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Continuous Parameter Estimation Model: Expanding the Standard Statistical Paradigm

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Abstract

Our classical paradigm of statistics considers parameters such as means, standard deviations, correlations, and standard errors as discrete parameters. This paper shows we can expand the "Discrete Parameter Estimation Model" (DPEM) to consider most parameters as continuous, both in the sense of (a) the parameter varying continuously as a function of other variables and of (b) each case having a separate score on a continuum related to each parameter. The "Continuous Parameter Estimation Model" (CPEM) is a broader paradigm which includes DPEM and includes analyzing parameters such as correlations and standard errors as conditional, that is, to vary as a function of other variables, through the use of standard statistics. (As a paradigm paper, there are no unique derivations nor new statistical formulas, but there is an expanded perspective for how variables are conceptualized and how analyzes may proceed.)

Keywords : CPEM, DPEM.

1. Introduction

Our usual statistics model considers statistical estimates as either a single discrete value or as a set of discrete estimates. An example of a single discrete value is a standard deviation given for a sample; an example of a set of discrete estimates is when the sample is divided into males and females with the standard deviation or other parameter estimated separately for each group. The purpose of this paper is to expand the traditional "Discrete Parameter Estimation Model" (DPEM) to a "Continuous Parameter Estimation Model" (CPEM).

The discrete estimation of standard deviations as a function of age can illustrate the difference between DPEM and CPEM. With discrete estimation, a decision must be made as to how the sample is to be divided. In one study, it may be divided into 5-year groups and in another it may be divided into 10-year groups. Once the groups have been defined, the sample is split in the age groups and standard deviations are computed in each of the groups. Of course the results may vary depending upon whether age is divided by 5 year or 10 year groups.

The discrete method has several inherent limitations for a parameter such as the standard deviation and a variable such as age. These limitations include:

1) The groups are arbitrarily defined. Hopefully the choices are wise, but that depends upon prior knowledge. If the groups are too broad, then detail is lost. If they are too fine, then the estimates contain sampling errors as well as true differences.

2) It is difficult to evaluate the replication of results across studies because the division into categories, such as age, in one study may not be the same as in another study. This may be a problem for meta-analysis.

3) Any method of division usually is theoretically inappropriate because one would expect a continuous shift across, for example, age; the parameter does not really shift suddenly because one has a birthday even though such shifts happens in discrete estimation.

4) Few statistical packages give a ready significance test or confidence intervals for whether a set of variance parameters from multiple groups differ, whether any pair of parameters differ (although several such tests have been proposed), or whether the standard deviations are curvilinearly related to age.

These problems all occur because age is continuous. Any impact of age is expected to be continuous. But the traditional statistical paradigm, DPEM, gives discrete estimates of standard deviations.

Although all the limitations of using DPEM, when that parameter may be a function of a continuous variable, can be important, it is the theoretical limitation that should give one pause. DPEMing denies or makes a rough approximation of the underlying continuous function. Consider a case where an ability is related across ages 6 to 14, with the standard deviation increasing with age. The changes in the standard deviation would not be a discrete shift from day d to day $d + 1$. Instead there is continual change, the speed of which may vary from year to year. This represents a continuous parameter estimation situation for which a discrete estimation, such as computing the standard deviation for ages 6 to 9 and 10 to 12, would be recommended if and only if a continuous method does not exist. The same problem that is illustrated for variances applies to other statistics, such as skew, kurtosis, correlations, and residual errors (as noted below).

Using the DPEM could give a better or a poorer estimate depending on how the groups were constructed. If the groups were numerous, then the estimation would be close to the results from using the CPEM. But if broader age groups were made, the results would be a function of the luck of selecting the groups.

In addition to being more theoretically appropriate, a method of continuous parameter estimation for parameters such as variances would be useful for several reasons.

1) CPEMing would eliminate the theoretical problem of treating a continuous function as if it were discrete. Variances could be analyzed for changes just as easily as means are analyzed for change.

