \ldots, >_n)$ corresponds in the ordinal context to the family

$$\mathop{\times}_{i=1}^{n} F(>_i) = \{(u_1, u_2, \ldots, u_n) : u_i \in F(>_i) \text{ for } i = 1, \ldots, n\},$$

where $F(>_i)$ is the class of utility functions for $>_i$.

Conditions used in preceding theorems can be reformulated in the ordinal utility mode provided we restrict our attention to weak orders that have utility functions, which as noted is always the case when X is countable. With F as the set of all real valued functions on X, A3 can be replaced by the assumption that $\mathcal{P}$ is the set of all mappings from N into F. Let U be such a mapping, say $U = (u_1, u_2, \ldots, u_n)$ when $N = \{1, 2, \ldots, n\}$. Then A5 requires two parts: for all $U, U' \in \mathcal{P}$,

A5a. *$>_U$ is a weak order on X;*

A5b. *If $U = (u_1, \ldots, u_n)$ and $U' = (u'_1, \ldots, u'_n)$ are equivalent in the ordinal sense [for all $i \in N$ and all $x,y \in X$, $u_i(x) > u_i(y) \Leftrightarrow u'_i(x) > u'_i(y)$], then $>_U = >_{U'}$.*

The latter property makes explicit the sense in which Arrow's theorem and related results rely only on the ordering information contained in the $>_i$ and not on any extra-ordering properties of numerical representations or utility functions. Relaxations of A5 like A5* and A5** are straightforward in this reformulation. A4 would say that for each $i \in N$ there are $x,y \in X$ and a $U \in \mathcal{P}$ such that $u_i(x) > u_i(y)$ and $y \gtrsim_U x$. A6 becomes: For all $U \in \mathcal{P}$ and all $x,y \in X$, if $u_i(x) > u_i(y)$ for all i, then $x >_U y$. And A7 gets changed to: For all $U, U' \in \mathcal{P}$ and all $x,y \in X$, if $(u_i(x) > u_i(y) \Leftrightarrow u'_i(x) > u'_i(y)$, $u_i(x) < u_i(y) \Leftrightarrow u'_i(x) < u'_i(y))$ for all $i \in N$, then $x >_U y \Leftrightarrow x >_{U'} y$. Since this obviously implies A5b we can drop A5b when this version of A7 is used. Changes for other conditions are left to the reader.

Given A5a, it is often assumed that the $>_U$ also have utility functions, which are usually referred to as social utility functions or social welfare functions. Suppose W_U is a social welfare function for the utility profile U, so that, for all $x,y \in X$,

$$W_U(x) > W_U(y) \Leftrightarrow x >_U y.$$

The dependence of W on U is also written as $W[u_1, \ldots, u_n]$, so that $W_U(x) = W[u_1(x), \ldots, u_n(x)]$.

This type of formulation is often found in the literature of welfare economics, where each $x \in X$ provides a composite description of every individual's consumption bundle. If it is assumed that each person's utility depends only on his own bundle, say x^i for individual i, with $x = (x^1, \ldots, x^n)$, then $W[u_1(x), \ldots, u_n(x)]$ reduces to $W[u_1(x^1), \ldots, u_n(x^n)]$. If, in addition, individuals' tastes are presumed to be fixed, so that there is only one relevant U (up to ordinal equivalence), then W specifies one form of the classical *Bergson-Samuelson social welfare function* [11, 127, 128, 129]. It is usually assumed that such a function is continuous and monotonically increasing in its arguments, among other things. The paper by Chipman and Moore [32] provides good coverage of the topic.

A large part of the motivation for the single-profile impossibility theorems in the preceding section—see the titles of [109, 114, 119]—came from the desire to show that a Bergson-Samuelson social welfare function cannot escape the problem uncovered by Arrow. Granting the palatability of A3*, the question of how well this effort has succeeded would appear to hinge on the acceptability of A9*. If impossibility is denied, then A9* must fail, and consequently there is some sense in which W must go beyond the ordinal independence notions of conditions like A9 and A9*.

Social choice impossibility theorems of the Arrow type have been developed by Inada [87, 88] for the economic commodity-bundles setting. He adopts the Bergson-Samuelson form $W_U(x) = W[u_1(x^1), \ldots, u_n(x^n)]$ but allows the u_i to vary in certain ways so that his theorems are multiprofile results. Restrictions on the u_i follow traditional lines, and his main analyses are carried out on the basis of marginal-rate-of-substitution functions rather than the u_i directly. Under suitable independence and regularity conditions, [87] shows that either there is a dictator or the social welfare function is imposed, i.e., it does not change as individuals' preferences change. In [88], a choice function approach is considered along with the welfare function approach.

Another line of development in the ordinal setting has been pursued by Chichilnisky [27, 28]. Her work relies heavily on advanced concepts in analysis and topology, and I can do no more than sketch its outlines here. The most interesting aspect of this work is its avoidance of an interprofile independence condition. Instead of independence, Chichilnisky uses continuity of the function that maps preference profiles into social preferences. This is also an interprofile condition; it says intuitively that small changes in individuals' preferences or ordinal utilities cause only small changes in social preferences or utilities.

Individual and social preferences in [27, 28] are essentially characterized as equivalence classes of continuous ordinal utility functions on a set X that has nice structural properties. Let $N=\{1,\ldots,n\}$ with $n\geqslant 2$, let S denote the "space of preferences" with members $p, p_1, p_2, \ldots$, and let ϕ denote the social choice mapping from S^n into S. It is assumed that ϕ is continuous under an appropriate topology for S. Then [27] shows that ϕ must violate either anonymity or unanimity. *Anonymity* says that $\phi(p_{\sigma(1)}, \ldots, p_{\sigma(n)})=\phi(p_1,\ldots,p_n)$ for all $(p_1,\ldots,p_n)\in S^n$ and all permutations σ on N; *unanimity* requires $\phi(p,\ldots,p)=p$ for all $p\in S$. A related result for individual cardinal utilities appears in [29]. On the other hand, [28] shows that if ϕ satisfies the Pareto condition A6 and a "weak positive association" condition, then it can be continuously deformed into a dictatorial map. This means that there is an $i\in N$ and a continuous function f from $S^n\times[0,1]$ into S such that, for all $(p_1,\ldots,p_n)\in S^n$, $f(p_1,\ldots,p_n,0)=\phi(p_1,\ldots,p_n)$ and $f(p_1,\ldots,p_n,1)=p_i$. Given the other conditions, this shows that the Pareto and nondictatorship conditions are topologically equivalent.

8. CARDINAL UTILITY AND IMPOSSIBILITY

Thus far we have ignored aspects of individual preferences or values that transcend elementary comparisons between alternatives in X. For example, within Arrow's formulation [4], or the ordinal utility approach, no distinction is made between

1: i barely prefers x to y, likes both, and passionately dislikes z, and

2: i likes x, passionately dislikes y and z, and barely prefers y to z.

Both cases are recorded as $x >_i y >_i z$.

In the present section, we shall consider strength of preference or preference intensity information of the type suggested by this example. Utilities based on comparisons between probability distributions over alternatives—often referred to as lotteries, gambles, or prospects—will also be used for representations of individual (and in one case social) preferences. At the same time, we shall continue to ignore comparisons between individuals of the sort which say that i prefers x to y more strongly than j prefers y to x, or that i is better off in social state x than is j in state y. These kinds of interpersonal comparisons will be considered in the next section.

Most of the section is based on so-called cardinal utility functions. However, before we get into the full cardinal case, we shall examine one version of utility that lies between ordinal utility and cardinal utility.

An intermediate case

Let $\mathscr{A}$ be the set of all nonempty subsets of a finite set X. To go beyond $>_0$ on X, we consider a binary preference relation $>_0$ on $\mathscr{A}$ with the interpretation that $A >_0 B$ if the lottery that assigns equal probability $1/|A|$ to each $a \in A$ is preferred to the lottery that assigns equal probability $1/|B|$ to each $b \in B$. We shall say that $>_0$ on $\mathscr{A}$ has an *averaging representation* if there exists a real valued function u on X such that, for all $A, B \in \mathscr{A}$,

$$A >_0 B \Leftrightarrow \frac{1}{|A|} \sum_{a \in A} u(a) > \frac{1}{|B|} \sum_{b \in B} u(b).$$

This is a specialized form of expected utility model with utility function u. When it holds, we refer to u as an *averaging utility* for $>_0$ on $\mathscr{A}$.

In the present case, two averaging utilities are equivalent if and only if they induce the same $>_0$ on $\mathscr{A}$ by way of averaging representations. Since an order-preserving transformation of u on X will preserve $>_0$ on singleton comparisons but need not do so for larger subsets of X, equivalence classes for averaging utilities will generally be smaller than those for ordinal utilities. At the same time, since a positive linear transformation of u [$v = \alpha u + \beta$, α and β real, $\alpha > 0$] is in the same averaging equivalence class as u, but such a class may contain other types of transformed functions, equivalence classes for averaging utilities will generally be larger than those for cardinal utilities.

The following impossibility theorem [53] applies to this intermediate form of utility.

THEOREM 12 *Suppose* C *on* $\mathcal{A} \times \mathcal{P}$ *is a social choice function that satisfies* A1 *with* $|N| \geq 3$, A2 *with* X *finite,* $|X| \geq 4$ *and* $\mathcal{A} = \{A \subseteq X : A \neq \varnothing\}$, *and*

A3′. *$\mathcal{P}$ is the set of all functions from N into the set of weak orders on $\mathcal{A}$ that have averaging utilities.*

Then C must violate one of the following:

A4*. *For every $i \in N$ there exist $x,y \in X$ and a $P \in \mathcal{P}$ for which* $\{x\} >_i \{y\}$ *and* $C(\{x, y\}, P) = \{y\}$,

A5′. *For all $P \in \mathcal{P}$ and all $A,B \in \mathcal{A}$, if $A \subset B$ then* $A \cap C(B, P) \subseteq C(A, P)$,

A6′. *For all $P \in \mathcal{P}$, all $x \in X$, and all $A \in \mathcal{A}$, if $A >_i \{x\}$ for all $i \in N$, then* $x \notin C(A \cup \{x\}, P)$,

A7. *For all $x,y \in X$ and all $P,P' \in \mathcal{P}$, if* $(\{x\} >'_i \{y\} \Leftrightarrow \{x\} >_i \{y\}, \quad \{y\} >_i \{x\} \Leftrightarrow \{y\} >'_i \{x\})$ *for all $i \in N$, then* $C(\{x, y\}, P) = C(\{x, y\}, P')$,

A8′. *For all distinct $x,y \in X$, all $P,P' \in \mathcal{P}$, and all $i \in N$, if* $>_j = >'_j$ *for all* $j \in N \setminus \{i\}$, $>'_i$ *is obtained from an averaging representation for* $>_i$ *by interchanging the values of* $u_i(x)$ *and* $u_i(y)$, $\{y\} >_i \{x\}$, *and* $x \in C(\{x, y\}, P)$, *then* $C(\{x, y\}, P') = \{x\}$.

