THE OK CORRAL AND THE POWER OF THE LAW (A CURIOUS POISSON-KERNEL FORMULA FOR A PARABOLIC EQUATION)

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1. *Introduction*

Two lines of gunmen face each other, there being initially m on one side, n on the other. Each person involved is a hopeless shot, but keeps firing at the enemy until either he himself is killed or there is no one left on the other side. Let $\mu(m,n)$ be the expected number of survivors. Clearly, we have boundary conditions:

$$\mu(m,0) = m, \quad \mu(0,n) = n. \tag{1.1}$$

We also have the equation

$$\mu(m,n) = \frac{m}{m+n}\mu(m,n-1) + \frac{n}{m+n}\mu(m-1,n) \quad (m,n \geqslant 1). \tag{1.2}$$

This is because the probability that the first successful shot is made by the side with m gunmen is $m/(m+n)$. On using the recurrence relation (1.2) together with the boundary condition (1.1), the computer produces Table 1 below, in which

$$m = 8192+k, \quad n = 8192-k, \quad d(m,n) = \sqrt{(m^2-n^2)} = 128\sqrt{(2k)}.$$

TABLE 1

k	$\mu(m,n)$	$d(m,n)$
0	904.382093	0
8	914.922249	512
32	1059.405803	1024
128	2024.198251	2048
512	4093.080505	4096
2048	8191.520732	8192
8192	16384.000000	16384

The 'deterministic' solution $d(m,n)$ is produced by the following standard reasoning (Lanchester's combat model). We consider the time-evolution

$$\frac{dx(t)}{dt} = -y(t), \quad \frac{dy(t)}{dt} = -x(t), \tag{1.3}$$

of an obvious deterministic analogue of our system. We see that $x(t)^2 - y(t)^2$ is constant: the orbits are hyperbolas. If $x(0) > y(0)$, then the final value x_F for x (with $y_F = 0$) will satisfy

$$x_F^2 - 0^2 = x(0)^2 - y(0)^2, \quad \text{so } x_F = d(x(0), y(0)).$$

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We would expect this method to give good results if m is significantly bigger than n; and Table 1 confirms this.

Some quantification that the approximation is good when the m and n values are not close is provided by the fact that

$$\sum_{m+n=N} d(m,n) \sim \frac{2N^2}{3}, \quad \sum_{m+n=N} \mu(m,n) = \frac{(N+1)(2N+1)}{3} \quad (N \geqslant 2). \tag{1.4}$$

The first of these results is obvious. Direct proof of the second is left as an exercise; an indirect proof is given at (3.5) below.

However, when the initial values m and n are almost equal, random forces will play a dominant part in deciding which hyperbola will be followed finally. We consider the case when $m = n$. The computer produces Table 2, in which

$$K := 3^{-1/4}\pi^{-1/2}\Gamma(3/4) = 0.52532558. \tag{1.5}$$

TABLE 2

m	$\mu(m,m)$	$2Km^{3/4}$
2048	319.556354	319.857107
4096	537.627362	537.933399
8192	904.382093	904.692518

The accuracy is surprising.

That a '3/4' power law looks plausible was first observed in simulation studies which also produce results of surprising accuracy; and it was these simulations which prompted the investigation begun here.

It is proposed to give a more rigorous treatment in a later paper which will, amongst other things, consider questions of accuracy of approximation and of simulation, and seek to interpolate between the 'deterministic' and 'stochastic' extremes considered here.

For now, we make a heuristic study of things.