2) The analysis would have more power because artificially creating a discrete variable from a continuous variable ignores information.

3) Because there is no arbitrary division of the continuous variable into artificial groups, any statistical differences in results which are a function of how the sample is divided are eliminated; each study can identify the phenomena as well as another study of the same population.

4) Research studies can be more readily compared because there would be no differences in how the continuous variable was artificially changed into a discrete variable.

5) The procedure suggested below allows standard statistical procedures to be used to provide parameter estimates, significance tests if desired, confidence intervals, and conditional estimates. Since these are standard procedures, it requires no new statistic procedures or programs and can be done with current computer programs.

CPEM is a broader paradigm that includes DPEM, the classical approach. Indeed the statistics discussed here are so basic each formula is in a thousand statistics texts across multiple disciplines. It is the use of those formulae that changes in going from DPEM to CPEM. (For references to the statistics noted in this paper, see any standard statistics text. The author has, despite using both standard search engines and consultation with associates, not found any direct prior work or usage of the CPEM.)

The purpose of this paper is to present a CPEM model. This allows for continuous estimation of parameters, including variances and correlations. The paper shows how standard statistic procedures can be applied to these. Examples of continuous parameter estimation (which can be called CPEMing) are given for several statistics. Hopefully this paper will provide understanding of CPEM basics and encourage applying the model to other parameters as well.

2. Continuous Parameter Estimation Model (CPEM)

To explain the CPEM, an illustration will be used with two variables, age (*X*) and math (*Y*). Table 1 contains each case's identification in column 1. Assume the first score (column 2) is age and the second (column 3) is a scale of mathematical ability whose standard deviation increases with age. The standard deviation is low with young children because almost everyone fails all the problems. But as the children increase in age, the spread becomes larger. Some children are still unable to do two-digit addition whereas others are doing more complex multiplication and division. Then the standard deviation of the scores for the older children will be higher than for the younger children.

Table 1. CPEM Parameters for Math (*Y*): variance, skew, and kurtosis, and as a function of age (*X*).

Scores by Case

	Observed Scores										
ID	*X*	*Y*	y=(Y-MY)	y^2	Z_y	Z_{y^2}	Z_{y^3}	Z_{y^4}	Z_X	Z_XZ_y	E^2
1	6	1	-1.25	1.56	-0.96	0.93	-0.89	0.86	-1.34	1.29	0.09
2	6	2	-0.25	0.06	-0.19	0.04	-0.01	0.00	-1.34	0.26	0.22
3	6	1	-1.25	1.56	-0.96	0.93	-0.89	0.86	-1.34	1.29	0.09
4	7	1	-1.25	1.56	-0.96	0.93	-0.89	0.86	-0.45	0.43	0.55
5	7	2	-0.25	0.06	-0.19	0.04	-0.01	0.00	-0.45	0.09	0.00
6	7	3	0.75	0.56	0.58	0.33	0.19	0.11	-0.45	-0.26	0.63
7	8	1	-1.25	1.56	-0.96	0.93	-0.89	0.86	0.45	-0.43	1.40
8	8	3	0.75	0.56	0.58	0.33	0.19	0.11	0.45	0.26	0.13
9	8	4	1.75	3.06	1.35	1.81	2.44	3.29	0.45	0.60	1.27
10	9	1	-1.25	1.56	-0.96	0.93	-0.89	0.86	1.34	-1.29	2.62
11	9	3	0.75	0.56	0.58	0.33	0.19	0.11	1.34	0.77	0.01
12	9	5	2.75	7.56	2.12	4.48	9.49	20.08	1.34	2.84	2.13
Descriptive Statistics											
Mean	7.50	2.25	0.00	1.68	0.03	1.00	0.67	2.33	0.00	0.49	0.76
SD	1.118	1.299	1.299	1.949	1.000	1.154	2.809	5.419	1.00	0.989	0.856
SE Mean	.34	.39	.39	.59	.30	.35	.85	1.63	.30	.30	.26

Note. Calculations by Excel. E^2 is the squared error of estimation $(Z_Y - r_{xy} \times Z_x)^2$

Columns 4 and 5 are the deviations from the mean of the mathematics scale, *y* (i. e., *Y* – Mean *Y*), and the squares of the deviations from the mean, y^2 (i. e., $(Y\text{-Mean }Y)^2$) These are presented in statistics texts as a step in calculating the overall variance but y^2 is, in CPEMing, of special interest. Note that y^2 has a score for each case.