Here A4* is the usual no-vetoer condition, A5′ is a passive intraprofile contraction condition mentioned in Sections 4 and 5 (it implies A5** on single alternatives), A6′ is a Pareto condition, A7 is the usual binary independence condition, and A8′ is a binary strong monotonicity condition that is weaker than A8 in the presence of A7. It may be noted that the choice function C is *not* presumed to be representable by an averaging representation through maximum utilities for alternatives, and it need not satisfy weak order on X in terms of binary choices for each profile.

The earlier multiprofile theorem that is most easily compared to Theorem 12 is the Mas-Colell and Sonnenschein [102] result for acyclic $>_P$, Theorem 4A.

Cardinal utility

Suppose u is a utility function for a weak order $>_0$ on X that satisfies one or more additional conditions. Let $\mathscr{C}$ denote these other conditions. We then say that u is a *cardinal utility function* (with respect to $\mathscr{C}$) if, for every utility function v for $>_0$, v satisfies $\mathscr{C}$ if and only if there are real numbers α and β with $\alpha > 0$ such that, for all $x \in X$,

$$v(x) = \alpha u(x) + \beta.$$

This is often abbreviated by saying that u is unique up to positive ($\alpha > 0$) linear ($v = \alpha u + \beta$) transformations.

There are many routes to cardinal utility: eight are mentioned in [59]. The two most familiar in economics are the von Neumann-Morgenstern expected utility theory [147], in which a binary preference relation $>_0$ on the set of all lotteries $p, q, \ldots$ on X is to satisfy

$$p >_0 q \Leftrightarrow \sum p(x)u(x) > \sum q(x)u(x),$$

and the approach of comparable preference differences or intensities of preference first axiomatized by Frisch [66], Lange [100], and Alt [2]. The latter route accords most closely with the intuitive version in economic theory during the second half of the nineteenth century and with the example given at the beginning of this section. Axiom systems for these and other interpretations of cardinal utility appear in [52].

As before, let F denote the set of all real valued functions on X. Functions $u, v \in F$ are said to be *cardinally equivalent* if they are related by a positive linear transformation, i.e., if there are numbers $\alpha > 0$ and β such that $v(x) = \alpha u(x) + \beta$ for all $x \in X$. We shall write this as $u \approx v$. Similarly, u and v are cardinally equivalent on a nonempty subset A of X, written $u \approx_A v$, if their restrictions to A are related by a positive linear transformation:

$$u \approx_A v \text{ if there are } \alpha > 0 \text{ and } \beta \text{ such that, for all } x \in A,\ v(x) = \alpha u(x) + \beta.$$

Equivalence extends to profiles of utility functions in the natural way. Let $\mathscr{U}$ denote the set of all functions from N into F. We write

$U(i) = u_i$ for the utility function assigned to $i \in N$ by the utility profile $U \in \mathcal{U}$. When $N = \{1, 2, \ldots, n\}$, we may represent U as $(u_1, u_2, \ldots, u_n)$, just as P is represented by $(>_1, >_2, \ldots, >_n)$. Primes apply in the usual way, e.g. $U' = (u'_1, u'_2, \ldots, u'_n)$. With this notation,

$$U \approx_A U' \text{ if } u_i \approx_A u'_i \text{ for all } i \in N.$$

When $A = X$, the subscript is dropped, so $U \approx U'$ if $u_i \approx u'_i$ for all $i \in N$.

Impossibility for social orders

Our first impossibility theorem for individual cardinal utilities is from Sen [136, Theorem 8.2*] and Roberts [123, Theorem 3]. The asymmetric social relation on X associated with the utility profile U will be denoted by $>_U$, similar to $>_P$, with

$$x >_U y \text{ if } x \neq y \text{ and } C(\{x, y\}, U) = \{x\}.$$

THEOREM 13 *Suppose C on $\mathcal{A} \times \mathcal{U}$ is a social choice function that satisfies*

C1. *N is a nonempty finite set,*
C2. *X has at least three elements, and $\mathcal{A}$ contains every two-element subset of X,*
C3. *$\mathcal{U}$ is the set of all functions from N into the set of real valued functions on X.*

Then C cannot satisfy all of the following:

C4. *For every $i \in N$ there exist $x,y \in X$ and a $U \in \mathcal{U}$ such that $u_i(x) > u_i(y)$ and $y \gtrsim_U x$,*
C5a. *For all $U \in \mathcal{U}$, $>_U$ is a weak order on X,*
C5b. *For all $U, U' \in \mathcal{U}$, if $U \approx U'$ then $>_U = >_{U'}$,*
C6. *For all $x,y \in X$ and all $U \in \mathcal{U}$, if $u_i(x) > u_i(y)$ for all $i \in N$, then $x >_U y$,*
C7. *For all $U, U' \in \mathcal{U}$, and all $\{x, y\} \in \mathcal{A}$, if U and U' are identical on $\{x, y\}$, then $>_U$ and $>_{U'}$ are identical on $\{x, y\}$.*

Conditions C1–C4, C5a, and C6 are essentially the same as their counterparts in Theorem 1. Condition C5b allows the social orders $>_U$ and $>_{U'}$ to differ even when each u_i is ordinally equivalent to its

u_i' counterpart, provided at least one u_i is not cardinally equivalent to u_i'. But C7 in conjunction with C5b forces $>_U$ and $>_{U'}$ to be the same when U and U' are ordinally equivalent, and this puts us back into Arrow's context. All that then remains to apply Theorem 1 to prove Theorem 13 is to show that A7 follows from the conditions of Theorem 13. In fact, it follows from C5b and C7 [37]. Given that $(x >_i y \Leftrightarrow x >_i' y,\ y >_i x \Leftrightarrow y >_i' x)$ for all i, apply positive linear transformations separately to each u_i' so that u_i' on $\{x, y\}$ equals u_i on $\{x, y\}$. By C7, the social relations on $\{x, y\}$ must be the same for U and the transformed U'; by C5b, the original U' and its transformed version have the same social relations on $\{x, y\}$, so the conclusion of A7 follows.

Of the two interprofile conditions in Theorem 13, C5b and C7, C5b preserves the cardinal character of the formulation but C7 does not. If C7 is dropped, consistency obtains. For example, a social order can be defined for each cardinal equivalence class of profiles in $\mathcal{U}$ by a sum of the u_i under specified normalizations for that class. Since the potential normalizations are very diverse, additional conditions might be imposed to limit their range. To do so would tend to raise issues of interpersonal comparisons, which we consider in the next section.

Impossibility for cardinal social utilities

Assuming that weak orders on X have utility functions, each $>_U$ in the preceding subsection could have been replaced by a social utility function f_U with $x >_U y \Leftrightarrow f_U(x) > f_U(y)$, for all $x, y \in X$. If we let $\approx^o$ denote *ordinal equivalence*, so that $f \approx^o g$ if, for all $x, y \in X$, $f(x) > f(y) \Leftrightarrow g(x) > g(y)$, then C5b and C7 would read as follows for all $U, U' \in \mathcal{U}$ and all $\{x, y\} \in \mathcal{A}$:

C5b. *If* $U \approx U'$ *then* $f_U \approx^o f_{U'}$,
C7. *If* $U =_{\{x,y\}} U'$, *then* $f_U \approx^o_{\{x,y\}} f_{U'}$.

In C7, $=_A$ denotes equality on A, and $\approx^o_A$ denotes ordinal equivalence on A.

Condition C5b says that cardinally equivalent profiles map into *ordinally* equivalent social utilities. We now consider the case in which cardinally equivalent profiles map into *cardinally* equivalent social utilities. That is, C5b is replaced by $U \approx U' \Rightarrow f_U \approx f_{U'}$.

The next theorem is from Kalai and Schmeidler [91] and Hylland [85]. According to [91], it was motivated by Samuelson's conjecture [128] to the effect that cardinal utilities do not offer a viable escape route from Arrow's impossibility. Condition C5a is implicit in the assumption of social utility functions, and C5b (cardinal version) is not listed since it is implied by C2 and C7*: $U \approx U' \Rightarrow (U \approx_A U'$ whenever $|A| = 3) \Rightarrow (f_U \approx_A f_{U'}$ whenever $|A| = 3) \Rightarrow f_U \approx f_{U'}$.

THEOREM 14 *Suppose C on $\mathscr{A} \times \mathscr{U}$ is a social choice function with a social utility function f_U for each $U \in \mathscr{U}$, that satisfies* C1, C2 *with* $|X| \geq 4$, C3, *and* C6. *Then one of the following must be violated*:

C4*. *For every $i \in N$ there is a $U \in \mathscr{U}$ with $f_U \not\approx u_i$.*

C7*. *For all three-element $A \subseteq X$, and all $U, U' \in \mathscr{U}$, if $U \approx_A U'$ then $f_U \approx_A f_{U'}$.*

The cardinal no-dictator condition C4* does not prohibit i from being a dictator in the usual sense since we can have f_U not cardinally equivalent to u_i even though $f_U \approx^o u_i$, i.e., even when i is an absolute dictator in the ordinal sense. Hence C4* is a very weak existential condition. Its interprofile companion, C7*, is quite strong since three-element subsets, as opposed to the usual two-element subsets, allow the full force of cardinal utility to come into play.

The proof of Theorem 14 is split between [91] and [85] since the latter shows that a continuity assumption used in the former is not needed. Like many other proofs, it uses Arrow's theorem. First, the conditions other than C4* are shown to imply the existence of an ordinal dictator via Arrow's theorem. It is then proved that the ordinal dictator is in fact a cardinal dictator, thus violating C4*.