2. *Heuristic explanation of Table* 2

For the discrete situation, let us now count time as being equal to the number of people killed. We pretend that, throughout the evolution, $m-n$ stays small relative to $m+n$. We are going to consider replacing the pair $(m^2-n^2, m+n)$ (chosen to reflect the hyperbola property) by a pair of continuous variables (u, v). We note that $m+n$ decreases deterministically at rate 1, so we assume the same for v. At the first step,

$$m^2-n^2 \text{ is changed by } 2n-1 \text{ with probability } \frac{m}{m+n},$$

$$\text{by } 1-2m \text{ with probability } \frac{n}{m+n}.$$

The variance of the change is therefore

$$\frac{m}{m+n}(2n-1)^2 + \frac{n}{m+n}(1-2m)^2 - \frac{(n-m)^2}{(m+n)^2} \approx 4mn \approx v^2.$$

We can therefore think of the random evolution of the pair (u, v) as a diffusion process in which v decreases deterministically and u has variance parameter v^2, so that the generator $\mathcal{G}$ is given by

$$\mathcal{G} := -\frac{\partial}{\partial v} + \frac{1}{2} v^2 \frac{\partial^2}{\partial u^2}.$$

Now we think of (m, n) as approximated by a pair of continuous variables (x, y), and of $\mu(m, n)$ as approximated by $H(x, y) = h(u, v)$, where $u = x^2 - y^2$ and $v = x + y$. The true boundary representing the final situation when either $x(t)$ or $y(t)$ is zero is $u = \pm v^2$, which has a cusp at $(0, 0)$. However, we fix on the idea that the random orbit $(x(t),\ y(t))$ will eventually be essentially hyperbolic, $u = \text{constant} = x_F^2$ or y_F^2, and we trust that we can run the diffusion process *further*, until $v = 0$, with little further change in u. We therefore wish to solve $\mathcal{G}h = 0$ with boundary condition $h(r, 0) = \surd(|r|)$ on the axis $v = 0$. The sensible thing to do is to write $h(u, v) = f(u, w)$, where $w := v^3/3$. To clarify the situation, let us summarize notations:

$$\mu(m, n) \approx H(x, y) = h(u, v) = f(u, w) = g(a, b),$$

the last one being utilized later.

We obtain the heat equation

$$\frac{\partial f}{\partial w} = \frac{1}{2} \frac{\partial^2 f}{\partial u^2}$$

in the half-plane $\{(u, w):\ -\infty < u < \infty,\ \ w \geqslant 0\}$ with boundary condition $f(r, 0) = \surd(|r|)$ on the axis $w = 0$. Hence, with $w = v^3/3$, and with the substitution $s = r^2/(2w)$ so that $dr = w^{1/2}(2s)^{-1/2}\, ds$, we have, using the well-known formula for the solution of the heat equation,

$$\begin{aligned} H(\tfrac{1}{2}v, \tfrac{1}{2}v) = f(0, w) &= \int_{-\infty}^{\infty} \frac{\exp(-r^2/2w)}{\surd(2\pi w)} \surd(|r|)\, dr \\ &= 2 \int_0^{\infty} \frac{e^{-s}}{\surd(2\pi w)} (2ws)^{1/4} w^{1/2} (2s)^{-1/2}\, ds \\ &= 2 \frac{2^{-3/4} w^{1/4}}{\surd \pi} \int_0^{\infty} e^{-s} s^{-1/4}\, ds \\ &= 2 \frac{(\tfrac{1}{2}v)^{3/4}}{3^{1/4} \surd \pi} \Gamma(3/4), \end{aligned}$$

so that

$$\mu(m, m) \approx 2Km^{3/4}. \tag{2.1}$$

Since we have assumed moderately strongly that $m - n$ remains small relative to $m + n$ throughout the evolution, and since we have in effect replaced a cusped boundary by a straight line, the accuracy of Table 2 seems astonishing: it is even reasonable to conjecture a much deeper result, namely, that the difference between the two sides of (2.1) converges to a constant as $m \to \infty$. A similar phenomenon was observed in connection with the related 'Mabinogion sheep problem' in Williams [**2**]; but there, rigorous treatment was far easier. For a fascinating development of that Mabinogion problem, which shows that diffusion approximation may not quite be what one expects, see Chan [**1**].

