DISCUSSION:

THE TOTAL EVIDENCE THEOREM FOR PROBABILITY KINEMATICS*

PAUL R. GRAVES

Department of Philosophy
University of California, Irvine

L. J. Savage and I. J. Good have each demonstrated that the expected utility of free information is never negative for a decision maker who updates her degrees of belief by conditionalization on propositions learned for certain. In this paper Good's argument is generalized to show the same result for a decision maker who updates her degrees of belief on the basis of uncertain information by Richard Jeffrey's probability kinematics. The Savage/Good result is shown to be a special case of the more general result.

L. J. Savage (1954) and I. J. Good (1967) have each shown that if degrees of belief are updated by conditionalization, the expected utility of information obtained at negligible cost is never negative. In this paper I will demonstrate that this result generalizes to a decision theory based on Richard Jeffrey's (1965) Probability Kinematics. The Savage/Good result will be seem to be a special case of the more general result for Probability Kinematics. My strategy will be first to outline Good's argument and then restate it, making the necessary changes to generalize the result. Although my exposition will in broad outline follow Good, there is an important difference in detail.

Good treats the problem atemporally. He considers a decision maker who has a body of evidence and is trying to determine if she should use all of her available evidence, or possibly discount some part of the evidence if that maximizes her expected utility. Good argues that the decision maker cannot increase her expected utility by ignoring a piece of information she already possesses, the atemporal *total evidence principle*. But the problem of assessing the utility of information that a decision maker does not now have, but can acquire in the future at negligible cost, has an essential temporal facet. Good assimilates the temporal case to the atemporal case, and then argues atemporally. I want to make the temporal facet of the argument explicit. What follows is therefore not strictly Good's argument, but rather an explicitly temporal extension of his argument.

*Received April 1986; revised August 1986.

Philosophy of Science, 56 (1989) pp. 317–324.

With this disclaimer in mind, I will now turn to outlining the argument. Good's argument implicity assumes:

(1) r mutually exclusive and exhaustive hypotheses, $K_1, \ldots, K_r$;
(2) For each hypothesis K_i, a prior degree of belief $P(K_i)$;
(3) An exhaustive partition E of the possible experimental[1] outcomes $E = [e_1, \ldots, e_n]$;
(4) For each hypothesis K_i and experimental outcome e_k, a conditional degree of belief, $P(K_i/e_k)$;
(5) None of the e_k is such that if it were the actual experimental outcome, this would change the conditional probabilities $P(K_i/e_k)$ for any i, k;
(6) A choice C among s possible acts or classes of acts $A_1, \ldots, A_s$;
(7) A system of utilities for each possible combination of actions A_j and hypotheses K_i, $U(A_j \;\&\; K_i)$;
(8) The cost of experimentation is negligible.

With this set of assumptions, Good's argument proceeds as follows: The prior expected utility of an action is the average of its utilities for each of the K_i, weighted according to the prior degree of belief in each of the K_i:

$$U(A_j) = \mathrm{SUM}_i P(K_i) U(A_j \;\&\; K_i) \tag{I}$$

To maximize expected utility, the decision maker should select the value of j which maximizes the value of (I). Thus, the prior expected utility of a choice of actions is equal to the expected utility of the action which maximizes (I).

$$\begin{aligned} U(C) &= \mathrm{MAX}_j \mathrm{SUM}_i P(K_i) U(A_j \;\&\; K_i) \\ &= \mathrm{MAX}_j \mathrm{SUM}_k \mathrm{SUM}_i P(K_i) P(e_k/K_i) U(A_j \;\&\; K_i) \end{aligned} \tag{II}$$

The posterior expected utility of an act A_j, given the experimental outcome e_k is:

$$U(A_j \;\&\; e_k) = \mathrm{SUM}_i P(K_i/e_k) U(A_j \;\&\; K_i) \tag{III}$$

For any experimental outcome e_k, the expected utility of a choice among the possible actions will again be maximized by the A_j that maximized (III).

$$U(C \;\&\; e_k) = \mathrm{MAX}_j \mathrm{SUM}_i P(K_i/e_k) U(A_j \;\&\; K_i) \tag{IV}$$

[1]By "experimental outcome" I here mean only the decision maker coming to have a degree of belief equal to 1 for one of the e_k. She may employ any means that seems reasonable to her to achieve this degree of belief. Any such means will count as "experimentation" for the present purposes.