Single-profile theorems

Roberts [124, p. 448] notes that single-profile analogues of multiprofile cardinal utility theorems can be developed in much the same way that single-profile analogues of multiprofile ordinal utility theorems were developed in Section 6. For example, A3* would be replaced by a domain condition which posits the existence of a utility profile U^* whose u_i^* components are unique up to positive linear transformations such that, for any utility profile U' on

$\{a_1, a_2, a_3\}$ there exist $x_1, x_2, x_3 \in X$ such that U^* restricted to $\{x_1, x_2, x_3\}$ is cardinally equivalent to U' under a relabeling of the a_j. Likewise, A9* would be replaced by an independence-neutrality condition for the cardinal context at hand (e.g., Theorem 13 or Theorem 14) that is powerful enough to generate extensions to arbitrary utility profiles, much like Theorem 11 for the ordinal case.

9. INTERPERSONAL COMPARISONS

The comparison of different individuals' preferences, utilities, or welfares has an interesting and turbulent history in social choice and welfare economics [4, 10, 33, 65, 84, 89, 105, 111, 115, 120, 121, 139]. Views have ranged from the total impossibility of making meaningful comparisons to the complete and precise ability to compare all individuals' utilities across all social states.

The purpose of the present section is to consider the effects of different presumptions about the degree of interpersonal comparability on social choice possibility and impossibility. Many of our results will be stated as possibility theorems, i.e., as characterizations of the forms that social utilities take under specified conditions. In each such case, an impossibility theorem arises when one or more additional conditions that contradict the characterization are proposed.

When interpersonal comparisons are considered, the question arises as to what agency is empowered to make such comparisons. Common answers refer to various ethical principles and/or to an extrasituational planner or planning group. The roles of mediators, judges, and juries are also relevant in many practical contexts. I shall not explore these possibilities, but wish only to note that a factor exogenous to the usual confines of a social choice function may lie behind interpersonal comparisons. At the conclusion of the section, we shall examine one formulation in which every individual in N is an interpersonal comparer.

In contrast to the preceding sections, we shall begin with the most rigid form of intrapersonal utility, namely ratio-scale utility, and then proceed to cardinal utility and thence to ordinal utility.

Ratio-scale utility

Suppose u is a utility function for a weak order $>_0$ on X that satisfies one or more additional conditions in a set $\mathscr{C}$, including *positivity*: $u(x) > 0$ for all $x \in X$. We say that u is a *ratio-scale utility function* (with respect to $\mathscr{C}$) if, for every utility function v for $>_0$, v satisfies $\mathscr{C}$ if and only if there is a positive number α such that, for all $x \in X$,

$$v(x) = \alpha u(x).$$

This is sometimes abbreviated by saying that ratio-scale utilities are unique up to positive multiplicative transformations or up to similarity transformations. A generalization of our definition allows zero as well as positive utilities for alternatives in X, but only the strictly positive version will be considered here.

Ratio-scale equivalence will be denoted by $\approx^r$. Thus, for any two utility functions u and v from X into the set of positive numbers, and for any nonempty $A \subseteq X$,

$$u \approx^r_A v \text{ if, for some } \alpha > 0,\ v(x) = \alpha u(x) \text{ for all } x \in A.$$

As usual, the subscript A is omitted when $A = X$. The name "ratio-scale" [142] refers to the fact that $v \approx^r u$ if and only if, for all $x,y \in X$,

$$\frac{v(x)}{v(y)} = \frac{u(x)}{u(y)}.$$

Ratio-scale measurement has deep roots in the physical sciences (length, absolute temperature, and so forth), where it is sometimes referred to as extensive measurement [99]. It has also been used in psychophysical measurement [142] and in the scaling of utilities [67].

Ratio-scale utilities can be viewed as specializations of cardinal utilities in which a zero point is either derived from a natural state (zero wealth, bankruptcy, death), or chosen by convention, or arises as an adjunct of the measurement process used to elicit relative intensities of preference. Any cardinal utility function that is bounded below can be converted to a ratio-scale utility function by fixing the infimum of the $u(x)$ values at 0. For certain mathematical purposes, it is sometimes useful to convert from one form to the other by exponentiation (cardinal to ratio) or by taking logarithms (ratio to cardinal) even though these transformations change the difference properties of the scales.

We shall consider two types of utility profile equivalence for individual ratio-scale utility functions. Let U and U' be mappings from N into the set of positive real valued functions on X. *Weak equivalence,* denoted by $U \approx^r U'$, is defined by

$$U \approx^r U' \text{ if, for all } i \in N,\ u_i \approx^r u_i'.$$

This conforms to our earlier definitions of equivalence for ordinal (Section 7) and cardinal ($\approx$) profiles. It means that for each $i \in N$ there is a positive α_i such that $u_i' = \alpha_i u_i$. *Strong equivalence,* denoted by $\approx^{rs}$, requires all α_i to be the same:

$$U \approx^{rs} U' \text{ if, for some } \alpha > 0,\ u_i' = \alpha u_i \text{ for all } i \in N.$$

Weak equivalence is involved in DeMeyer and Plott [43]; strong equivalence, identified as CRS, is used by Roberts [123].

Although the weak equivalence implication $U \approx^r U' \Rightarrow >_U = >_{U'}$ might appear to preclude all interpersonal comparisons, that is not the case. For example, the inequality $u_i(x)/u_i(y) > u_j(z)/u_j(w)$, which is preserved under $\approx^r$, could be interpreted to mean that i prefers x to y more intensely than j prefers z to w. At the same time, weak equivalence precludes interpersonal comparisons between single alternatives, such as i prefers x more strongly than j prefers y. However, strong equivalence involves such comparisons since $u_i(x)/u_j(y)$ does not vary as U ranges over a strong equivalence class.

The following characterization theorem is from [43]. Closely-related results appear in [1].

THEOREM 15A *Suppose C on $\mathscr{A} \times \mathscr{U}$ is a social choice function, with a positive social utility function f_U for each $U \in \mathscr{U}$, that satisfies* C1, C2 *with X finite, and*

C3$^+$. *$\mathscr{U}$ is the set of all functions from N into the set of positive real valued functions on X.*

Suppose further that the following hold:

C6$^=$. *For all $x,y \in X$ and all $U \in \mathscr{U}$, if $u_i(x) = u_i(y)$ for all $i \in N$, then $f_U(x) = f_U(y)$,* C7$^+$. *For all $i \in N$, all $x \in X$, and all $U, U' \in \mathcal{U}$, if $u_j = u'_j$ for all $j \in N \setminus \{i\}$, $u_i(y) = u'_i(y)$ for all $y \in X \setminus \{x\}$, and $u_i(x)/u'_i(x) = \gamma$, then there is a number $g(\gamma)$ such that*

$$f_{U'}(x)/f_{U'}(y) = g(\gamma)[f_U(x)/f_U(y)]$$

for all $y \in X \setminus \{x\}$. Moreover, g is a continuous function on $\{\gamma : \gamma > 0\}$ and is independent of i and x.

Then there is a positive real valued function c on $\mathcal{U}$ and a number k (positive, zero, or negative) such that, for all $U \in \mathcal{U}$ and all $x \in X$,

$$f_U(x) = c(U)[u_1(x)]^k[u_2(x)]^k \cdots [u_n(x)]^k$$

when $N = \{1, \ldots, n\}$.

Condition C6$^=$ is a Pareto-equality condition. Condition C7$^+$, which DeMeyer and Plott refer to as equal and continuous responsiveness, is a partial independence condition that incorporates notions of continuity, monotonicity, anonymity, and neutrality. It is a very powerful composite condition, as revealed by the conclusion of the theorem, which is established in part by a functional-equation derivation which shows that $g(\gamma\delta) = g(\gamma)g(\delta)$ and hence that $g(\gamma) = \gamma^k$ for some real k and all $\gamma > 0$.

It follows directly from the form of f_U in the conclusion of Theorem 15A that, for all $U, U' \in \mathcal{U}$, $U \approx^r U' \Rightarrow f_U \approx^r f_{U'}$. Hence the social utility functions are ratio-scale functions under the weak equivalence relation between profiles. If $k = 0$, then every f_U is constant; if $k > 0$, then a form of positive responsiveness obtains; and if $k < 0$, then negative responsiveness applies. Anonymity holds in all cases, as does a ratio form of neutrality. Moreover, if $k > 0$, then the usual Pareto conditions hold and, if $n > 1$, there is no dictator or vetoer.

Roberts's work [123, p. 433], which is presented for a partial independence condition in the cardinal utility context, can be modified for ratio scales to yield $[u_1(x)]^{k_1}[u_2(x)]^{k_2} \cdots [u_n(x)]^{k_n}$ instead of $[u_1(x)]^k \cdots [u_n(x)]^k$ in the conclusion of Theorem 15A. Impossibility obtains for either this nonanonymous generalization or for the original anonymous form when the usual binary independence axiom is imposed along with a condition that prevents every f_U from being constant.

The other characterization theorem we shall present for ratio-scale utilities is Theorem 6 in [123]. Its format is similar to our earlier Theorem 13.

THEOREM 15B *Suppose C on $\mathcal{A} \times \mathcal{U}$ is a social choice function that satisfies* C1 *with* $|N| \geq 3$, C2, C3$^+$, C5a, C6, *and* C7. *If, in addition, it satisfies*

C5b^{rs}. *For all* $U, U' \in \mathcal{U}$, *if* $U \approx^{rs} U'$ *then* $>_U = >_{U'}$,

C10. *For all permutations* σ *on* N, *and all* $U, U' \in \mathcal{U}$, *if* $u_i' = u_{\sigma(i)}$ *for all* $i \in N$, *then* $>_U = >_{U'}$,

C11. *For all* $U, U' \in \mathcal{U}$ *and all* $I \subseteq N$, *if* $u_i = u_i'$ *for all* $i \in I$, *and* u_i *and* u_i' *are constant for all* $i \in N \backslash I$, *then* $>_U = >_{U'}$,

then there exists a number k such that, for all $U \in \mathcal{U}$ *and all* $x, y \in X$,

$$\text{if } k \neq 0,\ \sum_{i \in N} [u_i(x)]^k > \sum_{i \in N} [u_i(y)]^k \Rightarrow x >_U y;$$

$$\text{if } k = 0,\ \sum_{i \in N} \log[u_i(x)] > \sum_{i \in N} \log[u_i(y)] \Rightarrow x >_U y.$$

In view of C5b^{rs} and the summation representations, which are inapplicable under weak equivalence, Theorem 15B applies to the case of strong ratio equivalence. The reason that the conclusions are not $\Leftrightarrow$, but go only one way, is that there is nothing in the theorem's conditions that forces $x \sim_U y$ whenever $\sum [u_i(x)]^k = \sum [u_i(y)]^k$. These sums may of course be taken to be f_U values. A continuity condition [123, p. 428] will then give $f_U(x) > f_U(y) \Leftrightarrow x >_U y$.

