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A new secretary problem with rank-based selection and cardinal payoffs

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Abstract

We present an extension of the secretary problem in which the decision maker (DM) sequentially observes up to n applicants whose values are random variables $X_1, X_2, \ldots, X_n$ drawn i.i.d. from a uniform distribution on [0, 1]. The DM must select exactly one applicant, cannot recall released applicants, and receives a payoff of x_t, the realization of X_t, for selecting the tth applicant. For each encountered applicant, the DM only learns whether the applicant is the best so far. We prove that the optimal policy dictates skipping the first sqrt(n)-1 applicants, and then selecting the next encountered applicant whose value is a maximum.

Keywords: Optimal stopping; Sequential search; Secretary problem

1. Introduction

Suppose a decision maker (DM) observes a sequence of up to n applicants whose values are random variables $X_1, X_2, \ldots, X_n$ drawn i.i.d. from a uniform distribution on [0, 1]. As in the standard secretary problem, the DM has two choices for each applicant: accept or reject. The DM's *payoff* for selecting an applicant with $X_t = x_t$ is itself x_t. Once an applicant is selected the problem terminates; if reached, the nth applicant must be accepted; and, once rejected, an applicant cannot be recalled. Importantly, however, at each stage t the DM only observes an indicator of X_t, where $I_t = 1$ if and only if $x_t = \max\{x_1, x_2, \ldots, x_t\}$; otherwise, $I_t = 0$. In other words, the DM only learns whether each observed applicant is the best so far. Her objective is to maximize her expected payoff.

Thus, the current problem is quite similar to the classical secretary problem (for reviews of secretary problems see Ferguson, 1989; Samuels, 1991). The only difference is that in our problem the DM's payoff is equal to the selected applicant's underlying "true" value, whereas in the classical secretary problem the DM earns a payoff of 1 if she selects the best overall applicant and earns nothing otherwise. Our motivation for this problem is the intuition that in some sequential search situations, DMs might pay close attention to rank-based information, but, ultimately, derive utility from the cardinal (true) value of the selected observation and not from its rank. Consider a trader (hirer) who wants to sell an asset when its price (applicant) is at its maximum during some period of time $[t_{\min}, t_{\max}]$. Though the price ranks are salient in deciding when to sell, presumably she will derive utility that is strictly increasing in cardinal selling price. The nothing-but-the-best payoff scheme of the classical secretary problem fails to capture this.

Supposing our trader does make her selling decisions at each point in time t solely on the basis of the rank of the current price with respect to the previous prices, how can she maximize her expected selling price? And how well might she do given that her earnings will ultimately be based on the price at which the asset is sold and not on the rank of the asset price?

2. The optimal policy

Since the applicants' values are i.i.d. draws from a uniform distribution on [0, 1], the expected value of the tth applicant given that $x_t = \max\{x_1, x_2, \ldots, x_t\}$ is given by

$$E_t = E(X_t | I_t = 1) = \frac{t}{t+1}. \qquad (1)$$

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Note that this is a standard result on the nth order statistic for the uniform distribution.

By the principle of optimality, since $\mathrm{d}E_t/\mathrm{d}t > 0$, if it is optimal to select an applicant with $I_t = 1$, then it is optimal to select an applicant with $I_{t+k} = 1$, $\forall k \geqslant 0$. Obviously, for $1 \leqslant t < n$, it is never optimal to select an applicant for which $I_t = 0$. Let us, therefore, define with smallest t at which it is optimal to select an applicant with $I_t = 1$ for a problem with n applicants as $c_n^\star$. We refer to $c_n^\star$ as the optimal *cutoff*; and the value of the optimal policy for a problem of size n is $V_n^\star \equiv V_n(c_n^\star)$.

Optimal policies and their values can easily be obtained from the proof of the following proposition.

Proposition 1. $c_n^\star \in \{\lfloor n^{\frac{1}{2}} \rfloor, \lceil n^{\frac{1}{2}} \rceil\}$.

Proof. Given a problem of size n, the expected payoff for some arbitrary cutoff $1 \leqslant c \leqslant n$ can be obtained by

$$\begin{aligned} V_n(c) &= \sum_{t=c}^{n-1} \left[\prod_{s=c}^{t-1} \left(\frac{s-1}{s} \right) \right] \left(\frac{1}{t+1} \right) + \left[\prod_{s=c}^{n-1} \left(\frac{s-1}{s} \right) \right] \frac{1}{2} \\ &= \sum_{t=c}^{n-1} \left(\frac{c-1}{t-1} \right) \left(\frac{1}{t+1} \right) + \left(\frac{c-1}{n-1} \right) \frac{1}{2} \\ &= \frac{1}{c+1} + (c-1) \left(\frac{1}{c(c+2)} + \cdots + \frac{1}{(n-2)n} \right) \\ &\quad + \left(\frac{c-1}{n-1} \right) \frac{1}{2} \\ &= \frac{2cn - c^2 + c - n}{2cn}. \end{aligned} \tag{2}$$

Differentiating the last line of Eq. (2) with respect to c, thereby obtaining $\partial V/\partial c = (-c^2 + n)/(2c^2 n)$, and noting that $\partial^2 V/\partial c^2 < 0$ for all permissible values of c, we find that V reaches its maximum at $c = n^{1/2}$. Since V is convex in c, $c_n^\star$, which is integer-valued, must be in $\{\lfloor n^{1/2} \rfloor, \lceil n^{1/2} \rceil\}$. Thus, the proposition is proved. $\square$

3. Conclusion

The secretary problem has received considerable attention by statisticians and applied mathematicians. One reason for this is the problem's surprising optimal policy. Under it, the DM skips the first $r_n^\star - 1$ applicants and then takes the next applicant whose value is a maximum. What is surprising is that $r_n^\star \to e^{-1} n$ as $n \to \infty$, and that in the limit the best overall applicant is selected with probability e^{-1}. A proof of this can be found in Gilbert and Mosteller (1966).

The problem introduced in the current note has a similar, to us, non-intuitive solution: skip the first $c_n^\star - 1$ applicants and then take the next applicant whose value is a maximum, where $c_n^\star = n^{1/2}$. Aside from the formal derivation of this result, we do not yet have an dinner table explanation that expresses why it obtains.

Seale and Rapoport (1997) found that subjects in an experimental study of the classical secretary problem tended to terminate their search earlier than is dictated by the application of the optimal policy. Studies of a number of related problems have tended to find the same general result, namely that subjects do not search enough (e.g., Seale and Rapoport, 2000; Zwick et al., 2003). For most values of n, the optimal cutoffs for the current problem appear considerably earlier than the corresponding cutoffs for the classical secretary problem. Given the strictness of the payoff function for the classical problem, which returns a positive payoff if and only if the best of the n applicants is selected, one wonders whether the results obtained in these experimental studies are an artifact of the problem's unusual payoff scheme. Our aforementioned trader may try to dump her asset when its price is its greatest during some interval, but it seems unlikely that her utility for selling at some prices slightly below the maximum would be zero. Compared to the classical secretary problem, it seems to us that the payoff scheme presented here is more natural.

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