The columns of Table 1 following y^2 include the second, third, and fourth powers of the *Z* scores for *Y* given in column Z_y. Note that these also have a separate score for each case.

Table 1 contains the means and standard deviations of all the columns. The mean of y^2, y^2/N, has the same definition as the variance of the math scores. Because the square root of the variance is by definition the standard deviation, the mean of squared deviations of the column of y^2 (1.69) equals the square of the standard deviation of Y (the square of the standard deviation of column *Y*, 1.299, is 1.69). The means of Z^3 and Z^4 are the sample skew and kurtosis. (Textbook formulas may be more complex because their equations are for computing variance, skew, and kurtosis directly from the data instead of Z scores, but Z scores are more instructive for this paper. Formulas are also more complex when corrected for small sample bias in estimating the population skew and kurtosis.)

By definition, the mean of the column Z_xZ_y, $\sum Z_xZ_y/N$, is the correlation between X and Y. To compute the correlation for this example, column Z_x contains the Z scores for X. Each case's Z_x score was multiplied by Z_y to give the column Z_xZ_y. The mean of Z_xZ_y is the correlation, and indicates that X and Y correlate .49. Note that each case has a separate score, Z_xZ_y.

Because the correlation between X and Y, .49, is known, we can estimate Z_y from Z_x and compute the error $(Z_y - r_{xy}Z_x)$ and square that for the final column of Table 1 which contains the squared error for each case. (The conclusions would be the same if these were the squared residuals from estimating the raw scores of Y from the raw scores of X.) The mean of E^2, $\sum E^2/N$, is the variance of the errors of estimation; its square root is the standard deviations of the Z score errors of estimation, known in the literature as the "standard error of estimate." The mean of E^2 has a separate score for each case.

From the data in Table 1, two types of analyses can be computed. The first are tests and confidence intervals for the means of each column. In addition to the raw score mean, these include the mean variance, mean skew, mean kurtosis, and the mean Z_xZ_y (which is the correlation in the sample from which the means and *SD*s were computed to calculate the Zs). Confidence intervals for each mean can be computed and examined; for example, the standard deviation of the mean E^2 gives confidence intervals for the squared standard error. Because the mean of Z_xZ_y is by definition the correlation between X and Y, it's confidence intervals can be examined to determine if they include zero (the classical test of the mean for significance).

For example, does the confidence interval for the mean for skew include zero? If not, then the data are skewed. Does the confidence interval for the mean of kurtosis include that of the normal curve, 3? If not, then the data are either too flat or too peaked. If the answer to either question is "no", then the data depart from normality. (Of course if there are no hypotheses as to expected skew or kurtosis, then both skew and kurtosis would be tested simultaneously for significance to protect the family wide alpha level for Type I errors (Hotelling's T^2)).

The means and confidence intervals are the standard statistical paradigm formulas. Textbooks often include formulas that do not require calculating all the columns of Table 1, but the logic is the same.

CPEM points out that the variance, skew, kurtosis, correlation, and the standard error of estimate are all means and it allows for a second, new set of analyzes from scores like those in Table 1 in addition to the means. Because we have separate scores for each case for y^2 (variance), Z_y^3 (skew), Z_y^4 (kurtosis), Z_xZ_y (correlations), and E^2 (errors of estimation), each set of scores can be analyzed just as any other variable. For example, to test if the variances differ across groups compared to the grand mean, the mean y^2 is computed for each group in a simple ANOVA. To test if the variances differ from one group to another in addition to differences from mean differences, new y^2s would be computed as deviations from the mean of each group and then compared (the y^2 in the table contains variation from both the between group variance and the within group variance). The skew and kurtosis indices can likewise be dependent variables to evaluate if the data may be non-normal; for example, whether the data are non-normal for males but not for females is evaluated by using the skew and kurtosis columns of Table 1 as dependent variables with gender as the independent variable.