There are four interprofile conditions in the theorem. One is C7 (independence) from Theorem 13. The other three, all of which have $>_U = >_{U'}$ for their conclusions, are the newly specified conditions. Anonymity is stated as C10, and C11 is a separability condition which says that social preferences shall not depend on individuals who are totally indifferent over X (u_i and u_i' constant). The latter condition gives rise to an independence axiom of Debreu [40] (see also [52, Chapter 5]) that is applied to obtain the additive representation. The requirement of at least three individuals is needed at this point. A stronger assumption [52, p. 65] can be used when $|N| = 2$.

Roberts [123, p. 432] notes interesting things that happen with the representation of Theorem 15B when $k=1$, $k\to\infty$, and $k\to-\infty$. Since these special cases obtain more generally within the setting of strong cardinal equivalence, which is examined in the next subsection, I shall discuss them there.

Strong cardinal equivalence

We shall consider two strong forms of cardinal equivalence for profiles U that conform to C3. Both go well beyond the weak cardinal equivalence $\approx$ of the preceding section. The first form [37, 70, 74, 89, 123, 136] is

$$U \approx^s U' \text{ if, for some } \alpha > 0 \text{ and real } \beta_i \text{ for each } i \in N,$$

$$u_i' = \alpha u_i + \beta_i \text{ for all } i \in N.$$

As usual, the principle for social comparisons that will be used under this equivalence relation is $U \approx^s U' \Rightarrow >_U = >_{U'}$. Since the same $\alpha > 0$ applies to all individuals, this says that intrapersonal utility differences are interpersonally comparable. Thus, if $U \approx^s U'$, then $u_i(x) - u_i(y) > u_j(z) - u_j(w)$ if and only if $u_i'(x) - u_i'(y) > u_j'(z) - u_j'(w)$. However, algebraic values or levels of individual utilities are not interpersonally comparable since origins of the utility functions can be set anywhere (different β_i) within a profile equivalence class.

The second form [37, 44, 70, 103, 105, 106, 123, 136] is the even stronger relation defined by

$$U \approx^{ss} U' \text{ if, for some } \alpha > 0 \text{ and real } \beta,$$

$$u_i' = \alpha u_i + \beta \text{ for all } i \in N.$$

The corresponding principle is $U \approx^{ss} U' \Rightarrow >_U = >_{U'}$. This asserts that algebraic values of individual utilities as well as intrapersonal utility differences are interpersonally comparable. It is quite close to the strong ratio equivalence $\approx^{rs}$. An ordinal weakening of $\approx^{ss}$ that maintains algebraic-value comparability but not difference comparability will be considered in the next subsection.

Our first two theorems for strong cardinal equivalence deal with $\approx^s$. The following is from [123].

THEOREM 16A *Suppose C on $\mathscr{A} \times \mathscr{U}$ is a social choice function that satisfies* C1, C2, C3, C5a, C6, C7, *and*

C5b^s. *For all $U, U' \in \mathscr{U}$, if $U \approx^s U'$ then $>_U = >_{U'}$.*

Then there exist nonnegative w_i for all $i \in N$ at least one of which is positive such that, for all $U \in \mathscr{U}$ and all $x, y \in X$,

$$\sum_{i \in N} w_i u_i(x) > \sum_{i \in N} w_i u_i(y) \Rightarrow x >_U y.$$

As with Theorem 15B, Roberts's conditions here are not strong enough to imply that $x \sim_U y$ whenever the weighted utility sums are equal. If anonymity, C10, is used, then all w_i are equal and, apart from the equality situation, we obtain the utilitarian social welfare function. Theorem 16A does not rule out the possibility of a dictator since it allows all but one w_i to equal 0. As we know from Theorem 13, if C5b^s is replaced by C5b for weak cardinal comparability, then there must be a dictator.

Several people [37, 103, 106] have noted conditions that imply the usual utilitarian form. The following is the version in [37].

THEOREM 16B *Suppose C on $\mathscr{A} \times \mathscr{U}$ is a social choice function that satisfies* C1, C2, C3, C5a, C5b^s, C7, C10, *and*

C6*. *For all $x, y \in X$ and all $U \in \mathscr{U}$, if $u_i(x) \geqslant u_i(y)$ for all $i \in N$ then $x \gtrsim_U y$; if, in addition, $u_i(x) > u_i(y)$ for some $i \in N$, then $x >_U y$.*

Then, for all $U \in \mathscr{U}$ and all $x, y \in X$,

$$x >_U y \Leftrightarrow \sum_{i \in N} u_i(x) > \sum_{i \in N} u_i(y).$$

Thus, under anonymity, the strengthening of C6 to the strongest Pareto condition C6* yields a precise representation of $>_U$ by unweighted utility sums.

Our first result under the super-strong cardinal equivalence relation $\approx^{ss}$ for utility profiles is also from Roberts [123]. We say that a real valued function h on Euclidean n-space $\mathbb{R}^n$ is *homogeneous of degree* 1 if, for all real $\gamma > 0$ and all $t = (t_1, \ldots, t_n)$ in $\mathbb{R}^n$, $h(\gamma t) = h(\gamma t_1, \ldots, \gamma t_n) = \gamma h(t)$. Let $N = \{1, \ldots, n\}$.

THEOREM 17A *Suppose C on $\mathcal{A} \times \mathcal{U}$ is a social choice function that satisfies* C1, C2, C3, C5a, C6, C7, *and*

C5b^{ss}. *For all $U, U' \in \mathcal{U}$, if $U \approx^{ss} U'$ then $>_U = >_{U'}$.*

Then, with $\mu(x) = [u_1(x) + \cdots + u_n(x)]/n$ for all x and U, there exists a homogeneous of degree 1 *function h on $\mathbb{R}^n$ such that, for all $U \in \mathcal{U}$ and all $x, y \in X$,*

$$\mu(x) + h(u_1(x) - \mu(x), \ldots, u_n(x) - \mu(x)) > \mu(y) + h(u_1(y) - \mu(y), \ldots, u_n(y) - \mu(y)) \Rightarrow x >_U y.$$

This is a very nice result because its representation $f_U(x) = \mu(x) + h(u_1(x) - \mu(x), \ldots, u_n(x) - \mu(x))$ allows interesting specializations without being too general. It does this with explicit reference to the average individual utility or mean welfare μ and to individuals' deviations from the mean. Hence, it encompasses various social utility or welfare measures based on the mean and a distributional measure of dispersion or variation. Examples appear in Roberts [123, p. 431].

Also note that, when $h = k \min[u_i(x) - \mu(x)]$, i.e.,

$$f_U(x) = \mu(x) + k \min_i [u_i(x) - \mu(x)],$$

we obtain a utilitarian measure when $k = 0$ and the Rawlsian [120] measure $\min u_i(x)$ when $k = 1$. Mixtures of these two obtain when $0 < k < 1$.

To prepare for our other super-strong theorem, let $(r_1(x), \ldots, r_n(x))$ be a rearrangement of the components of $(u_1(x), \ldots, u_n(x))$ in ascending order: $r_1(x) \leqslant r_2(x) \leqslant \cdots \leqslant r_n(x)$. We define two lexicographic orders based on utility vectors for each $U \in \mathcal{U}$ as follows:

$x >^{\ell}_U y$ if $(r_1(x), \ldots, r_n(x)) \neq (r_1(y), \ldots, r_n(y))$ and $r_j(x) > r_j(y)$ for the *smallest* j at which $r_j(x) \neq r_j(y)$;

$x >^{L}_U y$ if $(r_i(x)) \neq (r_i(y))$ and $r_j(x) > r_j(y)$ for the *largest* j at which $r_j(x) \neq r_j(y)$.

Thus $x >^{\ell}_U y$ if the individual with the smallest utility at x is better off than the individual with the smallest utility at y, or if these two have equal utilities and the individual with the next smallest utility at x is better off than the individual with next smallest utility at y, and so forth. Similarly, $x >^L_U y$ if the individual with the largest utility at x is better off than the individual with the largest utility at y, or if . . . , and so forth.

When $>_U = >^\ell_U$ for all $U \in \mathcal{U}$, social preferences are said to obey the lexicographic maximin rule, or *leximin rule*. This rule, which seeks to make the worst-off person as well off as possible, comes from Rawls's work [120]. When $>_U = >^L_U$ for all $U \in \mathcal{U}$, social preferences are said to obey the lexicographic maximax rule, or *leximax rule*. In most contexts this is a very objectionable principle, and it is often ruled out by a suitable equity axiom.

The following theorem, from Deschamps and Gevers [44], has precursors in [37, 79, 136, 143].

THEOREM 17B *Suppose C on $\mathcal{A} \times \mathcal{U}$ is a social choice function that satisfies* C1, C2, C3 *confined to bounded real valued functions,* C5a, C5b^{ss}, C6*, C7, C10, *and* C11. *Then either $>_U = >^\ell_U$ for all $U \in \mathcal{U}$, or $>_U = >^L_U$ for all $U \in \mathcal{U}$, or else $\mu(x) > \mu(y) \Rightarrow x >_U y$ for all $x,y \in X$ and all $U \in \mathcal{U}$.*

In other words, the conditions of the theorem imply either the leximin rule, the leximax rule, or a utilitarian rule. In the latter case the social relation between x and y is left open when $\sum u_i(x) = \sum u_i(y)$. Related theorems which uniquely characterize the leximin rule appear in [37, 44].