Here the essential temporal facet of the problem appears. The argument's next step requires the decision maker to determine her prior expectation of her posterior degrees of belief for each of the e_k. If she is going to conditionalize on the experimental outcome, her posterior degree of belief in one of the e_k must be one. But before the experiment, she has no way to determine which of the e_k is true, so her prior expectation of her posterior degree of belief in any of the e_k must be represented by some number between zero and one. This uncertainty in the outcomes is represented by taking the average of the expected utilities for each of the possible experimental outcomes weighted according to the prior degrees of belief in each of the experimental outcomes. Thus the expected utility of a choice of actions after experiments is:

$$U(C \;\&\; E) = \mathrm{SUM}_k P(e_k)\mathrm{MAX}_j \mathrm{SUM}_i P(K_i/e_k)U(A_j \;\&\; K_i) \tag{V}$$

By Bayes' Law this is equivalent to:

$$U(C \;\&\; E) = \mathrm{SUM}_k\, P(e_k)\mathrm{MAX}_j \mathrm{SUM}_i\, P(e_k/K_i)P(K_i)/P(e_k)\; U(A_j \;\&\; K_i) \tag{VI}$$

which is equivalent to:

$$U(C \;\&\; E) = \mathrm{SUM}_k \mathrm{MAX}_j \mathrm{SUM}_i P(K_i)P(e_k/K_i)\; U(A_j \;\&\; K_i). \tag{VII}$$

Since (VII) differs from (II) only in the order of the SUM_k and MAX_j operators, and since $\mathrm{SUM}_k\mathrm{MAX}_j f(k,j)$ is always at least as great as $\mathrm{MAX}_j\mathrm{SUM}_k f(k,j)$, (Good 1967, p. 320) this suffices to show the expected utility of a choice of actions posterior to experimentation is always at least as great as the expected utility of the choice of actions prior to experimentation, provided that the cost of the experiment is negligible. Hence the expected utility of new information is never negative. With a few minor changes, Good's argument can be generalized from a decision theory based on updating degrees of belief by conditionalization to a decision theory based on updating degrees of belief by Probability Kinematics.

The first change that must be made is to generalize the characterization of experimental outcomes. Good's argument presupposes that by experiments we learn for certain which of the e_k is true and we conditionalize on this information. The generalized theory allows for cases where the experimental[2] result is not the certainty of one of the e_k, but rather a new coherent probability distribution which assigns a degree of belief for each

[2]Here again "experimentation" is intended in the very weak sense as anything which brings about a new coherent system of degrees of belief in the decision maker.

element e_k of E. I will refer to the generalized product of experiment as a 'result', and I will reserve 'outcome' to mean learning from experiments in the more restricted sense used in Good's argument. Thus, learning e_1 for certain is an experimental *outcome*. Learning that e_1 and e_2 are equally likely and no other e_k is possible is an experimental *result*, since it is a coherent assignment of degrees of belief to the elements of E. An outcome is the special case of a result where the degree of belief for one of the e_k is 1. I will suppose that the decision maker has a prior system of degrees of belief associating some degree of belief $P_0(e_k)$ with each of the e_k of E, and the result of experimentation is a new, possibly identical, system of degrees of belief for each of the e_k of E.

There are several ways to represent these systems. I will represent a system of degrees of belief as the conjunction that exhaustively assigns degrees of belief to each of the elements e_k of the partition E of the possible experimental outcomes. I will use R_m to represent the m-th of the t possible experimental results, and R_0 to represent the prior system of degrees of belief, the result of not experimenting. R may be thought of roughly as a function from classes of experimental observations to systems of degrees of belief, where the observations within any class are equivalent in the sense that any observation in a class yields the same posterior degrees of belief for each of the e_k as any other observation in that class. Each conjunct of R_m will be a statement of the form $P_m(e_k) = z_{km}$, where $P_m(e_k)$ is the degree of belief assigned to the k-th element of E by R_m, and z_{km} is some number between 0 and 1 inclusive. Thus:

$$R_m = [P_m(e_1) = z_{1m} \,\&\, \ldots \,\&\, P_m(e_n) = z_{nm}] \tag{A}$$

$$R_0 = [P_0(e_1) = z_{1,0} \,\&\, \ldots \,\&\, P_0(e_n) = z_{n0}] \tag{A$'$}$$

Even though R_m is a proposition, this is consistent with Jeffrey's claim ([1965] 1983, p. 165) that what is learned from experiments need not be expressible as a proposition, because R_m represents not what is learned, but rather the effect that learning has on our degrees of belief. Again, the sort of experimental outcomes Good employs are just those special cases of these generalized results where the q-th conjunct is of the form $P_m(e_q) = 1$, and the rest of the conjuncts are of the form $P_m(e_k) = 0$.