These examples used a nominal independent variable but, because CPEM has separate scores for each case for these variables, correlations can be also computed with any other variables we choose. For example, what is the correlation of education or intelligence with the columns of Table 1? Do more educated people have greater variance, or a different skew or kurtosis? Does the variable correlate differently with other variables; if so, can it be estimated more or less accurately? CPEM provides for ready examination of questions such as these. The correlations among the variables of Table 1 are in Table 2.

Table 2. Correlations among columns of Table 1.

Variables		1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	
X	1.	1.00										
Y	2.	.49	1.00									
y	3.	.49	1.00	1.00								
y^2	4.	.43	.58	.58	1.00							
Z_y	5.	.47	.99	.99	.55	1.00						
Z_y^2	6.	.43	.58	.58	1.00	.55	1.00					
Z_y^3	7.	.45	.83	.83	.90	.82	.90	1.00				
Z_y^4	8.	.42	.66	.66	.96	.65	.96	.96	1.00			
Z_x	9.	1.00	.49	.49	.43	.47	.43	.45	.42	1.00		
Z_xZ_y	10.	-.05	.50	.50	.69	.49	.69	.70	.72	-.05	1.00	
E^2	11.	.64	.20	.20	.62	.17	.62	.45	.54	.64	-.12	1.00

Many of these statistics are best computed sequentially. For example, it may be that y^2 differs significantly between males and females. Then all estimates using Z scores would calculate the Zs using each separate gender's estimated y^2 to compute the standard deviation.

Among CPEM's tests are those associated with the column of Z_xZ_y. This allows analysis of the conditions under which the correlation shifts. If males differ from females when Z_xZ_y is used as the dependent variable, then there is better prediction for one gender than another. If X is a possible item for a scale and Y is a criterion, using Z_xZ_y as the dependent variable and ethnicity as the independent variable tests whether the item is equally predictive for all ethnic groups. (Note that if each gender has the same mean and SD, then this is analyzing the correlations directly; otherwise it is analyzing an index related to the correlation since $\sum Z_xZ_y/N$ is only the correlation in the total sample from which the Zs were computed. A complete analysis would also use appropriate x^2s and y^2s to evaluate whether the correlation was shifting due to differences in the variances which would be indicative of restriction of range.)

Correlating the errors from the raw score estimation of Y from X with X itself is a valuable analysis. Correlating X with E_y^2 addresses the question of the degree to which the prediction is equally accurate for all levels of X. If X is uncorrelated with the squared residuals, E_y^2, then the prediction is the same at all levels of X (a condition known as homoscedastic). But if there is a correlation, then X is not equally predictive of Y at all levels (a condition known as heteroscedastic). A simple positive correlation means that the error of estimation increases as X increases, and so X is most predictive at the low end. If the correlation were negative instead of positive, then X would predict poorly at the low end but better at the high end.

The errors may be a curvilinear function of X. One example is when there is an end effect for the lowest scores restricting the variations in scores. Then a polynomial regression (computed by using powers of X as predictors in a multiple regression) of X to the E_y^2 would be computed. If the correlation is curvilinear (found when Z_X^2 or Z_X^3 is significant), computing samples of the relationship is usually the basis for interpretation.