Ordinal interpersonal comparisons

We now drop the cardinal aspects of $\approx^{ss}$ to consider ordinal interpersonal comparisons of individuals' welfare levels. Strong ordinal equivalence between profiles is defined by

$$U \approx^{os} U' \text{ if, for all } i,j \in N \text{ and all } x,y \in X,$$
$$u_i(x) > u_j(y) \Leftrightarrow u_i'(x) > u_j'(y).$$

The associated interprofile principle for social choice is

C5b^{os}. *For all $U, U' \in \mathcal{U}$, $U \approx^{os} U' \Rightarrow >_U = >_{U'}$.*

Instead of working with C5B^{os}, which we use initially since it fits into our previous format and interprofile theme, several authors, beginning with Suppes [144] and including [79, 122, 136, 143], work with *extended weak orders* $>^e$ on $X \times N$. Others [37, 70, 123] adopt the utility profile formulation. The intent of a weak order $>^e$ on $X \times N$ is to provide an interwoven ordering of intraindividual preferences and interpersonal preference comparisons. Given such an $>^e$, it identifies an equivalence class of utility profiles in $\mathscr{U}$ under $\approx^{os}$ by means of the correspondence

$$u_i(x) > u_j(y) \Leftrightarrow (x, i) >^e (y, j), \text{ for all } (x, i), (y, j) \in X \times N.$$

Conversely, every such equivalence class identifies an extended weak order $>^e$ on $X \times N$ in the obvious way. The $>^e$ format will be used later when *profiles* of extended weak orders are considered.

Our first theorem for strong ordinal equivalence, from d'Aspremont and Gevers [37], nullifies the utilitarian option in the conclusion of Theorem 17B by replacing C5b^{ss} with C5b^{os} in its hypotheses. A similar characterization of the leximin rule under strong ordinal equivalence is [79, Theorem 7.2].

THEOREM 18A *Suppose the hypotheses of Theorem* 17B *hold with* C5b^{os} *in place of* C5b^{ss}. *Then either* $>_U = >_U^{\ell}$ *for all* $U \in \mathscr{U}$, *or* $>_U = >_U^L$ *for all* $U \in \mathscr{U}$.

The next result is from Roberts [122]. In comparison with Theorem 18A, it drops C11 (separability) and weakens C6* to the original Parteo condition C6. As before, let $(r_1(x), \ldots, r_n(x))$ be a rearrangement of the components of $(u_1(x), \ldots, u_n(x))$ in ascending order. Assuming that $N = \{1, \ldots, n\}$, we say that *position* $d \in \{1, \ldots, n\}$ is *dictatorial* if, for all $x, y \in X$ and all $U \in \mathscr{U}$, $r_d(x) > r_d(y) \Rightarrow x >_U y$. It is important to note that this is a dictatorship of a position in rankings, not of an individual in N. When leximin applies, position 1 (worst-off) is dictatorial; when leximax applies, position n (best-off) is dictatorial.

THEOREM 18B *Suppose* C *on* $\mathscr{A} \times \mathscr{U}$ *is a social choice function that satisfies* C1, C2, C3, C5a, C5b^{os}, C6, C7, *and* C10. *Then there is a dictatorial position.*

When anonymity, C10, is dropped, a more general array of possibilities than the positional dictatorships arises, but it is still fairly limited [122, Theorems 2 and 3]. On the other hand, if C10 is retained and C6 is strengthened to C6*, then there will be a partial lexicographic hierarchy of dictatorial positions [70, Theorem 5].

And if, in addition, C11 is added, then we are back to Theorem 18A, where only leximin and leximax are possible.

We now turn to the extended profile case in which every individual in N has an extended weak order on $X \times N$ and these weak orders are to be used as the basis on which a social weak order on X is determined. This format has been adopted by Suppes [144] and Varian [146] in somewhat different settings. In the following theorem from Roberts [122], profile P is tantamount to $(>^e_1, \ldots, >^e_n)$, and P' to $(>^{e'}_1, \ldots, >^{e'}_n)$.

THEOREM 19 *Suppose C on $A \times \mathcal{P}$ is a social choice function that satisfies* C1, C2, *and*

C3^e. $\mathcal{P}$ is the set of all functions from N into the set of weak orders on $X \times N$.

Then C cannot satisfy all of the following:

C4^e. *For every $i \in N$ there exist $x, y \in X$ and a $P \in \mathcal{P}$ such that $(x, i) >^e_i (y, i)$ and $y \gtrsim_P x$,*
C5^e. *For every $P \in \mathcal{P}$, $>_P$ is a weak order on X,*
C6^e. *For all $x, y \in X$ and all $P \in \mathcal{P}$, if $(x, i) >^e_i (y, i)$ for all $i \in N$, then $x >_P y$,*
C7^e. *For all $x, y \in X$ and all $P, P' \in \mathcal{P}$, if $>^e_i$ and $>^{e'}_i$ are identical on $\{x, y\} \times N$ for each $i \in N$, then $>_P$ and $>_{P'}$ are identical on $\{x, y\}$.*

Conditions C4^e and C6^e are essentially the same as A4 and A6, respectively, C5^e maps extended profiles into weak orders on X, and C7^e is a binary independence condition that differs from A7 in that the individual orders in the extended form consider N as well as $\{x, y\}$.

It may be wondered what happens to Theorem 19 when C3^e is restricted by requiring the same set of intrapersonal orders within every extended ordering in an admissible profile [136], i.e., when everybody's extended order accurately reproduces the intrapersonal order of everyone else. The only differences between extended orders in an admissible extended profile would then be the ways they interweave the $>_i$ on X to reflect their interpersonal judgements. According to Roberts [122, pp. 418-419], this would allow a social choice function that satisfies the other conditions of the theorem but, when $C5^e$, $C6^e$, and $C7^e$ hold, the nondictatorial possibilities would probably not be very appealing. I am not aware of more definitive results on this matter.

10. INFINITE NUMBERS OF INDIVIDUALS

The second proof of Theorem 1, which was modified for partially ordered social preferences in Theorems 2A and 2B, showed that N must be infinite when $N \neq \varnothing$ and A2 through A7 hold. This suggests that, when A1 is replaced by

A1$^+$. *N is a nonempty set,*

the conditions of Theorem 1 will be consistent. The present section shows that this is indeed true. Moreover, we shall see precisely how consistency obtains for infinite N and will note several related results.

Since infinite sets of individuals are absurd in practice, it may be wondered whether anything can be gained beyond satisfaction of mathematical curiosity by their consideration. I believe that there is. In particular, the work discussed here has provided deeper insights into the general nature of Arrow's problem that have helped us to better understand the structure of social choice when N is finite. A case in point is Brown's analysis [25, 26] of voting rules that was guided by questions originally posed for infinite sets of individuals.

Unless it is noted otherwise, we shall assume throughout this section that C denotes a social choice function on $\mathscr{A} \times \mathscr{P}$ that satisfies A1$^+$, A2, *and* A3. As usual, $>_P$ will denote an asymmetric binary relation on X for each $P \in \mathscr{P}$ such that $x >_P y$ if and only if $x \neq y$ and $C(\{x, y\}, P) = \{x\}$.

Decisive coalitions

Decisive coalitions will play a central role throughout the section. A *coalition* is any subset of N. The term is intended to be descriptive, not political. We say that coalition $I \subseteq N$ is *decisive for x over y* if, for all $P \in \mathscr{P}$, $x >_P y$ whenever $x >_I y$, where $x >_I y$ means that $x >_i y$

for all $i \in I$. A *decisive coalition* is a coalition that is decisive for x over y for all distinct x and y in X. Condition A6, i.e., $x >_N y \Rightarrow x >_P y$, says that N is a decisive coalition.

Families of decisive coalitions will often be denoted by $\mathscr{F}$, with or without affixes. The family of all decisive coalitions for C is denoted by $\mathscr{F}(C)$.

Decisive coalitions illustrate one important difference between the acyclic social preferences condition A5** and its weak order and partial order counterparts, A5 and A5*. By a proof that is entirely similar to step 2 in the first proof of Arrow's theorem (replace $\{i\}$ by I), it follows from A6 and A7, and either A5 or A5*, that if I is decisive for x over y for some two distinct alternatives in X, then I is decisive for *all* pairs and is therefore a decisive coalition. Consequently, $\mathscr{F}(C)$ tells us a great deal about C when A5 or A5* is assumed along with A6 and A7.

The same thing is not true under A5**. That is, given A5**, A6, and A7, I can be decisive for some pairs of alternatives but not for others. An example of this appears immediately after Theorem 4C. In that example, every coalition of $|N| - 1$ individuals is decisive for the pairs in $\mathscr{A}_1$, but only N is decisive for the pairs in $\mathscr{A}_2$.

This difference between A5** and {A5, A5*} will be removed if a neutrality condition like A9 is adopted. Given {A5**, A6, A9}, $\mathscr{F}(C)$ will not be empty, and every coalition that is decisive for an x over a $y \neq x$ will be in $\mathscr{F}(C)$.

According to A3 and the definition of decisive coalition, if $I \in \mathscr{F}(C)$ and $I \subset J \subseteq N$, then $J \in \mathscr{F}(C)$. Hence every superset of a decisive coalition is also decisive. We shall say that $I \in \mathscr{F}(C)$ is a *minimal* decisive coalition if there is no $J \in \mathscr{F}(C)$ that is a proper subset of I. When N is finite, $\mathscr{F}(C)$ equals its minimal elements and their supersets. When N is infinite, $\mathscr{F}(C)$ might have no minimal decisive coalitions. Examples of this follow.

Examples

Throughout this subsection, $N = \{1, 2, 3, \ldots\}$, the set of positive integers. Two examples will illustrate the theme of the section.

First, let $\mathscr{F}_0$ be the family of all subsets of N that contain all but a finite number of individuals in N. Every $I \in \mathscr{F}_0$ is countably infinite with finite *complement* $I^c = N \backslash I$. Clearly, $\mathscr{F}_0$ has no minimal member.

Define the binary part of a social choice function C by using $\mathscr{F}_0$ as $\mathscr{F}(C)$:

$$x >_P y \text{ if and only if } x >_I y \text{ for some } I \in \mathscr{F}_0.$$

Then C satisfies A6 (Pareto) and A7 (binary independence), but A5 fails since some $>_P$ are not negatively transitive. In particular, if P has $y >_i x >_i z$ for all odd $i \in N$ and $x >_i z >_i y$ for all even $i \in N$, then $x \sim_P y$, $y \sim_P z$, and $x >_P z$.

On the other hand, since the intersection of any two sets in $\mathscr{F}_0$ is also in $\mathscr{F}_0$, every $>_P$ is transitive and therefore A5* holds. Moreover, no individual is a vetoer, so A4* holds. Hence, under the change from A1 to A1$^+$, all conditions of Theorem 2A hold, so its impossibility result for partially ordered social preferences disappears when N is allowed to be infinite.