The second change that must be made in Good's argument is a generalization of (5). Good's argument requires that E is a sufficient partition in that each of the e_k preserves all of the conditional degrees of belief $P(K_i/e_k)$. Jeffrey's Probability Kinematics also requires that each of the conditional degrees of belief $P(K_i/e_k)$ remain unchanged by the experimental results (Jeffrey 1965, p. 169). Therefore the R_m must be restricted to possible experimental results which preserve the conditional degrees of belief $P(K_i/e_k)$. For all i, k, m:

$$P(K_i/e_k \,\&\, R_m) = P(K_i/e_k). \tag{B}$$

This entails, for all i, k, m:

$$P(K_i/e_k \,\&\, P_m(e_k) = z_{km}) = P(K_i/e_k) \quad \text{(B}'\text{)}$$

which Brian Skyrms (1980, Appendix 2) has shown to be a sufficient condition for updating degrees of belief by Probability Kinematics. Thus (B) formalizes the assumption that the decision maker expects to update her degrees of belief by Probability Kinematics.

For each of these possible experimental results R_m, the decision maker has a prior degree of belief, $P_0(R_m)$. At the very least she thinks all of R_m equally probable, or more typically she has various degrees of belief for each of the R_m. Since the R_m exhaust the possible experimental results,

$$\text{SUM}_m\, P(R_m) = 1 \quad \text{(D)}$$

Miller's Principle, together with (D), yields an important relation between the prior degree of belief for any of the e_k, and the expectation of the posterior degree of belief for that e_k. The decision maker has a prior degree of belief, $P_0(e_k)$, for each of the e_k in E, and for each of the possible experimental results R_m, a prior expectation of her posterior degree of belief for each of the e_k given that R_m, that is $P_m(e_k)$. But since the experimental results are unknown, and rational action depends on the probability of each of the e_k, what is needed is the prior expectation of the unconditional posterior degree of belief in each of the (e_k). Since SUM_m $P(R_m) = 1$, these prior expectations of posterior degrees of belief for each of the e_k must be the weighted average of the degrees of belief in the e_k for each of the possible experimental results. That is,

$$P(e_k) = \text{SUM}_m P(R_m)\, P_m(e_k) \quad \text{(E)}$$

The problem is to determine the relationship between these prior expectations of posterior degrees of belief for the e_k and the prior degrees of belief for the e_k. The answer to this problem is given by *Miller's Principle*:

$$P(A/P_0(A) = r) = r \quad \text{(M)}$$

Michael Goldstein (1983) and Bas van Fraassen (1984) have shown that a decision maker whose degrees of belief did not satisfy Miller's Principle would be incoherent because her prior degree of belief in some e_k differs from her expectation of her posterior degree of belief. In one sense it is reasonable to expect the posterior degree of belief to differ from the initial degree of belief. A decision maker's posterior degree of belief that a tossed coin comes up heads will be either 1 or 0. But her prior expectation of her posterior degree of belief must be based on a mixture of her degrees of belief for each of the possible experimental outcomes. If the decision maker believed this mixture differed from her prior degree of belief in any of the e_k, rationality would require her to adjust either her prior degrees of belief, or her expectations of her future degrees of belief. Thus, although each of the e_k may have a new probability $P_m(e_k)$, the only expectations of posterior degrees of belief the decision maker can coherently assign to the e_k, prior to experiment, are the weighted average of her prior partial degrees of belief for each of the possible experimental results. The relevant instance of Miller's Principle is:

$$P(e_k/P_0(e_k) = z_{k0}) = z_{k0} \tag{F}$$

This, together with (E) entails that the weighted average of the degrees of belief in e_k for each of the possible results is equal to the prior degree of belief in e_k.

$$\text{SUM}_m\, P(R_m)\, P_m(e_k) = P_0(e_k) \tag{G}$$

If the decision maker believed this average differed from her prior degree of belief, Miller's Principle would require her to adjust her prior degrees of belief, either of the results $P(R_m)$, or of the $P_0(e_k)$.[3] Given these changes in the background assumptions we can now state the generalized argument.