In the example, X correlates .64 with the E_y^2. This means that the errors of prediction increase with increases in X (see Table 1, comparing X with the squared residual column). The sample does not have a homogeneous error variance; instead the errors of estimation are a function of age. The regression of E_y^2 is given by the following raw score regression equation:

$$\text{Raw Math Score } E_y^2 = .75 \text{ X Age} - 3.937$$

This formula is used to compute the estimated squared error for any age group. Because the estimate is for an age group, it is the mean for that group. The estimated standard deviation, that is, the Standard Error of Estimate or *SE* of estimate, is the square root of the estimated squared error. For example, the estimated squared error at age 7 is .75 X 7 - 3.937 = 1.313; the *SE* of estimate is the square root of the variance, 1.15, which is smaller than the standard deviation of the entire sample. Note that testing the older half against the younger half may not be significant despite the fact that the correlation of age with the math squared deviations is significant. This possible result is because CPEM uses all the data from all ages and hence is more sensitive to predicted relationships than discretized estimates based on nominalized continuous variables.

Because correlations can be analyzed as a function of other variables, CPEM can be useful in analysis of tests and their items. Items are selected based on their means, standard deviations, and correlations in the classical test theory and with variations of these in item response theories, and CPEMing provides for item and test statistics conditional upon other variables. Reliabilities and validities use correlations, and so conditional reliabilities and validities are useful parts of CPEM. For example, an end effect means a test is both less valid and less reliable at the end of the test where the effect is found. CPEM allows for analysis of the shift in reliability and validity as a function of end effect. A test score for a particular person can be interpreted taking into account the conditional validities and reliabilities specific to that person, including end effects.

Because variables such as y^2 and Z_xZ_y are just variables within our set of variables, any analysis of theoretical interest that can be processed by the usual statistical methods can be used. This includes regression analysis and ANOVA. It also includes partialing out other variables and multivariate analyzes.

3. The Question of Normality

But wait! How can it be appropriate to compute correlations with variables raised to the second, third, and fourth powers? These variables cannot be distributed normally if the raw scores are normally distributed.

The first point regarding normality is that CPEM allows more powerful ways of examining distributions to evaluate whether they are normal. Analyses of variance, skew, and kurtosis conditional on other variables, possible with CPEM, provide information about the distributions of variables and of error terms. Simulation studies on the impact of normality may be more informative by CPEMing.

If a CPEM parameter such as y^3 or Z_xZ_y differs as a function of another variable, as in the example with age, the raw score distribution and uncorrected residuals may be disturbed accordingly. When, for example, y^3 is regressed on age, the impact of age as a disturbance in the distribution is eliminated from the residuals of y^3 and these corrected values are more likely to be normally distributed across samples. In this case, CPEM actually increases the possibly of meeting the assumption of normal distribution of error.

There are two reasons the phrase "normally distributed variables" is used. One reason for desiring normal distributions is because the maximum value of a correlation --or a difference between means--may be reduced if there are variations in distributions. Two variables with different distributions cannot correlate 1.0 with each other (although they may be able to correlate -1.0 if one variable's distribution is the mirror image of the other). Only if all variables have the same symmetric distribution can the possible correlations range from –1.0 to 1.0. However that is not an absolute requirement because high correlations can occur despite some varying distributions, and it may be superseded by other concerns. For example, polynomial regression uses powers of X to predict Y; if X is normal, the square and cube of X cannot be normal. Yet polynomial regression is an accepted procedure for examining possible curvilinear relationships.

Despite differences in distributions that prevent correlations from being 1.0, the use of squares and cubes under conditions preventing them from correlating 1.0 attests to the fact that this is not a disqualifying problem. Indeed as long as a correlation is large enough to be judged useful, the attenuating factors have not wiped out the relationship. It is primarily when a correlation is too low to be useful that one investigates possible attenuating factors such as distributions. CPEM analyses follow in this tradition.

The second reason for assuming normality is that using the standard procedures for confidence intervals and significance tests does include a normality assumption: the assumption that the standard deviations of the errors of that parameter (a) are normally distributed across samples and (b) can be estimated from the current sample. Note that the raw scores can be distributed in any manner whatsoever so long as that does not affect the distributions of the errors upon which the standard errors are based. As an example, consider the test of a mean of Y against zero. Its standard error is $SD_y/(N\text{-}1)^{.5}$. This standard error is an estimate of the standard deviation of means across samples randomly drawn from the same population, but the observed SD_y is a function of y^2. The assumption is y^2 is a sufficiently accurate population estimate not only for the standard deviation but also for the standard deviation of means across samples (called the standard error of the mean).