What about Theorem 1 and A5? The underlying reason that A5 fails for $\mathscr{F}(C) = \mathscr{F}_0$ is that there are $I \subseteq N$ such that neither I nor its complement I^c are in $\mathscr{F}_0$. As shown above, this will contradict negative transitivity for some $>_P$. To satisfy A5, we enrich $\mathscr{F}_0$ to a larger family $\mathscr{F}_1$ so that, for all $I \subseteq N$, either I or I^c (but not both) is in $\mathscr{F}_1$. Then new $>_P$ relations, based on the enriched set of decisive coalitions, are defined by

$$x >_P y \text{ if and only if } x >_I y \text{ for some } I \in \mathscr{F}_1.$$

In making the additions to $\mathscr{F}_0$, we must ensure that the intersection of any two sets in $\mathscr{F}_1$ is nonempty, for otherwise some $>_P$ will not be asymmetric. But this is not enough to guarantee negative transitivity, for we also need $I \cap J \in \mathscr{F}_1$ whenever $I, J \in \mathscr{F}_1$. Otherwise, with $I, J \in \mathscr{F}_1$ and $I \cap J \notin \mathscr{F}_1$, so $(I \cap J)^c \in \mathscr{F}_1$, a profile with

$$z >_{I \setminus J} x >_{I \setminus J} y$$

$$y >_{J \setminus I} z >_{J \setminus I} x$$

$$x >_{I \cap J} y >_{I \cap J} z$$

along with $z >_{(I \cup J)^c} x$ gives $x >_P y >_P z >_P x$.

It easily checked that every new $>_P$ will be a weak order if $N \in \mathscr{F}_1$, $(I \in \mathscr{F}_1,\ I \subset J) \Rightarrow J \in \mathscr{F}_1$, exactly one of I and I^c is in $\mathscr{F}_1$ for each $I \subseteq N$, and $I \cap J \in \mathscr{F}_1$ whenever $I, J \in \mathscr{F}_1$. The question remains as to whether $\mathscr{F}_0$ can be enriched to yield an $\mathscr{F}_1$ with these properties. If it can, then all conditions of Theorem 1 hold (including A4 since all finite coalitions are excluded from $\mathcal{F}_1$) under the change from A1 to A1$^+$.

It is far from obvious that such $\mathcal{F}_1$ exist, but they do, and in great abundance. However, it is not possible to demonstrate one by an elementary formula or construction. The standard way to prove existence is to consider the class of all enrichments $\mathcal{F}$ of $\mathcal{F}_0$ that satisfy the properties of the preceding paragraph except perhaps for I or I^c in $\mathcal{F}$ for every $I \subseteq N$. It is then shown that if there is an I such that neither I nor I^c is in $\mathcal{F}$, then one of the two can be added to $\mathcal{F}$ without violating the other properties, provided its supersets are also added if they are not already in $\mathcal{F}$. A technical axiom, called by various names, including Zorn's lemma, the axiom of choice, and the well-ordering principle [52, 93], is then used to show that the class of enrichments has a maximal member which, by the extension property just noted, must contain either I or I^c for every $I \subseteq N$. Any such maximal member serves as a suitable $\mathcal{F}_1$.

Filters

The families of decisive coalitions just considered are examples of what are known as filters ($\mathcal{F}_0$) and ultrafilters ($\mathcal{F}_1$). We now define these terms and will then see how they illuminate Arrow's theorem and other results.

A family $\mathcal{F}$ of subsets of N is a *prefilter* if

1. $N \in \mathcal{F}$,
2. $J \in \mathcal{F}$ whenever $I \in \mathcal{F}$ and $I \subset J$,
3. The intersection of any positive finite number of members of $\mathcal{F}$ is nonempty.

A *filter* is a prefilter $\mathcal{F}$ that satisfies

3′. $I \cap J \in \mathcal{F}$ whenever $I, J \in \mathcal{F}$.

An *ultrafilter* is a filter $\mathcal{F}$ that satisfies

4. For all $I \subseteq N$, either $I \in \mathcal{F}$ or $I^c \in \mathcal{F}$.

By properties 1 and 3, the empty coalition $\varnothing$ is not a member of any prefilter. Moreover, an alternative definition of a filter is obviously given by 1,2,3′, and $\varnothing \notin \mathcal{F}$. In a manner of speaking, an ultrafilter is a complete or total filter.

The following theorems show how these three types of coalitional structures tie into the three basic types of passive intraprofile conditions for social preference considered earlier. After making a few comments on the theorems, we shall address the matter of dictators and other specialized power configurations.

THEOREM 20A *If C satisfies* A5, A6, *and* A7, *then* $\mathcal{F}(C)$ *is an ultrafilter. If* $\mathcal{F}$ *is an ultrafilter on* N, *then there is a* C *with* $\mathcal{F}(C) = \mathcal{F}$ *that satisfies* A5, A6, *and* A7.

THEOREM 20B *If C satisfies* A5*, A6, *and* A7, *then* $\mathcal{F}(C)$ *is a filter. If* $\mathcal{F}$ *is a filter on* N, *then there is a* C *with* $\mathcal{F}(C) = \mathcal{F}$ *that satisfies* A5*, A6, *and* A7.

THEOREM 20C *If C satisfies* A5**, A6, *and* A9, *and if either* $|X|$ *is as large as the number of minimal decisive coalitions in* $\mathcal{F}(C)$ *when* N *is finite, or if* X *is infinite when* N *is infinite, then* $\mathcal{F}(C)$ *is a prefilter. If* $\mathcal{F}$ *is a prefilter on* N, *then there is a* C *with* $\mathcal{F}(C) = \mathcal{F}$ *that satisfies* A5**, A6, *and* A9.

Theorem 20A, which was motivated by an example in [51], was established independently by Kirman and Sondermann [98] and Hansson [83]. Theorem 20B is from [83], and Theorem 20C is essentially due to Brown [25, 26].

Theorems 20A and 20B delineate the structures of decisive coalition sets that arise from Theorems 1 and 2A when their existential conditions (A4, A4*) are ignored and A1 is generalized to A1$^+$. Similar results can be obtained for other configurations of conditions in Section 5.

For example, it can be shown that the conditions of Theorem 2B, including strong monotonicity A8, $|N| \geqslant 3$, and A5*, but excluding A4, imply that $\mathcal{F}(C)$ is an ultrafilter. Border [22] obtains a result similar to Theorem 5B under A1$^+$ and a restriction of profiles to a specialized structure. He concludes that either $x \sim_P y$ for all x, y, and P, or the set $\mathcal{F}(C)$ of decisive coalitions is an ultrafilter, or the set of "antidecisive" coalitions is an ultrafilter. Chichilnisky and Heal [30] extend the analysis of [27] to the infinite-N context.

A sketch of the proof of Theorem 20C may be instructive. For the first part, let C satisfy A5**, A6, and A9, with N finite and X as numerous as the number of minimal decisive coalitions in $\mathcal{F}(C)$. (The proof for infinite X is simpler and will be omitted.) Property 1 for $\mathcal{F}(C)$ follows from A6, and property 2 is immediate from the definitions. To verify property 3 for a prefilter, suppose to the contrary that some nonempty finite collection of members of $\mathcal{F}(C)$ is empty. Let $I_1, \ldots, I_m$ be the minimal decision coalitions of $\mathcal{F}(C)$ included in the members of this collection, with all I_j distinct and $\cap I_j = \emptyset$. Since $|X| \geq m$ by assumption, let $x_1, \ldots, x_m$ be distinct alternatives in X, take $x_j >_i x_{j+1}$ for all $i \in I_j$ when $j \leq m-1$, and take $x_m >_i x_1$ for all $i \in I_m$. Assign a weak order to each individual that contains the ordered pairs just defined for the coalitions I_j that contain the individual. It is always possible to do this since $\cap I_j = \emptyset$. Then, since each I_j is in $\mathcal{F}(C)$, we conclude that $x_1 >_P x_2 >_P \cdots >_P x_m >_P x_1$, which contradicts A5**.

For the second part of Theorem 20C, let $\mathcal{F}$ be a prefilter on N. Define the $>_P$ by $x >_P y$ if and only if $x >_I y$ for some $I \in \mathcal{F}$. Then $\mathcal{F}(C) = \mathcal{F}$ follows immediately, as do A6 and A9. By property 3, if some $>_P$ has a cycle, then some i must be in every decisive coalition used to generate the cycle, so some $>_i$ will be cyclic, contrary to A3.

Special power structures

A prefilter (filter, ultrafilter) $\mathcal{F}$ is *free* if the intersection of all $I \in \mathcal{F}$ is empty, and is *fixed* or *principal* if $\cap_{\mathcal{F}} I \neq \emptyset$. For the examples given earlier with $N = \{1, 2, 3, \ldots\}$, $\mathcal{F}_0$ is a free filter, and every maximal extension $\mathcal{F}_1$ of $\mathcal{F}_0$ is a free ultrafilter. The relevance of these concepts to social choice is suggested by the following elementary facts.

Fact 1. Every free prefilter, filter, and ultrafilter must contain an infinite number of sets.

Fact 2. Every fixed ultrafilter consists of a singleton $\{i\}$ and its supersets.

Fact 3. Every finite filter consists of a coalition I and its supersets.

Fact 4. Every finite prefilter consists of a family of supersets of $\cap_{\mathcal{F}} I$.

We note also that the conclusion of Fact 3 need not hold for a fixed filter when N is infinite. For example, let $\mathcal{F}$ with $N = \{1, 2, 3, \ldots\}$ be the subfamily of $\mathcal{F}_0$ whose sets contain 1. Then $\mathcal{F}$ is a fixed filter with $\cap_{\mathcal{F}} I = \{1\}$, but $\{1\}$ and many sets that contain 1 are not in $\mathcal{F}$.

Given a social choice function C, let

$$I_C = \bigcap_{I \in \mathscr{F}(C)} I.$$

We refer to I_C as the *power core* of C. Assuming that $\mathscr{F}(C)$ is at least a prefilter, Fact 1 says that C can have an empty power core only if N is infinite. Aspects of nonempty power cores will be considered in the ensuing paragraphs. We begin with ultrafilters and then comment on filters and prefilters.