We end this part of the paper by checking out one further aspect of the accuracy of the diffusion approximation for the current problem. Let

$$a = \tfrac{1}{2}(x-y), \quad b = \tfrac{1}{2}(x+y), \quad g(a,b) = H(x,y) \approx \mu(m,n).$$

Then

$$\mathscr{G} = \frac{1}{2}\frac{a}{b}\frac{\partial}{\partial a} - \frac{1}{2}\frac{\partial}{\partial b} + \frac{1}{8}\frac{\partial^2}{\partial a^2}.$$

By symmetry, $\partial g/\partial a = 0$ when $a = 0$. Hence,

$$\begin{aligned}\mu(m+1,m-1) \approx g(1,m) &\approx g(0,m) + \frac{1}{2}\frac{\partial^2 g}{\partial a^2}(0,m)\\ &= H(m,m) + 2\frac{\partial}{\partial m}H(m,m)\\ &\approx \left(1+\frac{3}{2m}\right)\mu(m,m),\end{aligned}$$

using the formula (2.1). This argument leads us to conjecture that

$$r(m) := 2m\left\{\frac{\mu(m+1,m-1)}{\mu(m,m)} - 1\right\} \to 3,$$

and the values

$$r(2048) = 3.0018039, \quad r(4096) = 3.0011988, \quad r(8192) = 3.0007752,$$

provide strong support.

3. *The method of characteristics*

Define, for $0 < \alpha < 1$ and $0 < \beta < 1$,

$$M(\alpha,\beta) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mu(m,n)\,\alpha^m\beta^n. \tag{3.1}$$

The recurrence relation (1.2) leads to

$$\alpha\frac{\partial M}{\partial\alpha} + \beta\frac{\partial M}{\partial\beta} = S(\alpha) + S(\beta) + \alpha\beta\frac{\partial M}{\partial\alpha} + \alpha\beta\frac{\partial M}{\partial\beta},$$

where, from the boundary condition (1.1), we have

$$S(\alpha) := \sum_{m=0}^{\infty} m^2\alpha^m = \frac{\alpha(1+\alpha)}{(1-\alpha)^3}.$$

Hence

$$\frac{\alpha}{1-\alpha}\frac{\partial M}{\partial\alpha} + \frac{\beta}{1-\beta}\frac{\partial M}{\partial\beta} = R(\alpha,\beta) := \frac{S(\alpha)+S(\beta)}{(1-\alpha)(1-\beta)}. \tag{3.2}$$

Thus

$$\frac{d}{dt}M(\alpha(t),\beta(t)) = R(\alpha(t),\beta(t)), \tag{3.3a}$$

where $(\alpha(\cdot),\beta(\cdot))$ is a 'characteristic curve' in the unit square, satisfying

$$\alpha'(t) = \frac{\alpha(t)}{1-\alpha(t)}, \quad \beta'(t) = \frac{\beta(t)}{1-\beta(t)}. \tag{3.3b}$$

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For some constants c_1 and c_2, we shall have, when $t+c_1$ and $t+c_2$ are in $(-\infty, -1)$,

$$\ln(\alpha(t))-\alpha(t) = t+c_1, \quad \ln(\beta(t))-\beta(t) = t+c_2. \tag{3.4}$$

The fact that we cannot write $\alpha(t)$ in closed form is a nuisance. A much more serious difficulty is that all characteristic curves emanate from $(0,0)$ (when $t=-\infty$). (We know the value of M on the bottom and left edges of the unit square. On the top and right edges, $M=\infty$.)

This method quickly yields

$$M(\alpha,\alpha) = \frac{4}{3(1-\alpha)^3} - \frac{1}{(1-\alpha)^2} - \frac{1}{3}, \tag{3.5}$$

which is equivalent to the second result at (1.4) (and, indeed, led us to (1.4)). However, further explicit calculations by this route become complicated.

The fundamental problem is: *how do we see the* 3/4-*power law in* (3.2) *and* (3.3)? It would seem that we have to consider $\mathbb{C}$-valued functions α and β to resolve this.

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References

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