The first requirement, that the distribution of the statistic computed from many samples is normally distributed, is the easily made. Although the raw distribution of CPEM parameters may include variables that are not normally distributed, the central limit theorem indicates that estimates of means across samples will be normally distributed regardless of the variable's distribution. The principle is similar to those that allow flipping of coins to illustrate the normal curve even though each flip gives a dichotomous result: if you flip enough of them, the distribution becomes normal. It is also illustrated by the fact that adding together six values from uncorrelated rectangular distributions gives scores that are normally distributed. The central limit theorem requires assuming that the samples differ only randomly. Then the statistic will average to the population mean with the means across the samples forming a normal distribution. This appears to be the case with CPEM variables such as such as y^2 and E^2.

The second requirement, that the standard error of our statistic across samples be estimated from terms within our sample, may be problematic because the estimate can be affected by the observed variable distributions. The question then becomes that of whether the sample based estimate of standard errors is affected sufficiently to give a misleading estimate of the standard deviation across samples. For example, some CPEM variables are skewed. Then the mean is not equally distant from the minimum and the maximum scores, which may affect the standard deviation and so the standard error of the mean. The more problematic of these is when the skewed is caused by an end effect.

To illustrate end effects, note that if one sample has a proportion of .95 instead of the population value of .99, there is no way a 1.04 can occur in another sample to average out the .95 to obtain the population value of .99. So the errors are unlikely to be normally distributed.

To examine the impact of an end effect on estimation of standard errors, 500 samples were run for *N*s of 50, 100, and 1000 with a known skewed variable: population proportions of .95, .98, and .99. The *SE*s were computed within each sample by the standard formula for the sample based *SE*s of the mean, and averaged. To check whether the distortion from the end effect produced problems for the sample based *SE*s, *SE*s were computed empirically by computing the observed *SE*s across the 500 samples for each of the three sample sizes for each of the three levels of percentages.

Table 3 contains the average sample based standard errors and the empirical standard errors computed from the multiple samples. Note that the sample based standard errors are close to the ones computed across samples even for the most skewed variable, .99. The sample based *SE*s are generally conservative, so using the sample based standard error of the mean would make slightly less Type I errors than by using the actual standard error (from the empirical simulations). The similarity increases as the *N* increases.

Another example is the *SE* of correlations. If we wish, we could test the significance of the mean of Z_xZ_y by that mean's *SE* based on the observed standard deviation of the Z_xZ_y column (i. e., (*SD* of Z_xZ_y) / $(N\text{-}1)^{.5}$. We also have an exact procedure, Fisher's *Z* technique, which is independent of any distribution in Table 1. It produces a *SE* that can be compared to the *SE* of the mean based on Z_xZ_y. Using the correlation mentioned above and testing the correlation against 0.0 gives a *t* of 1.63 when the *SE* is computed from the data and 1.59 by the Fisher *Z* test. Once again the test using the standard error of the mean is close to the traditional test. The point is not that the traditional tests be abandoned but that treating the CPEM parameters as we do other variables is reasonable despite obvious skew.

It appears that the standard error of the mean of many CPEM variables can be used and these results may generalize by extension to ANOVA and multiple regression standard errors. The usual statistical tests appear to be

Table 3. Standard errors computed by formula and by simulations (500) for end effect data.

Proportion		Standard Error	
		By Formula	Observed
	N = 50		
.95		.0308	.0298
.98		.0198	.0192
.99		.0141	.0132
	N = 100		
.95		.0218	.0219
.98		.0140	.0130
.99		.0099	.0099
	N = 1000		
.95		.0069	.0068
.98		.0044	.0044
.99		.0031	.0032

usable with CPEM. However, until more work is done, it is wise to have a large N if CPEMing is planed.