Suppose first that $\mathscr{F}(C)$ is an ultrafilter: see Theorem 20A. If $\mathscr{F}(C)$ is fixed then, by Fact 2, the power core consists of a single individual, who is a dictator. By Fact 1, this must be the case when N is finite (Arrow's theorem). Hence the only way to avoid a dictator under {A5, A6, A7} is for N to be infinite with $\mathscr{F}(C)$ free, in which case I_C is empty. Suppose that the power core is empty. Then there is a basic imbalance of power in N even though there is no dictator. The reason is that anonymity must fail. In the present context, we say that C is *anonymous* if $>_P = >_{P'}$ whenever there is a one-to-one mapping σ from N onto itself such that $>'_i = >_{\sigma(i)}$ for all $i \in N$. To see why anonymity cannot hold when $\mathscr{F}(C)$ is a free ultrafilter [83], we note that there must be an $I \in \mathscr{F}(C)$ such that I and I^c have the same cardinality. Therefore, there is a σ with $\sigma(I) = I^c$ and $\sigma(I^c) = I$. Let P be a profile with $x >_I y$ and $y >_{I^c} x$. Since $I \in \mathscr{F}(C)$, $x >_P y$. Moreover, with P' as defined through σ, $y >'_I x$, so $y >_{P'} x$, and this contradicts anonymity. Consequently, Arrow's dictator reappears under {A5, A6, A7} and infinite N when C is anonymous.

Further remarks on power imbalances in free ultrafilters appear in the next subsection.

Suppose now that $\mathscr{F}(C)$ is a filter: see Theorem 20B. If $\mathscr{F}(C)$ is fixed, then every individual in the power core is a vetoer. For suppose to the contrary that $x >_i y$ and $y >_P x$ for some $i \in I_C$ and some x,y, and P. Let P' be like P on $\{x, y\}$ and have $x >_i z$ along with $x >_j z$ and $y >_j z$ for all $j \neq i$. Then $y >_{P'} x$ by A7, $x >_{P'} z$ by A6, and therefore $x >_{P'} z$ by A5*. Since the relation in P' between y and z for i is arbitrary, $N \setminus \{i\}$ is decisive for y over z and is therefore in $\mathscr{F}(C)$. But this contradicts $i \in I_C$.

It follows that if $\mathscr{F}(C)$ is a fixed filter then I_C is an oligarchy if and only if it is in $\mathscr{F}(C)$. By Fact 3, this must be true if N is finite. When

$I_C \notin \mathscr{F}(C)$ for infinite N, and all individuals in I_C prefer x to y, some individuals not in the power core must prefer x to y to obtain $x >_P y$.

When $\mathscr{F}(C)$ is a free filter with N infinite, no individual is a vetoer. Moreover, C can be anonymous. An example is $\mathscr{F}_0$. However, when C is anonymous, every decisive coalition must have the same cardinality as N, so only "large" sets can be decisive and the complement of every decisive set must have cardinality less than $|N|$. Hence anonymous C that satisfy $\{A5^*, A6, A7\}$ are quite similar to the Pareto rule for finite N which says that $x >_P y$ if and only if $x >_N y$.

Finally, suppose that $\mathscr{F}(C)$ is a prefilter: see Theorem 20C. We consider only the case in which N is finite and the fixed prefilter (Facts 1 and 4) does not contain the power core, i.e., is not a filter. In this case Brown [26] refers to the power core as the *collegium*. Because $I_C \notin \mathscr{F}(C)$, the power core is not a decisive coalition and the collegium is not an oligarchy. An example with $N = \{1, 2, 3, 4, 5\}$ is $\mathscr{F}(C)$ equal to $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$ and their supersets. The collegium is $\{1,2\}$. If the $>_P$ are defined from $\mathscr{F}(C)$ as in the final paragraph of the preceding subsection, then x is socially preferred to y if and only if the two individuals in the collegium and at least one other prefer x to y.

Anonymity cannot hold when N is finite and $\mathscr{F}(C)$ is a prefilter that is not a filter. For if anonymity holds when $\mathscr{F}(C)$ is a prefilter, then every i must be in the power core, so $I_C = N$ and $\mathscr{F}(C) = \{N\}$.

Power in free ultrafilters

Further analyses of the distribution of power over N when $\mathscr{F}(C)$ is a free ultrafilter have been made by Kirman and Sondermann [98], Schmitz [132], and Armstrong [3]. Since much of this work is quite technical, I shall only identify some of its major themes.

Both Schmitz [132] and Armstrong [3] generalize Theorem 20A, but in different ways involving A3. Schmitz shows that the theorem remains valid when $\mathscr{P}$ is any set of weak-order profiles such that any subprofile of linear orders on three alternatives can be found in some $P \in \mathscr{P}$. Armstrong considers a *Boolean algebra* $\mathscr{N}$ of subsets of N, so $N \in \mathscr{N}$, $I \in \mathscr{N} \Rightarrow I^c \in \mathscr{N}$, and $I, J \in \mathscr{N} \Rightarrow I \cup J \in \mathscr{N}$. He then takes $\mathscr{P}$ as the set of all $\mathscr{N}$-measurable weak-order profiles, where P

is $\mathcal{N}$*-measurable* if, for all $x,y \in X$, $\{i : x >_i y\}$ is in $\mathcal{N}$. Theorem 20A remains valid when ultrafilters are defined in an appropriate way with respect to the coalitional algebra $\mathcal{N}$.

Analyses of power distributions are carried out in [3, 98, 132] under additional structure for N that goes well beyond A1$^+$. At minimum, it is assumed that there is a finitely-additive measure space $(N, \mathcal{N}, \eta)$, where $\mathcal{N}$ is a Boolean algebra of subsets of N and η is a *nonnegative measure* on $\mathcal{N}$ into $[0, \infty]$, i.e., $\eta(I) \geq 0$ for all $I \in \mathcal{N}$, $\eta(N) > 0$, and $\eta(I \cup J) = \eta(I) + \eta(J)$ [with $\infty + n = \infty + \infty = \infty$] for all disjoint I and J in $\mathcal{N}$. This is the structure used in [3]. Schmitz [132] and Kirman and Sondermann [98] assume also that $\mathcal{N}$ is a *σ-algebra* (the union of a countable collection of sets in $\mathcal{N}$ is in $\mathcal{N}$) and that η is *countably additive* ($\eta(I_1 \cup I_2 \cup \cdots) = \eta(I_1) + \eta(I_2) + \ldots$ when the I_j are mutually disjoint coalitions in $\mathcal{N}$), and [98] assumes further that $\eta(N) < \infty$ and that the measure space is *atomless*. This means that there is no *η-atom*, i.e., no $I \in \mathcal{N}$ such that $\eta(I) > 0$ and $\eta(J) = 0$ for every $J \subset I$ in $\mathcal{N}$.

We interpret $\eta(I)$ as a measure of the *a priori* weight, numerosity, or importance of coalition I. It can be used to characterize the powers of decisive coalitions in $\mathcal{F}(C)$. Examples of egalitarian measures are the counting measure $\eta(I) = |I|$ for $N = \{1, 2, 3, \ldots\}$, and uniform (Lebesgue) measure on $N = [0, 1]$. The first of these has η-atoms and $\eta(N) = \infty$; the second is atomless with $\eta(N) = 1$.

We shall say that C *behaves dictatorially* within this formulation if for every $\varepsilon > 0$ there is a nonempty $I \in \mathcal{N} \cap \mathcal{F}(C)$ such that either $\eta(I) < \varepsilon$ or I is an η-atom with $\eta(I) \geq \varepsilon$. The corresponding A4-type condition is

A4$^+$. *C does not behave dictatorially.*

Thus A4$^+$ is violated if there are measurable decisive coalitions with arbitrarily small weights, or if there are η-atoms with positive weights that are decisive coalitions.

The following result [132], which generalizes and extends Proposition 5 in [98], implies that some C based on free ultrafilters behave dictatorially, while others do not.

THEOREM 21 *Suppose C on* $\mathcal{A} \times \mathcal{P}$ *is a social choice function that satisfies* A1$^+$, A2, *and* A3 *with* $(N, \mathcal{N}, \eta)$ *a countably-additive measure space and* $\mathcal{N}$ *a σ-algebra of subsets of N.*

(A) *If* $\eta(N) < \infty$, *then* A4$^+$, A5, A6, *and* A7 *cannot all hold*;

(B) *If* $\eta(N) = \infty$ *and if* $I_j \in \mathcal{N}$ *and* $\eta(I_j) < \infty$ *for all* I_j *in some countable partition of* N, *then* A4$^+$, A5, A6 *and* A7 *are consistent.*

Conclusion (A) is illustrated by Lebesgue measure η on $[0, 1]$. If A5, A6, and A7 hold, then either $\mathscr{F}(C)$ is a fixed ultrafilter with a standard dictator i with $\eta(\{i\}) = 0$, or else $\mathscr{F}(C)$ is a free ultrafilter that contains decisive coalitions in $\mathcal{N}$ of arbitrarily small measure. In the latter case there is a technical construct referred to as an "invisible dictator" which, under a certain transformation of the problem, behaves much like a standard dictator.

Conclusion (B) is illustrated by the counting measure η on $N = \{1, 2, 3, \ldots\}$, with $\mathcal{N}$ the set of all subsets of N. In this case, every free ultrafilter $\mathscr{F}_1$ that includes $\mathscr{F}_0$ satisfies A4$^+$. See also the example in [51]. The invisible dictator concept also applies here. What it shows is that there is vastly more imbalance in power than that suggested earlier by the failure of anonymity: see [3, 98, 132] for details.

Generalizations of Theorem 21 appear in [3]. The three-part summary on p. 73 of [3] is especially useful.

11. RELATED THEOREMS

We conclude by noting results that were motivated by Arrow's theorem but are not directly in the mainstream of social choice functions as considered in this monograph. Four aggregation contexts will be considered. They concern equivalence relations, probability distributions, decisions under uncertainty, and strategic voting.

Equivalence relations

An *equivalence relation* $\approx_0$ on X is a reflexive ($x \approx_0 x$), symmetric ($x \approx_0 y \Rightarrow y \approx_0 x$) and transitive binary relation on X. We consider the aggregation of equivalence relations $\approx_1, \ldots, \approx_n$ on X into a consensus equivalence relation $\approx$ on X. Each $\approx_i$ could refer to a strong similarity relation on X for a particular criterion used to judge similarity. Then $\approx$ says which items in X are similar on the basis of $\approx_1$ through $\approx_n$.

Although there need be no underlying notions of asymmetric orderings behind the equivalence relations, the ensuing theorem will be stated in the choice function mode to show clearly the connections to Theorem 1. Profiles of equivalence relations will be denoted by E and E', with corresponding individual equivalence relations $\approx_i$ and $\approx_i'$. In terms of C, we define aggregate equivalence $\approx_E$ by

$$x \approx_E y \text{ if } x = y, \text{ or if } x \neq y \text{ and } C(\{x, y\}, E) = \{x, y\}.$$

The theorem is a slight specialization of the first part of Theorem 5 in [126].