In Jeffrey's Probability Kinematics, the probability of a hypothesis K_i on the basis of experimental result R_m is:

$$P(K_i/R_m) = \text{SUM}_k P(K_i/e_k)P_m(e_k) \tag{H}$$

Hence, the initial probability of K_i will be the value obtained from (H) for R_0, the result of not experimenting.

$$P(K_i) = P(K_i/R_0) = \text{SUM}_k P(K_i/e_k P_0(e_k) \tag{G}$$

Substituting this expression for $P(K_i)$ in (II) yields the prior expected utility of choice C.

$$U(C) = \text{MAX}_j \text{SUM}_i \text{SUM}_k P(K_i/e_k)P_0(e_k)U(A_j \;\&\; K_i) \tag{VIII}$$

By (G), (VIII) is equivalent to:

$$\begin{aligned} \text{U(C)} &= \text{MAX}_j \text{SUM}_i \text{SUM}_k P(K_i/e_k)\text{SUM}_m \\ &\quad P(R_m)P_m(e_k)U(A_j \;\&\; K_i) \\ &= \text{MAX}_j \text{SUM}_m \text{SUM}_i \text{SUM}_k P \\ &\quad (K_i/e_k)P(R_m)P_m(e_k)U(A_j \;\&\; K_i) \end{aligned} \tag{IX}$$

The posterior expected utility of an act A_i given a particular experimental result R_m is represented by using (H) to generalize (III) from ex-

[3]Notice that Miller's principle is required, for the same reasons, to argue from (IV) to (V).

perimental *outcomes* to experimental *results*:

$$U(A_j \ \& \ R_m) = \mathrm{SUM}_i \mathrm{SUM}_k P(K_i/e_k)P_m(e_k)U(A_j \ \& \ K_i) \quad \text{(X)}$$

The expected utility of a choice C among the possible actions given the experimental result R_m will equal the expected utility of the act A_j which maximizes (X), that is:

$$U(C \ \& \ R_m) = \mathrm{MAX}_j \mathrm{SUM}_i \mathrm{SUM}_k$$

$$P(K_i/e_k)P_m(e_k)U(A_j \ \& \ K_i) \quad \text{(XI)}$$

Thus, the expected utility of a choice C among possible actions after experimentation is the average of the expected utilities for the actions which maximize utility for each of the possible experimental results R_m, weighted according to the prior degrees of belief of the R_m,

$$U(C \ \& \ E) = \mathrm{SUM}_m P(R_m) \mathrm{MAX}_j \mathrm{SUM}_i \mathrm{SUM}_k$$

$$P(K_i/e_k)P_m(e_k)U(A_j \ \& \ K_i) \quad \text{(XII)}$$

But by algebra, this is equivalent to

$$U(C \ \& \ E) = \mathrm{SUM}_m \mathrm{MAX}_j \mathrm{SUM}_i \mathrm{SUM}_k P(K_i/e_k)P(R_m)$$

$$P_m(e_k)U(A_j \ \& \ K_i) \quad \text{(XIII)}$$

Since (XIII) differs from (IX) only in the order of the first two operators, and since $\mathrm{SUM}_m \mathrm{MAX}_j f(m,j)$ is always at least as great as MAX_j $\mathrm{SUM}_m f(m,j)$, the expected utility of a choice C among actions after experimentation is always at least as great as the expected utility of the choice prior to experimentation for a decision maker who updates her degrees of belief by Probability Kinematics, provided that the cost of experimentation is negligible. Since the theorem has been proved for decision makers with arbitrary degrees of belief in the various possible results R_m, in particular it has been shown for decision makers who only alter their degrees of belief on the basis of experimental results where the posterior probability of one of the e_k is one. Hence the total evidence principle for conditionalizers is a special case of the theorem for Probability Kinematics. Q.E.D.[4]

[4]One of the referees has suggested that I should consider the case where there are infinitely many possible experimental results, "i.e. any apportionment of final probability over members of the salient partition". The referee correctly points out that with arbitrarily many possible results "the general case can be approximated as closely as you please by the finite one", and hence the theorem is established for such approximations. If the $P(R_m)$ are thought of as discrete probabilities lying within probability intervals, then the general case should fall out smoothly as the limit of the SUM over the intervals.

REFERENCES

Goldstein, M. (1983), "The Prevision of a Prevision", *Journal of the American Statistical Association 78*: 817–819.
Good, I. J. (1967), "On the Principle of Total Evidence", *British Journal for the Philosophy of Science, 17*: 319–321.
Jeffrey, R. [1965] (1983), *The Logic of Decision*, 2nd ed. Chicago: University of Chicago Press.
Savage, L. J. [1954] (1972), *Foundations of Statistics*, rev. ed. New York: Dover Publications.
Skyrms, B. (1980), *Causal Necessity*. New Haven: Yale University Press.
van Fraassen, B. (1984), "Belief and the Will", *Journal of Philosophy 81*: 235–256.