4. A Further Illustration

In addition to the illustration of the variance differing as a function of age, another simple illustration of CPEM can be given for the testing of homogeneity of variances. Assume that one has two experimental conditions to reduce anxiety--standard desensitization of anxiety and a special Anti-Anxiety Program--and two control groups, one a non-treated control and the other a discussion group as a type of placebo control. To test for mean differences, one assumes the within group variances, that is, the within $y2$ means, are equal. The question is: do these groups differ in their variances on the outcome measure of anxiety?

To test for homogeneity of variances, each person's anxiety score would be subtracted from his or her own group mean, and that difference would be squared to give the y^2s. Using group as the independent variable in a simple ANOVA with the squared differences from the individual group means as the dependent variable provides a significance test as well as confidence intervals. Tests of differences between y^2 means overall or between selected groups can be computed with standard post hoc procedures, such as protected F tests or Scheffe tests. Thus CPEM provides a test of homogeneity of variances in ANOVA designs.

The regression version of testing for homogeneity of variance is to test for homoscedasticity. When the relationship is homoscedastic, the regression equation predicts equally well at all levels of the independent variable. When the relationship is heteroscedastic, then the accuracy of prediction varies. With DPEM heteroscedasticity is seldom checked, and so it is assumed that the prediction is equally good at all levels. CPEM allows this issue to be explicitly tested, and, if heteroscedasticity is found, allows for that information to be used as noted above. It also allows for other modeling of the heteroscedastic effect.

Heteroscadisticity can occur for many reasons, one of which is a curvilinear relationship between X and Y. And there may still be heteroscadisticity in addition to curvilinearity. A case can be made for expecting curvilinear heteroscedastic relationships when predicting behaviors from abilities. Abilities predict if a person can or cannot do the activity under the assumption that they are motivated to do so; under the condition of equal motivation the prediction would be expected to be linear and homoscedastic.

But people may have an ability, as perhaps shown by passing an algebra course, --as perhaps shown by passing an algebra course --but vary in their interest in using it; situations requiring algebra are avoided by some and sought out by others among those who can do algebra. With varying motivation low ability still predicts that they will not do algebra (both low and high motivation predict the same low usage as they can not do it even if they wish to). But the prediction among those who can do algebra is not from abilities but from motivation (as they can all do it if they wish). The model states that algebra ability would predict well among those with low ability as no one can do it but would not predict who among the algebra proficient would actually use it. Thus there should be a curvilinear relationship (a prediction for low scores and none for moderate and high scores) and the standard deviation of errors of prediction should be higher among those with ability than among those without. CPEMing can readily provide SE of estimates for all levels of even curvilinear predictors.

Using the example of the previous paragraph, a set of data was made where an algebra test predicts well those who will never do algebra but the test predicts algebra behavior less among those who can do algebra (but may choose not to). The prediction from algebra scores is curvilinear because it definitely predicts at the low end but flattens towards zero prediction at moderate high math scores (no curvilinear analysis was given for Table 1 because of the small *N*). It is expected to also be heteroscedastic because the prediction is good for low scores but poor for higher scores.

Analyzing the data by polynomial regression found a curvilinear relationship as expected, indicating that both algebra test scores and algebra scores squared were necessary to predict algebra usage well (see Table 4. A). The relationship is reasonably strong. The traditional discrete parameter estimation model gives the SE estimate presented in Table 3. B, following the curvilinear (polynomial) regression analysis.

CPEM analyzes for homogeneity of estimation were conducted by analyzing the squared errors of prediction. These are found by using the formula from 4. A to estimate each observed score, computing the residual by subtracting the estimates from the observed scores, and squaring the residuals. Analyzing for the *SE* estimate from the CPEM model (Table 3. C) uses the squared residuals as the dependent variable and algebra and algebra squared as the independent variables to produce an equation for the mean squared errors (which are converted to *SE* estimates by taking their square root). Because there was a relationship between test scores and the squared errors of prediction ($R = .43$), the errors of estimation are not homogeneous and the relationship is heteroscedastic.