THEOREM 22 *Suppose C is a social choice function on $\mathcal{A} \times \mathcal{E}$ that satisfies*

E1. *N is a nonempty finite set,*
E2. *X is a finite set with at least three elements, and $\mathcal{A}$ contains every two-element subset of X,*
E3. *$\mathcal{E}$ is the set of all functions from N into the set of equivalence relations on X,*
E5. *For all $E \in \mathcal{E}$, $\approx_E$ is an equivalence relation on X,*
E6. *For all $x,y \in X$ and all $E \in \mathcal{E}$, if $x \approx_i y$ for all $i \in N$ then $x \approx_E y$, and if not $(x \approx_i y)$ for all $i \in N$ then not $(x \approx_E y)$,*
E7. *For all $x,y \in X$ and all $E,E' \in \mathcal{E}$ if $x \approx_i y \Leftrightarrow x \approx_i' y$ for all $i \in N$, then $x \approx_E y \Leftrightarrow x \approx_{E'} y$.*

Then there is a nonempty $I \subseteq N$ such that, for all $x,y \in N$ and all $E \in \mathcal{E}$,

$$x \approx_E y \Leftrightarrow (x \approx_i y \text{ for all } i \in I).$$

Thus, under suitable passive (E5) and active (E6) intraprofile conditions, plus an interprofile independence condition E7, aggregate equivalence is governed by a sort of oligarchy I in N. If $i \notin I$, then $\approx_i$ is ignored in the aggregation. If $i \in I$ then $x \not\approx_E y$ whenever $x \not\approx_i y$. If an existential condition like the following is imposed, then we get an impossibility theorem:

E4. *For every $i \in N$ there is an $E \in \mathcal{E}$ and $x,y \in X$ such that $x \not\approx_i y$ and $x \approx_E y$.*

Since it is easy to imagine settings in which global equivalence should perhaps hold if and only if equivalence holds for every criterion, i.e., $I=N$, the impact of such an impossibility theorem may be minimal.

Probability distributions

Let S denote a finite set of three or more mutually disjoint and exhaustive events, or *states*, and for each $i \in N$ let p_i be a probability distribution on S, so that $p_i(s) \geq 0$ for all $s \in S$, and $\sum_S p_i(s) = 1$. We consider the aggregation of p_1 through p_n, as characterized by a profile $p = (p_1, \ldots, p_n)$, into a consensual probability distribution f_p on S.

There is a sizable literature on this type of aggregation, represented in small part by [23, 24, 35, 101, 107, 108, 148, 151, 152]. Since probability measurement can be viewed as a normalized form of ratio-scale measurement, our discussion of ratio scales in Section 9 can be recast in the probability context.

In the following theorem, $\mathcal{P}$ will be used to denote profiles of probability distributions, such as p with associated individual distributions p_i, and q with associated individual distributions q_i. The aggregate distribution based on profile p is denoted by f_p. The theorem is an application of Theorem 2 in [126].

THEOREM 23 *Suppose f maps $\mathcal{P}$ into real valued functions on S such that*

P1. *N is a nonempty finite set,*
P2. *S is finite with $|S| \geq 3$,*
P3. *$\mathcal{P}$ is the set of all functions from N into the set of probability distributions on S,*
P5. *For all $p \in \mathcal{P}$, f_p is a probability distribution on S,*
P6. *For all $p \in \mathcal{P}$ and all $s \in S$, if $p_i(s) = p_j(s)$ for all $i,j \in N$, then $f_p(s) = p_1(s)$,*
P7. *For all $p,q \in \mathcal{P}$ and all $s \in S$, if $p_i(s) = q_i(s)$ for all $i \in N$, then $f_p(s) = f_q(s)$,*
P7′. *f is continuous in p, i.e., if $p(1), p(2), \ldots$ in $\mathcal{P}$ converge to $p \in \mathcal{P}$, then $f_{p(1)}, f_{p(2)}, \ldots$ converge to f_p.*

Then there exist nonnegative numbers w_i that sum to 1 over N such that, for all $p \in \mathcal{P}$ and all $s \in S$,

$$f_p(s) = \sum_{i \in N} w_i p_i(s).$$

Condition P6 is a Pareto equality condition or unanimity condition within each state, and P7 is an independence condition which says that the consensus probability for state s shall depend only on the individuals' probabilities for that state. The final condition is a typical continuity axiom.

The conclusion of the theorem identifies f as a member of the class of weighted additive aggregators. If an obvious anonymity condition is added, we get $w_i = 1/|N|$ for each i. So long as $|N| \geqslant 2$, this clearly satisfies the existential condition which says that for every $i \in N$ there is a $p \in \mathcal{P}$ such that $f_p \neq p_i$.

Additional conditions are needed in the context of a theorem like Theorem 23 if an impossibility theorem is to be obtained. An example of such a condition [35] is that there exist three states $r,s,t \in S$ such that $f_p(r) = [f_p(r) + f_p(s)][f_p(r) + f_p(t)]$ whenever $p_i(r) = [p_i(r) + p_i(s)][p_i(r) + p_i(t)]$ for all $i \in N$. This is a special instance of the independence condition which says that, for all $A,B \subseteq S$, $f_p(A \cap B) = f_p(A)f_p(B)$ whenever $p_i(A \cap B) = p_i(A)p_i(B)$ for all $i \in N$. When $|N| \geqslant 2$, it is inconsistent with the conclusion of Theorem 23 and the assumption that $w_i > 0$ for at least two individuals.

Decision under uncertainty

Hylland and Zeckhauser [86] combine probability aggregation with utility aggregation to obtain an impossibility theorem in the setting of group decision making under uncertainty. As in the preceding subsection, they consider the set $\mathcal{P}$ of probability profiles. In addition, they work with a set $\mathcal{U}$ of utility profiles. Each $U \in \mathcal{U}$ assigns a utility function u_i on $X \times S$ to each $i \in N$, where X is a finite set of courses of action. A combined profile is then a pair $(p, U) \in \mathcal{P} \times \mathcal{U}$. It assigns a probability distribution p_i on S and a utility function u_i on $X \times S$ to each individual in N.

They assume that the p_i for p are combined to yield a consensus probability distribution f_p on S. Independently, the u_i for U are to be merged into a consensus utility function g_U on $X \times S$. Then, for every $(p, U) \in \mathcal{P} \times \mathcal{U}$, the choice set $C(X, (p, U))$ is to be a nonempty subset of courses of action $x \in X$ that maximize expected social utility $\sum_S f_p(s) g_U(x, s)$ and are not Pareto dominated in individual expected utilities.

Two more conditions are imposed on f. One is the weakening of

P6 which says that $f_p = p_1$ if $p_i = p_j$ for all $i,j \in N$. The other is the no-dictator condition mentioned above ($f_p \neq p_i$ for some p). Hylland and Zeckhauser then prove that these conditions on group decision making are inconsistent when each of N, S, and X is a finite set with at least two elements. Their theorem is quite strong in terms of the demands placed on probability aggregation, which are minimal. The power to obtain their inconsistency conclusion stems from the extra structure provided by $\mathcal{U}$ and the assumption that probabilities and utilities are aggregated independently of each other.

Strategic voting

Although strategic voting lies outside the main concerns of this monograph, I find it impossible not to acknowledge the seminal contributions of Gibbard [72] and Satterthwaite [130] to social choice impossibility. I will therefore conclude with a version of their basic theorem.

The question addressed by Gibbard and Satterthwaite is whether it is possible to design a social choice function that makes a *unique* choice from X for each profile P of weak orders on X that is not dictatorial and is strategyproof or nonmanipulable. We say that i *manipulates* C at P if there is a $P' \in \mathcal{P}$ with $>_j' = >_j$ for all $j \neq i$ such that, when $C(X, P) = \{y\}$ and $C(X, P') = \{x\}$, $x >_i y$. Then C is said to be *nonmanipulable* or *strategyproof* if there is no i and no $P \in \mathcal{P}$ such that i manipulates C at P. In other words, there is no situation (X, P) at which some individual can obtain a preferred outcome by "voting" a weak order $>_i'$ that differs from his true or sincere weak order $>_i$.

Their answer is given by our final theorem. For convenience, let $X^* = \cup_{\mathcal{P}} C(X, P)$, the effective range of C. Also let $\{x\} \gtrsim_i \{y\}$ mean the same thing as $x \gtrsim_i y$. Because the theorem uses $\mathcal{A} = \{X\}$, we designate its conditions in the manner used at the end of Section 5. See in particular Theorems 8A and 8B, where X is the only set assumed to be in $\mathcal{A}$.

THEOREM 24 *Suppose C on $\{X\} \times \mathcal{P}$ is a social choice function that satisfies* B1 *and* B3, *and has* $|C(X, P)| = 1$ *for every* $P \in \mathcal{P}$ *with* $|X^*| \geq 3$. *Then the following cannot both hold*:

B4°. *For every $i \in N$ there exists a $P \in \mathcal{P}$ and $x,y \in X^*$ such that $x >_i y$ and $C(X, P) = \{y\}$,* B7^o. *For all* $P, P' \in \mathscr{P}$ *and all* $i \in N$, *if* $>_j = >'_j$ *for all* $j \in N \backslash \{i\}$, *then* $C(X, P) \gtrsim_i C(X, P')$.

The effective range X^* is required to contain at least three alternatives, and the nondictatorship condition B4^o applies to the possible outcomes in X^*. The passive intraprofile condition of the theorem is the unique-choice requirement $|C(X, P)| = 1$. The only vestige of an active intraprofile condition is $|X^*| \geqslant 3$, but this may be more properly viewed as an existential condition, along with B4^o.

The theorem's interprofile condition is the nonmanipulability condition B7^o. Although this is unlike interprofile conditions discussed earlier, it may be compared to monotonicity conditions that vary one $>_i$ at a time.

The thrust of Theorem 24, that either there is a dictator or C is manipulable, is blunted somewhat by the requirement of unique choices. Relaxation of this requirement was considered by Kelly [94] and has subsequently been addressed by many others.

There is by now a large literature on strategic voting and the allied topics of incentive compatibility and implementation. A few references that discuss these matters and provide access to this literature are [36, 62, 76, 95, 117].

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