Section D of Table 4 contains the estimated algebra usage score for sample levels of the algebra test scores. It includes the CPEM *SE* estimates for confidence intervals. Section D shows that the actual situation is better described with CPEM than with DPEM. Note that the prediction is excellent for the lowest algebra score, is almost the same as the raw standard deviation for the middle algebra scores but is greater for the highest algebra scores. (The slight drop in predicted scores at the end is not significant.)

In Table 4. D the correlation at each level of the independent variable was estimated by solving the equation of 4. B for *r*, using the *SE* estimate for each level of the independent variable and the same sample standard

Table 4. Discrete and CPEM analysis of error variances in the presence of curvilinear prediction.

A. Curvilinear Regression

Algebra Usage = 1.452 X Algebra -.0979 X Algebra^2 -.2184

R= .76

B. Discrete Parameter Estimation of Errors Of Estimation

$SE_e = SD_Y \times (1 - r_{xy}^2)^{.5}$

$SE_e = 1.756 \times (1 - .76^2)^{.5}$

$SE_e = 1.14$

C. Continuous Parameter Estimation (CPE) of Errors of Estimation (Regression of $(Y-Y')^2$ on Algebra Test)

Mean Squared Residual = .268 X Arithmetic + 0.0

R= .43

D. Sample Predicted Scores With Their Standard Deviations of Errors (SEs)

Test	Algebra Predicted Scores	*SE* estimate Usage (with *r*)
1	1.1	.45 (.92)
3	3.3	.89 (.86)
5	4.6	1.14 (.76)
7	5.2	1.34 (.49)
9	4.9	1.54 (.35)

deviation (to counter restriction of range). The correlation is what would occur if a sample with the *SD* of the total sample showed the same relationship across the range as found within this sub-sample. As expected, the prediction gives an excellent standardized correlation for the lowest level of scores but a low standardized correlation at the highest level of the independent variable. This extra dividend of estimating correlations, standardized for restriction of range, for different levels illustrates how the CPEM perspective can give useful information.

CPEM results for the example are more complicated than discrete results because the situation is complicated; using the traditional paradigm loses considerable information so that a *SE* estimate of 1.14 would be used for all in Table 4. DPEM fails to tell the complete story: scoring low has a correlation of .9 with not using algebra with a low *SE* estimate, but scoring high has much less predictive power as seen in the high *SE* estimate. This is true even though the traditional analysis allowed for curvilinear prediction. For applied purposes, the equation for the prediction of the dependent variable and the equation for the prediction of the *SE* estimate can be computerized so that the appropriate mean and *SE* can be provided for any case as a function of that case's scores on the predicted (and other) variables.

Of course, if none of the additional variables possible by CPEMing show effects, then the analysis is the traditional one since DPEM is a special case of CPEM.

5. Conclusion[a]

The Continuous Parameter Estimation Model suggests a broader model of which traditional statistics (Discrete Parameter Estimation Model) is a special case. CPEM can be used for improved understanding of data. It provides tests of and estimation procedures for heterogeneous variances and standard errors and is also useful for evaluating the sources of the heterogeneity. Although a new conceptualization, it is another application of traditional statistics and so can be computed with any statistics package.

CPEM is recommended for every study on a trial basis because we have so little experience in asking the questions that can be addressed with it. This could involve a number of analyzes in one study. For example, all possible *ZxZy*s for a dozen variables would be examined. To protect the overall alpha level for the study, a multivariate test of an entire set of CPEM parameters are recommended to guide decisions on when the broader CPEM paradigm is useful.[b]

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[a] As a paradigm paper, readers often have an "ah ha" experience which then can be applied without further consulting of this paper; when you use the new insight gained, please (1) cite this paper and (2) send me a copy of your use of it. Thank you.

[b] No references are given for the formulas used in this paper because they are in every introductory statistics book. No references are given to previous discussions of this type model because I (and the several psychologists and statisticians who have reviewed the paper or attended lectures where it was presented) do not know of any previous model like this.